# Traffic on networks: modeling and analysis 

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# Traffic on networks: modeling and analysis 

Régis Monneau

(joint work with C. Imbert)

Paris-Est University

Tours; June 27, 2014

## Great thanks

for the whole organizing committee!!

## A mathematical point of view

$$
u_{t}+H\left(x, D u, D^{2} u\right)=0
$$

$H$ discontinuous in $x$

## A mathematical point of view

$$
u_{t}+H(x, D u)=0
$$

$H$ discontinuous in $x$

## A junction



## Hamilton-Jacobi equations

$$
\begin{aligned}
& \begin{cases}u_{t}^{i}+H_{i}\left(u_{x}^{i}\right)=0, & x>0, \quad i=1, \ldots, N \\
u^{i}=u^{j}=: u, & x=0, \\
u_{t}+F\left(u_{x}^{1}, \ldots, u_{x}^{N}\right)=0, & x=0\end{cases}
\end{aligned}
$$

(1) Motivation, Hamilton-Jacobi equations and optimal control
(2) Doubling of variables revisited
(3) General junction conditions
(9) Homogenization of networks

## Motivation

## The simple divergent road



$$
\gamma_{r}+\gamma_{1}=1
$$

LWR model (Lighthill, Whitham 1955 ; Richards 1956) :

$$
\rho_{t}+(f(\rho))_{x}=0
$$

## The fundamental diagram



## Some references

Existence for $N \geq 1$
[Garavello, Piccoli (2006)]
Uniqueness result for $N=2$ branches
[Garavello, Natalini, Piccoli, Terracina (2007)]
[Andreianov, Karlsen, Risebro (2011)]

Uniqueness for $N \geq 1$ for Hamilton-Jacobi equations
[Achdou, Camilli, Cutri, Tchou; (2013)] : particular Hamiltonians.
[Camilli, Schieborn ; (2013)] : eikonal equations.
[Imbert, M., Zidani (2013)] : particular junction condition [Imbert, M. preprint (2013-2014)] : general junction conditions
Generalizations
[Barles, Briani, Chasseigne ; (2013), (2014)] : hyperplane (higher dimensions). [Rao, Zidani ; (2013)], [Rao, Siconolfi, Zidani ; (2013)] : multi-domains.

## How to get Hamilton-Jacobi equations?

$-u^{i}(x, t)=$ continuous index of the car at $(x, t)$ on the branch $i$


## Getting HJ equations

- Take some (space) primitive of the density $\rho^{i}$ with prefactor $1 / \gamma_{i}$
- change some signs and the orientation of the incoming roads

$$
u_{t}^{i}+H_{i}\left(u_{x}^{i}\right)=0, \quad x>0, \quad i=e, l, r
$$

with

$$
\left\{\begin{array}{l}
\text { convex } \quad H_{i} \quad \text { if } f \text { concave } \\
H_{i}(p)=-\frac{1}{\gamma_{i}} f\left(\gamma_{i} p\right), \quad i=e \\
H_{i}(p)=-\frac{1}{\gamma_{i}} f\left(-\gamma_{i} p\right), \quad i=l, r
\end{array}\right.
$$

$$
f\left(\rho^{i}\left(0^{+}, t\right)\right)=\gamma_{i} f\left(\rho^{e}\left(0^{-}, t\right)\right) \quad \text { for } \quad i=l, r \quad \text { and } \quad \gamma_{e}=1
$$



What is the condition at $x=0$ ?


$$
u_{t}+F\left(u_{x}^{e}, u_{x}^{l}, u_{x}^{r}\right)=0
$$

## Optimal control

If

$$
\left\{\begin{array}{lll}
u_{t}+H\left(u_{x}\right)=0 & \text { for } & x \in \mathbb{R}, \quad t>0 \\
u(x, 0)=u_{0}(x) & \text { for } & x \in \mathbb{R}
\end{array}\right.
$$

with

$$
H(p)=L^{*}(p):=\sup _{q \in \mathbb{R}}\{p q-L(q)\}
$$

then

$$
u(x, t)=\left\{\begin{array}{l}
\inf ^{y \in \mathbb{R},} \\
X(0)=y \\
X(t)=x
\end{array}\right\}\left\{u_{0}(y)+\int_{0}^{t} d s L(\dot{X}(s))\right\}
$$




## Representation formula



$$
J=\bigcup_{i=e, l, r} J_{i}
$$

$$
u(x, t)=u_{o c}(x, t):=\inf _{\left\{\begin{array}{l}
y \in J, \\
X(0)=y \\
X(t)=x
\end{array}\right\}}\left\{u_{0}(y)+\int_{0}^{t} d s L(X(s), \dot{X}(s))\right\}
$$

## Representation formula



$$
J=\bigcup_{i=e, l, r} J_{i}
$$

$u(x, t)=u_{o c}(x, t):=\left\{\begin{array}{l}\inf _{y \in J,} \\ X(0)=y, \\ X(t)=x\end{array}\right\}\left\{u_{0}(y)+\int_{0}^{t} d s L(X(s), \dot{X}(s))\right\}$
with a Lagrangian

$$
L(x, p)= \begin{cases}L_{i}(p) & \text { if } \quad x \in J_{i} \backslash\{0\}, \quad i=e, l, r \\ L_{0} & \text { if } \quad x=0\end{cases}
$$

## Analogy with the swimming pool



## Junction condition with flux limiter $A$

$$
u_{t}(0, t)+F_{A}=0
$$

with

$$
\left\{\begin{array}{l}
F_{A}:=\max \left\{A, \quad \max _{i=e, l, r} H_{i}^{-}\left(u_{x}^{i}\left(0^{+}, t\right)\right)\right\} \quad \text { with } A=-L_{0} \\
H_{i}^{-}(p)=\sup _{q \leq 0}\left\{p q-L_{i}(q)\right\}
\end{array}\right.
$$



## Traffic interpretation of the junction condition

$$
u_{t}(0, t)=\text { passing flux }=\min \left(L_{0}, \quad \frac{1}{\gamma_{e}} f_{D}\left(\rho^{e}\right), \quad \min _{i=l, r} \frac{1}{\gamma_{i}} f_{S}\left(\rho^{i}\right)\right)
$$

Lebacque condition : minimum of Demand ( $f_{D}$ ) and Supply $\left(f_{S}\right)$


Riemann Solver of Garavello, Piccoli, Boundary condition of [Bardos, LeRoux, Nedelec (1979)]

Doubling of variables revisited

The standard doubling of variables method does not work!

$$
\sup \left\{u(x, t)-v(y, t)-\frac{(x-y)^{2}}{\varepsilon}-\ldots\right\} \text { with } x \text { and } y \text { on different branches! }
$$

The standard doubling of variables method does not work!
$\sup \left\{u(x, t)-v(y, t)-\frac{(x-y)^{2}}{\varepsilon}-\ldots\right\}$ with $x$ and $y$ on different branches!

Adaptation with the vertex test function $G$

$$
\sup \left\{u(x, t)-v(y, t)-\varepsilon G\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)-\ldots\right\}
$$

## Doubling of variables

$$
0<\eta \leq H\left(y,-G_{y}\right)-H\left(x, G_{x}\right)
$$

Contradiction if

$$
H\left(y,-G_{y}\right) \leq H\left(x, G_{x}\right)
$$

## Construction of the test function $G$

Set of particular solutions for $p=\left(p^{1}, \ldots, p^{N}\right)$

$$
u_{p, \lambda}(x, t)=-\lambda t+p^{i} x \quad \text { on } \quad J_{i}
$$

Germ

$$
\mathcal{G}_{A}=\left\{(p, \lambda) \quad \text { with } \quad F_{A}(p)=H_{i}\left(p^{i}\right)=\lambda \geq A\right\}
$$

## Construction of the test function $G$

Set of particular solutions for $p=\left(p^{1}, \ldots, p^{N}\right)$

$$
u_{p, \lambda}(x, t)=-\lambda t+p^{i} x \quad \text { on } \quad J_{i}
$$

Germ

$$
\mathcal{G}_{A}=\left\{(p, \lambda) \quad \text { with } \quad F_{A}(p)=H_{i}\left(p^{i}\right)=\lambda \geq A\right\}
$$

Let

$$
w=w_{p, \lambda}(x, t ; y, s)=u_{p, \lambda}(x, t)-u_{p, \lambda}(y, s)
$$

which solves

$$
H\left(y,-w_{y}\right)=\lambda=H\left(x, w_{x}\right)
$$

## Construction of the test function $G$

The function

$$
\widehat{G}(x, t ; y, s)=\sup _{(p, \lambda) \in \mathcal{G}_{A}} \quad\left\{w_{p, \lambda}(x, t ; y, s)\right\}
$$

satisfies

$$
H\left(y,-\widehat{G}_{y}\right)=H\left(x, \widehat{G}_{x}\right)
$$

where $\widehat{G}$ is smooth

## Properties of $G$



## Construction of the test function $G$

Let

$$
G^{0}(x, y)=A+\widehat{G}(x, 1 ; y, 0)
$$

s.t.

$$
G^{0}(x, y) \geq 0=G^{0}(x, x)
$$

## Construction of the test function $G$

Let

$$
G^{0}(x, y)=A+\widehat{G}(x, 1 ; y, 0)
$$

s.t.

$$
G^{0}(x, y) \geq 0=G^{0}(x, x)
$$

But $G^{0}$ is smooth except on the diagonal $\{x=y\}$.

Define $G=G^{\delta}$ as a smoothing of $G^{0}$ on the diagonal such that

$$
\left\{\begin{array}{l}
G(x, y) \geq 0 \quad \text { and } \quad 0 \leq G(x, x) \leq \delta \\
H\left(y,-G_{y}\right) \leq H\left(x, G_{x}\right)+\delta
\end{array}\right.
$$

## General junction conditions for $N=1$

## Monotonicity of $F$



$$
\begin{cases}u_{t}+H\left(u_{x}\right)=0, & x>0 \\ u_{t}+F\left(u_{x}\right)=0, & x=0\end{cases}
$$

## Monotonicity of $F$



$$
\begin{cases}u_{t}+F\left(u_{x}\right)=0, & x=0 \\ \phi_{t}+F\left(\phi_{x}\right) \leq 0, & x=0\end{cases}
$$

$F(\rho)$ non increasing in $\rho$

## Meaning of the junction condition

0

Junction condition not satisfied if the imposed flux is too large


Relaxation


## Relaxed junction condition

## 0

Relaxed junction condition for subsolutions

$$
\left\{\begin{array}{l}
u_{t}+F\left(u_{x}\right) \quad \leq 0, \quad x=0 \quad \text { (Boundary condition) } \\
\text { OR } \\
u_{t}+H\left(u_{x}\right) \quad \leq 0, \quad x=0 \quad \text { (PDE) }
\end{array}\right.
$$

## Relaxed junction condition

## 0

Relaxed junction condition for supersolutions

$$
\left\{\begin{array}{l}
u_{t}+F\left(u_{x}\right) \geq 0, \quad x=0 \quad \text { (Boundary condition) } \\
\text { OR } \\
u_{t}+H\left(u_{x}\right) \geq 0, \quad x=0 \quad \text { (PDE) }
\end{array}\right.
$$

Theorem (Existence, uniqueness, reduction to $F_{A}$-solutions)
If

$$
\left\{\begin{array}{l}
H \text { convex and coercive (i.e. } H(p) \rightarrow+\infty \text { as }|p| \rightarrow+\infty) \\
F \text { continuous and non increasing }
\end{array}\right.
$$

Then there exists a unique (viscosity) relaxed solution $u$ of

$$
\left\{\begin{array}{lc}
u_{t}+H\left(u_{x}\right)=0, & x>0 \\
u_{t}+F\left(u_{x}\right)=0, & x=0
\end{array}\right.
$$

with Lipschitz initial data

$$
u_{\mid t=0}=u_{0}
$$

$$
\left\{\begin{array}{l}
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u_{t}+F\left(u_{x}\right)=0, & x=0
\end{array}\right.
$$

with Lipschitz initial data

$$
u_{\mid t=0}=u_{0}
$$

Moreover there exists a real $A$ such that $u$ is also solution with $F$ replaced by

$$
F_{A}\left(u_{x}\right)=\max \left(A, \quad H^{-}\left(u_{x}\right)\right),
$$





Ramp metering on the A19 in Middlesbrough, UK.

## Reduction to a single test function



Check with test functions

$$
\varphi(x, t)=\psi(t)+\phi_{A}(x) \quad \text { with } \quad \phi_{A}(x)=p_{A}^{+} x
$$



## Application : reformulation of state constraint problems

For $H$ convex (or quasi-convex) and coercive,

$$
\begin{cases}u_{t}+H\left(u_{x}\right) \geq 0, & x \geq 0 \\ u_{t}+H\left(u_{x}\right) \leq 0, & x>0\end{cases}
$$

## (state constraint)

if and only if

$$
\begin{cases}u_{t}+H\left(u_{x}\right)=0, & x>0 \\ u_{t}+H^{-}\left(u_{x}\right)=0, & x=0\end{cases}
$$

( $F_{A}$ condition with $A=-\infty$ )

## Applications for $N \geq 1$ branches

- relation with Ishii solutions
- characterization of subsolutions on the junction point only
- restriction of solutions on a subdomain
- homogenization
- ...


## Homogenization of networks

(Numerical simulations done by G. Costeseque)

## A periodic network



$\varepsilon$

## Graph of $-\bar{H}$

$$
\text { Results for } \gamma=\left[\begin{array}{llll}
0.5 & 0.5 & 0.5 & 0.5
\end{array}\right]
$$



$$
\gamma_{h}=\gamma_{v}=1 / 2, \quad \text { no flux limiter }
$$

## Graph of $-\bar{H}$


$\gamma_{h}=\gamma_{v}=1 / 2, \quad$ with flux limiter $\quad A$


## $-\bar{A}(d, \theta)$

Flux limiter w.r.t. signals spacing and offset


# Conclusion : 

## what we know

## What we can do in general

Infinite networks with Hamiltonians $H\left(x, t, u_{x}\right)$ and $F\left(t,\left\{u_{x}^{i}\right\}_{i}\right)$

Hamiltonians $H\left(x, t, u_{x}\right)$ are convex in $u_{x}$, or quasi-convex


## Open problems

open problem : micro-macro limit?

weakly related to :
Achdou, Tchou, (preprint : 2014),
Galise, Imbert, M. (preprint 2014)

## Open problems

- traffic : time evolution/ optimization of the proportions $\gamma_{i}$ at each node


## Open problems

- traffic : time evolution/ optimization of the proportions $\gamma_{i}$ at each node
- more general non quasi-convex Hamiltonians in 1D
- general junction conditions in higher dimensions: submanifolds, multi-domains.
- mixed first order/second order PDEs


## Thank you




## The case with $N \geq 1$ branches

## Viscosity solution at the junction point



Junction

$$
J=\bigcup_{i=1, \ldots, N} J_{i}
$$

test function

$$
\phi \in C_{*}^{1} \Longleftrightarrow\left\{\begin{array}{l}
\phi \in C(J \times[0, T)), \\
\phi \in C^{1}\left(J_{i} \times(0, T)\right), \quad i=1, \ldots, N
\end{array}\right.
$$

## Viscosity solution at the junction point



## Definition (Relaxed subsolution at the junction point)

The USC function $u$ is a relaxed subsolution at the junction point, if

$$
u \leq \phi \quad \text { with equality at } P_{0}=\left(x_{0}, t_{0}\right) \text { with } x_{0}=0
$$

then at $P_{0}$ :

$$
\left\{\begin{array}{l}
\phi_{t}+F\left(\phi_{x}^{1}, \ldots, \phi_{x}^{N}\right) \leq 0 \\
O R \\
\phi_{t}+H_{i}\left(\phi_{x}^{i}\right) \leq 0 \quad \text { for some index } i
\end{array}\right.
$$

with

$$
\phi^{i}=\phi_{\mid J_{i} \times(0, T)}
$$

We rewrite

$$
\begin{cases}u_{t}^{i}+H_{i}\left(u_{x}^{i}\right)=0, & \text { for } \quad(x, t) \in(0,+\infty) \times(0, T), \quad i=1, \ldots, N \\ u_{t}+F\left(u_{x}^{1}, \ldots, u_{x}^{N}\right)=0, & \text { for } \quad(x, t) \in\{0\} \times(0, T)\end{cases}
$$

as

$$
\begin{equation*}
u_{t}+H\left(x, u_{x}\right)=0 \quad \text { on } \quad J \times(0, T) \quad \text { with } \quad J=\bigcup_{i=1, \ldots, N} J_{i} \tag{1}
\end{equation*}
$$

with

$$
H(x, p)=\left\{\begin{array}{ll}
H_{i}(p) & \text { convex coercive } \\
\text { if } \quad x \in J_{i} \backslash\{0\} \\
F\left(p^{1}, \ldots, p^{N}\right) & \text { non increasing }
\end{array} \text { if } x=0\right.
$$

The PDE

$$
\begin{equation*}
u_{t}+H\left(x, u_{x}\right)=0 \quad \text { on } \quad J \times(0, T) \tag{1}
\end{equation*}
$$

## Initial condition

$$
\begin{equation*}
u(x, 0)=u_{0}(x) \quad \text { for all } \quad x \in J, \quad \text { with } \quad u_{0} \in \operatorname{Lip}(J) \tag{2}
\end{equation*}
$$

## The PDE

$$
\begin{equation*}
u_{t}+H\left(x, u_{x}\right)=0 \quad \text { on } \quad J \times(0, T) \tag{1}
\end{equation*}
$$

Initial condition

$$
\begin{equation*}
u(x, 0)=u_{0}(x) \quad \text { for all } \quad x \in J, \quad \text { with } \quad u_{0} \in \operatorname{Lip}(J) \tag{2}
\end{equation*}
$$

## Theorem (Existence, uniqueness, reduction to $F_{A}$-solutions)

There exists a unique (viscosity) relaxed solution $u$ of (1)-(2).
Moreover there exists a real $A$ such that $u$ is also solution with $F$ replaced by

$$
F_{A}\left(u_{x}^{1}, \ldots, u_{x}^{N}\right)=\max \left(A, \max _{i=1, \ldots, N} H_{i}^{-}\left(u_{x}^{i}\right)\right)
$$

where

$$
A=F\left(\left(H_{1}^{+}\right)^{-1}(A), \ldots,\left(H_{N}^{+}\right)^{-1}(A)\right) \quad \text { or } \quad A=A_{0}:=\max _{i=1, \ldots, N}\left(\min _{\mathbb{R}} H_{i}\right)
$$

