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Bang–bang trajectories with a double switching time in the minimum time problem

L. Poggiolini M. Spadini G. Stefani



NetCo 2014 New trends in optimal control
23-27 June 2014 Tours, France.

The minimum time problem

$$T \rightarrow \min$$

$$\dot{\xi}(t) = f_0(\xi(t)) + \sum_{i=1}^m u_i f_i(\xi(t)),$$

$$\xi(0) \in N_0, \quad \xi(T) \in N_f,$$

$$u = (u_1, \dots, u_m) \in Q := [-1, 1]^m$$

The minimum time problem

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$$u = (u_1, \dots, u_m) \in Q := [-1, 1]^m$$

Assume $(\hat{T}, \hat{\xi}, \hat{u})$ is an admissible triplet satisfying PMP with an adjoint covector

$$\hat{\lambda}: t \in [0, \hat{T}] \mapsto \hat{\lambda}(t) \in T^*M$$

The state space M is a finite dimensional manifold

Problem

Give sufficient second order conditions for $\hat{\xi}$ to be a strong local optimal trajectory

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Definition ((time,state)-local optimality)

The trajectory $\widehat{\xi}$ is a (time,state)-local minimiser if there is a neighborhood $\widetilde{\mathcal{U}}$ of its graph such that $\widehat{\xi}$ is a minimiser among the admissible trajectories whose graph is in $\widetilde{\mathcal{U}}$.

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Give sufficient second order conditions for $\widehat{\xi}$ to be a strong local optimal trajectory

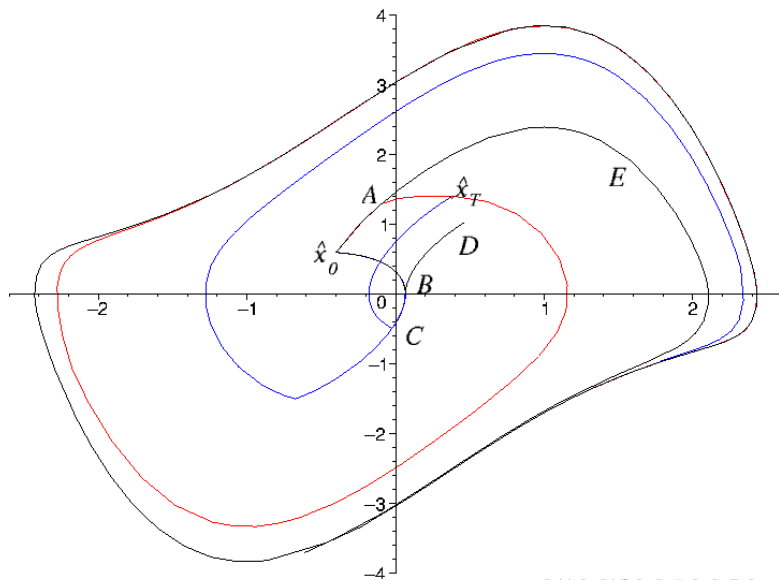
Definition ((time,state)-local optimality)

The trajectory $\widehat{\xi}$ is a (time,state)-local minimiser if there is a neighborhood $\widetilde{\mathcal{U}}$ of its graph such that $\widehat{\xi}$ is a minimiser among the admissible trajectories whose graph is in $\widetilde{\mathcal{U}}$.

Definition (state-local optimality)

The trajectory $\widehat{\xi}$ is a state-local minimiser if there are neighborhoods \mathcal{U} of its range $\widehat{\xi}([0, \widehat{T}])$, \mathcal{U}_0 of $\widehat{\xi}(0)$ and \mathcal{U}_f of $\widehat{\xi}(\widehat{T})$ such that $\widehat{\xi}$ is a minimiser among the admissible trajectories whose range is in \mathcal{U} , whose initial point is in $N_0 \cap \mathcal{U}_0$ and whose final point is in $N_f \cap \mathcal{U}_f$.

Example



Hamiltonians associated to the vector fields

$$F_i : \ell = (p, q) \in T^*M \mapsto \langle p, f_i(q) \rangle, \quad i = 0, 1, \dots, m$$

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Reference Hamiltonian

$$\widehat{F}_t = F_0 + \sum_{i=1}^m \widehat{u}_i(t) F_i$$

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Reference Hamiltonian

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Maximised Hamiltonian

$$F^{\max} : \ell \mapsto \sup_{u \in Q} \left\{ F_0(\ell) + \sum_{i=1}^m u_i F_i(\ell) \right\}$$

Hamiltonian vector fields

$$\vec{F}^{\max}, \vec{F}_t, \vec{F}_i$$

Pontryagin Maximum Principle (PMP)

there exist $\hat{\lambda}: [0, \hat{T}] \rightarrow T^*M$, $\rho_0 \in \{0, 1\}$

such that: $\|\hat{\lambda}(0)\| + \rho_0 \neq 0$

$$\pi \hat{\lambda}(t) = \hat{\xi}(t)$$

$$\frac{d}{dt} \hat{\lambda}(t) = \vec{F}_t(\hat{\lambda}(t))$$

$$\hat{F}_t(\hat{\lambda}(t)) = F^{\max}(\hat{\lambda}(t)) \equiv \rho_0$$

$$\hat{\lambda}(0) \Big|_{T_{\hat{\xi}(0)} N_0} = 0, \quad \hat{\lambda}(\hat{T}) \Big|_{T_{\hat{\xi}(\hat{T})} N_f} = 0$$

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$$\hat{\lambda}(0) \Big|_{T_{\hat{\xi}(0)} N_0} = 0, \quad \hat{\lambda}(\hat{T}) \Big|_{T_{\hat{\xi}(\hat{T})} N_f} = 0$$

if $\rho_0 = 1$, $\hat{\lambda}$ is called **normal extremal**

if $\rho_0 = 0$, $\hat{\lambda}$ is called **abnormal extremal**

Hamiltonian approach to (time, state)–local optimality 1/2

- the flow of an over-maximised Hamiltonian

$$H: \ell \in T^*M \mapsto H(\ell) \in \mathbb{R}$$

$$H(\ell) \geq F^{\max}(\ell) \quad H(\widehat{\lambda}(t)) = F^{\max}(\widehat{\lambda}(t))$$

$$\mathcal{H}: (t, \ell) \in [0, \widehat{T}] \times T^*M \mapsto \mathcal{H}_t(\ell) \in T^*M$$

$$\widehat{\lambda}(t) = \mathcal{H}_t(\widehat{\lambda}(0)), \text{ i.e. } \dot{\widehat{\lambda}}(t) = \overrightarrow{H}(\widehat{\lambda}(t))$$

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$$\widehat{\lambda}(t) = \mathcal{H}_t(\widehat{\lambda}(0)), \text{ i.e. } \dot{\widehat{\lambda}}(t) = \overrightarrow{H}(\widehat{\lambda}(t))$$

- a smooth function $\alpha: M \rightarrow \mathbb{R}$ such that $d\alpha(\widehat{\xi}(0)) = \widehat{\lambda}(0)$

$$\Lambda := \{(d\alpha(x), x)\}$$

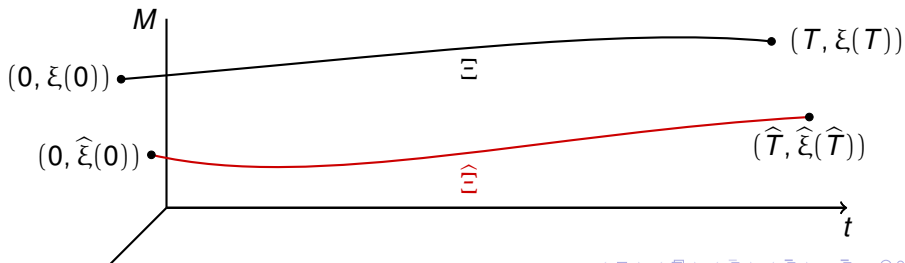
Hamiltonian approach to (time, state)–local optimality 2/2

$$\begin{array}{ccc}
 \bullet & [0, \widehat{T}] \times \Lambda & \xrightarrow{\text{id} \times \mathcal{H}} & [0, \widehat{T}] \times T^*M \\
 & & & \downarrow \text{id} \times \pi \\
 & & & [0, \widehat{T}] \times M \\
 & \swarrow (\text{id} \times \pi \mathcal{H})^{-1} & &
 \end{array}$$

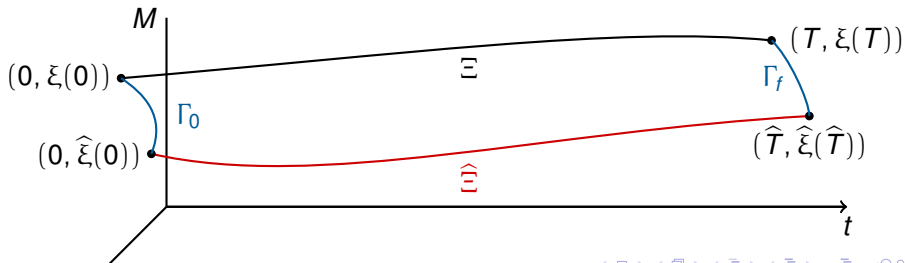
Property

$\mathcal{H}^*(p dq - H_t dt)$ is exact on $[0, \widehat{T}] \times \Lambda$

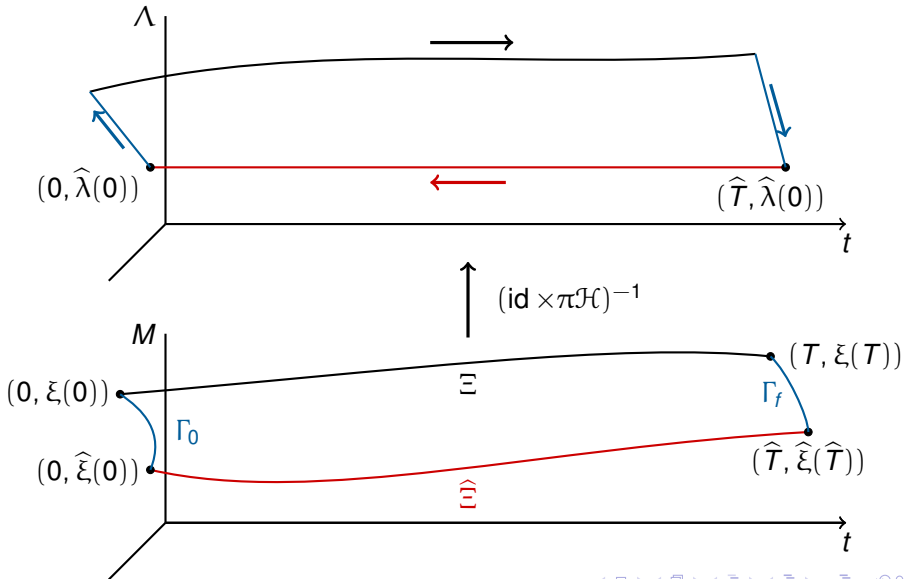
Lifting trajectories



Lifting trajectories



Lifting trajectories



$\mathcal{H}^*(p dq - H_t dt)$ is exact \implies

$$0 = \oint \omega = \int_{(\text{id} \times \pi \mathcal{H})^{-1}(\xi)} \omega + \int_{(\text{id} \times \pi \mathcal{H})^{-1}(\Gamma_f)} \omega - \int_{(\text{id} \times \pi \mathcal{H})^{-1}(\hat{\xi})} \omega + \int_{(\text{id} \times \pi \mathcal{H})^{-1}(\Gamma_0)} \omega$$

$\mathcal{H}^*(p \, dq - H_t \, dt)$ is exact \implies

$$0 = \oint \omega = \int_{(\text{id} \times \pi\mathcal{H})^{-1}(\xi)} \omega + \int_{(\text{id} \times \pi\mathcal{H})^{-1}(\Gamma_f)} \omega - \int_{(\text{id} \times \pi\mathcal{H})^{-1}(\hat{\xi})} \omega + \int_{(\text{id} \times \pi\mathcal{H})^{-1}(\Gamma_0)} \omega$$

$$0 \leq p_0(T - \hat{T}) + O((T - \hat{T})^2)$$

If $p_0 = 1 \implies \hat{\xi}$ is a (time, state)-local optimal trajectory

Hamiltonian approach to state–local optimality 1/2

- the flow of an over-maximised Hamiltonian

$$H: \ell \in T^*M \mapsto H(\ell) \in \mathbb{R}$$

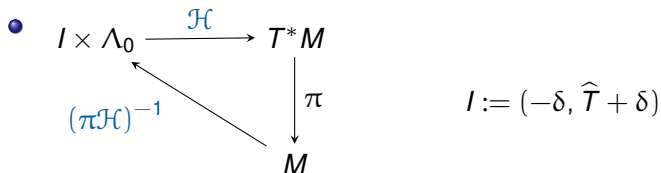
$$H(\ell) \geq F^{\max}(\ell) \quad H(\widehat{\lambda}(t)) = F^{\max}(\widehat{\lambda}(t))$$

$$\mathcal{H}: (t, \ell) \in (-\delta, \widehat{T} + \delta) \times T^*M \mapsto \mathcal{H}_t(\ell) \in T^*M$$

$$\widehat{\lambda}(t) = \mathcal{H}_t(\widehat{\lambda}(0))$$

- a smooth function $\alpha: M \rightarrow \mathbb{R}$ such that
 - $d\alpha(\widehat{\xi}(0)) = \widehat{\lambda}(0)$
 - $\Lambda = \{(d\alpha(x), x)\}$ is transverse to $\{H = p_0\}$ in $\widehat{\lambda}(0)$
- $\Lambda_0 := \Lambda \cap \{H = p_0\}$ is a $(n - 1)$ -dim manifold of T^*M

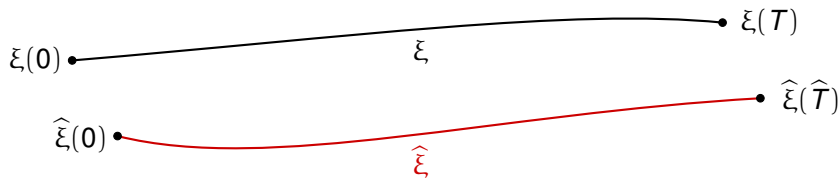
Hamiltonian approach to state–local optimality 2/2



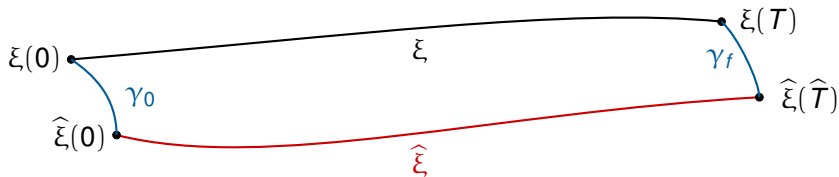
Property

$\mathcal{H}^*(p dq)$ is exact on $(-\delta, \widehat{T} + \delta) \times \Lambda_0$

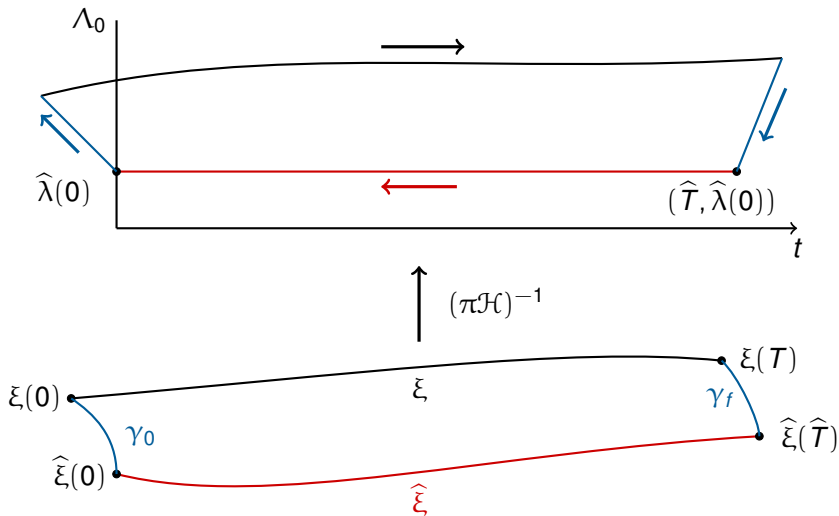
Lifting trajectories



Lifting trajectories



Lifting trajectories



$\mathcal{H}^*(p dq)$ is exact \implies

$$0 = \oint \omega = \int_{(\pi\mathcal{H})^{-1}(\xi)} \omega + \int_{(\pi\mathcal{H})^{-1}(\gamma_f)} \omega - \int_{(\pi\mathcal{H})^{-1}(\hat{\xi})} \omega + \int_{(\pi\mathcal{H})^{-1}(\gamma_0)} \omega$$

$\mathcal{H}^*(p \, dq)$ is exact \implies

$$0 = \oint \omega = \int_{(\pi\mathcal{H})^{-1}(\xi)} \omega + \int_{(\pi\mathcal{H})^{-1}(\gamma_f)} \omega - \int_{(\pi\mathcal{H})^{-1}(\hat{\xi})} \omega + \int_{(\pi\mathcal{H})^{-1}(\gamma_0)} \omega$$

$$\begin{aligned} p_0(T - \hat{T}) &\geq p_0(t_f - \hat{T}) + \alpha(q_f) - \alpha(\hat{\xi}(0)) + \frac{t_0^2}{2} h_1 \cdot h_1 \cdot \alpha(\bar{q}) \\ &= C(\xi(T)) - C(\hat{\xi}(\hat{T})) + \frac{t_0^2}{2} h_1 \cdot h_1 \cdot \alpha(\bar{q}) \end{aligned}$$

$$C(x) := p_0(\psi^{\mathbb{R}}(x)) + \alpha(\pi\psi^{\wedge_0}(x)) \quad (\psi^{\mathbb{R}}, \psi^{\wedge_0}) := (\pi\mathcal{H})^{-1}$$

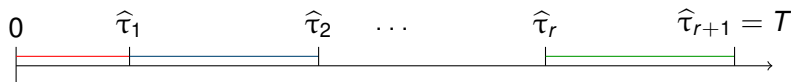
$$(t_f, q_f) := (\psi^{\mathbb{R}}, \pi\psi^{\wedge_0})(\xi(T))$$

$$(t_0, q_0) := (\psi^{\mathbb{R}}, \pi\psi^{\wedge_0})(\xi(0))$$

$$\bar{q} = \exp \bar{s} h_1(q_0)$$

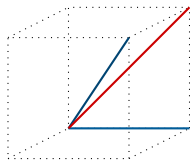
Bang-bang controls

$$\hat{u}(t) \in \text{Vertexes}(Q) \quad \forall t \in [0, \hat{T}] \setminus \{\hat{\tau}_j : j = 1, \dots, r\}$$



Simple, double and multiple switches

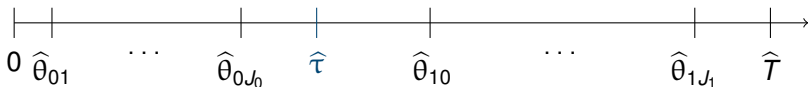
- **simple switch** $\hat{\tau}$ is a **simple switching time** if only one control component switches at time $\hat{\tau}$
- **double switch** $\hat{\tau}$ is a **double switching time** if two control components switch at time $\hat{\tau}$
- **j -multiplicity switch** $\hat{\tau}$ is a **j multiplicity switching time** if j control components switch at time $\hat{\tau}$



Bang-bang with multiple switches

Simple and double switches

- there exists only double switching time $\hat{\tau}$;
- a certain number of simple switching times $\hat{\theta}_{0j}$ occurs before $\hat{\tau}$ and a certain number of simple switching times $\hat{\theta}_{1j}$ occur after the double switch;



Simplified case

$$T \rightarrow \min,$$

$$\dot{\xi}(t) = f_0(\xi(t)) + u_1(t)f_1(\xi(t)) + u_2(t)f_2(\xi(t)),$$

$$\xi(0) \in N_0, \quad \xi(T) \in N_f,$$

$$|u_i(t)| \leq 1 \quad i = 1, 2, \text{ a.e. } t \in [0, T].$$

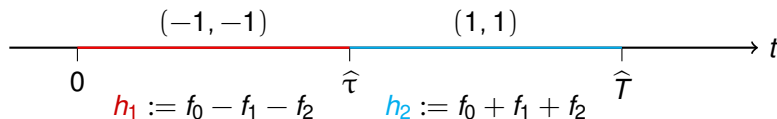
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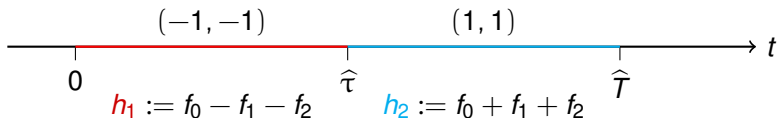
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$$|u_i(t)| \leq 1 \quad i = 1, 2, \text{ a.e. } t \in [0, T].$$



$$H_1(\ell) := \langle \ell, h_1(\pi\ell) \rangle \quad H_2(\ell) := \langle \ell, h_2(\pi\ell) \rangle$$

Assumptions: Regularity along the bang arcs

$$\text{PMP} \implies \begin{aligned} \hat{u}_1(t) F_1(\hat{\lambda}(t)) &= \hat{u}_1(t) \langle \hat{\lambda}(t), f_1(\hat{\xi}(t)) \rangle \geq 0 \\ \hat{u}_2(t) F_2(\hat{\lambda}(t)) &= \hat{u}_2(t) \langle \hat{\lambda}(t), f_2(\hat{\xi}(t)) \rangle \geq 0. \end{aligned}$$

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Regularity along the bang arcs

If $t \neq \hat{\tau}$, then \hat{u} is the only control that gives the maximised Hamiltonian:

$$\hat{u}_1(t)F_1(\hat{\lambda}(t)) = \hat{u}_1(t)\langle \hat{\lambda}(t), f_1(\hat{\xi}(t)) \rangle > 0$$

$$\hat{u}_2(t)F_2(\hat{\lambda}(t)) = \hat{u}_2(t)\langle \hat{\lambda}(t), f_2(\hat{\xi}(t)) \rangle > 0$$

$$\text{i.e.} \quad \begin{array}{lll} F_1(\hat{\lambda}(t)) < 0 & F_2(\hat{\lambda}(t)) < 0 & \text{if } t \in [0, \hat{\tau}) \\ F_1(\hat{\lambda}(t)) > 0 & F_2(\hat{\lambda}(t)) > 0 & \text{if } t \in (\hat{\tau}, \hat{T}] \end{array}$$

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Regularity along the bang arcs

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$$\implies \left. \frac{d}{dt} F_i(\hat{\lambda}(t)) \right|_{t=\hat{\tau}^-} \geq 0 \quad \left. \frac{d}{dt} F_i(\hat{\lambda}(t)) \right|_{t=\hat{\tau}^+} \geq 0$$

Assumptions: Regularity at the switching time 1/2

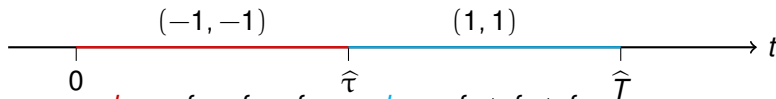
Regularity at the switching times

$$\left. \frac{d}{dt} F_i(\hat{\lambda}(t)) \right|_{t=\hat{\tau}^-} > 0 \quad \left. \frac{d}{dt} F_i(\hat{\lambda}(t)) \right|_{t=\hat{\tau}^+} > 0 \quad i = 1, 2.$$

Assumptions: Regularity at the switching time 1/2

Regularity at the switching times

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$$h_1 := f_0 - f_1 - f_2$$

$$h_2 := f_0 + f_1 + f_2$$

$$k_1 := f_0 + f_1 - f_2 = h_1 + 2f_1 = h_2 - 2f_2$$

$$k_2 := f_0 - f_1 + f_2 = h_1 + 2f_2 = h_2 - 2f_1$$

$$K_1(\ell) = \langle \ell, k_1(\pi\ell) \rangle$$

$$K_2(\ell) = \langle \ell, k_2(\pi\ell) \rangle$$

Assumptions: Regularity at the switching time 2/2

$$\langle \hat{\lambda}(\hat{\tau}), [h_1, k_1] (\hat{\xi}(\hat{\tau})) \rangle = \sigma \left(\vec{H}_1, \vec{K}_1 \right) (\hat{\lambda}(\hat{\tau})) > 0$$

$$\langle \hat{\lambda}(\hat{\tau}), [h_1, k_2] (\hat{\xi}(\hat{\tau})) \rangle = \sigma \left(\vec{H}_1, \vec{K}_2 \right) (\hat{\lambda}(\hat{\tau})) > 0$$

$$\langle \hat{\lambda}(\hat{\tau}), [k_1, h_2] (\hat{\xi}(\hat{\tau})) \rangle = \sigma \left(\vec{K}_1, \vec{H}_2 \right) (\hat{\lambda}(\hat{\tau})) > 0$$

$$\langle \hat{\lambda}(\hat{\tau}), [k_2, h_2] (\hat{\xi}(\hat{\tau})) \rangle = \sigma \left(\vec{K}_2, \vec{H}_2 \right) (\hat{\lambda}(\hat{\tau})) > 0$$

The maximised flow

- for $\nu = 1, 2$ let $\tau_\nu(\ell)$ be the unique solution to

$$2F_\nu \circ \exp \tau_\nu(\ell) \vec{H}_1(\ell) = (K_\nu - H_1) \circ \exp \tau_\nu(\ell) \vec{H}_1(\ell) = 0$$

in a neighborhood of $(\hat{\tau}, \hat{\lambda}(0))$;

The maximised flow

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in a neighborhood of $(\hat{\tau}, \hat{\lambda}(0))$;

- choose

$$\theta_1(\ell) := \min\{\tau_1(\ell), \tau_2(\ell)\};$$

The maximised flow

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in a neighborhood of $(\hat{\tau}, \hat{\lambda}(0))$;

- choose

$$\theta_1(\ell) := \min\{\tau_1(\ell), \tau_2(\ell)\};$$

- for $\nu = 1, 2$, let $\tau_\nu^2(\ell)$ be the unique solution to

$$\begin{aligned} 2F_{3-\nu} \circ \exp(\tau_\nu^2(\ell) - \tau_\nu(\ell)) \vec{K}_\nu \circ \exp \tau_\nu(\ell) \vec{H}_1(\ell) \\ = (H_2 - K_\nu) \circ \exp(\tau_\nu^2(\ell) - \tau_\nu(\ell)) \vec{K}_\nu \circ \exp \tau_\nu(\ell) \vec{H}_1(\ell) = 0 \end{aligned}$$

in a neighborhood of $(\hat{\tau}, \hat{\lambda}(0))$;

The maximised flow

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in a neighborhood of $(\hat{\tau}, \hat{\lambda}(0))$;

- choose

$$\theta_2(\ell) = \begin{cases} \tau_1^2(\ell) & \text{if } \tau_1(\ell) \leq \tau_2(\ell), \\ \tau_2^2(\ell) & \text{if } \tau_2(\ell) < \tau_1(\ell). \end{cases}$$

- $S_0 := \{(t, \ell) : t \in [-\delta, \theta_1(\ell)]\}$

$$\mathcal{F}_t^{\max}(\ell) = \exp t\vec{H}_1(\ell);$$

- $S_1 := \{(t, \ell) : \theta_1(\ell) = \tau_1(\ell), t \in (\theta_1(\ell), \theta_2(\ell))\}$

$$\mathcal{F}_t^{\max}(\ell) = \exp(t - \theta_1(\ell))\vec{K}_1 \circ \exp \theta_1(\ell)\vec{H}_1(\ell);$$

- $S_1^2 := \{(t, \ell) : \theta_1(\ell) = \tau_1(\ell), t \in (\theta_2(\ell), \hat{T} + \delta)\}$

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- $S_2 := \{(t, \ell) : \theta_1(\ell) = \tau_2(\ell), t \in (\theta_1(\ell), \theta_2(\ell))\}$

- $S_2^2 := \{(t, \ell) : \theta_1(\ell) = \tau_2(\ell), t \in (\theta_2(\ell), \hat{T} + \delta)\}$

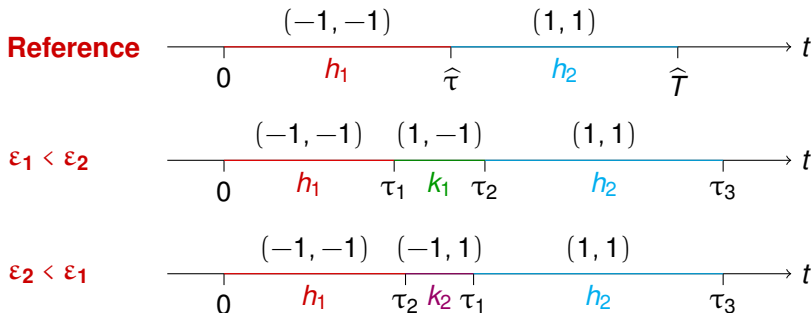
Finite-dimensional subproblem

Allow for different perturbed switching times of the control components

$$\tau_1 := \hat{\tau} + \varepsilon_1 \quad \tau_2 := \hat{\tau} + \varepsilon_2$$

and for variations of the final time

$$\tau_3 := \hat{T} + \varepsilon_3.$$



if $\varepsilon_1 < \varepsilon_2$

$T \rightarrow \min$

$$\dot{\xi}(t) = \begin{cases} h_1(\xi(t)) & t < \tau_1 \\ k_1(\xi(t)) & \tau_1 < t < \tau_2 \\ h_2(\xi(t)) & \tau_2 < t < \tau_3 \end{cases}$$

$$\xi(0) \in N_0, \quad \xi(T) \in N_f,$$

if $\varepsilon_1 < \varepsilon_2$

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$$\xi(0) \in N_0, \quad \xi(T) \in N_f,$$

if $\varepsilon_2 < \varepsilon_1$

$T \rightarrow \min$

$$\dot{\xi}(t) = \begin{cases} h_1(\xi(t)) & t < \tau_2 \\ k_2(\xi(t)) & \tau_2 < t < \tau_1 \\ h_2(\xi(t)) & \tau_1 < t < \tau_3 \end{cases}$$

$$\xi(0) \in N_0, \quad \xi(T) \in N_f,$$

Assumption: coercivity of the second order approximations

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Each second order approximation J''_{ν} $\nu = 1, 2$ is coercive on the half-space of the linearised constraints of the finite-dimensional subproblem.

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$$V_{0,\nu}^+ := \{(\delta x, a_1, b, a_2) \in T_{\hat{\xi}(0)} N_0 \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}:$$

$$\delta x + a_1 g_1(\hat{\xi}(0)) + b j_{\nu}(\hat{\xi}(0)) + a_2 g_2(\hat{\xi}(0)) \in T_{\hat{x}_0} \hat{N}_f\}$$

$$J''_{\nu}[\delta x, a_1, b, a_2] = 2 \delta x \cdot (a_1 g_1 + b j_{\nu} + a_2 g_2) \cdot \hat{\beta}(\hat{\xi}(0)) +$$

$$+ (a_1 g_1 + b j_{\nu} + a_2 g_2)^2 \cdot \hat{\beta}(\hat{\xi}(0))$$

$$+ a_1 b [g_1, j_{\nu}] \cdot \hat{\beta}(\hat{\xi}(0)) + a_1 a_2 [g_1, g_2] \cdot \hat{\beta}(\hat{\xi}(0)) + b a_2 [j_{\nu}, g_2] \cdot \hat{\beta}(\hat{x}_0),$$

$$\hat{N}_f := \hat{S}_{\hat{\tau}}^{-1}(N_f) \quad g_{\nu} := \hat{S}_{\hat{\tau}^*}^{-1} h_{\nu} \circ \hat{S}_{\hat{\tau}} \quad j_{\nu} := \hat{S}_{\hat{\tau}^*}^{-1} k_{\nu} \circ \hat{S}_{\hat{\tau}}$$

$$\hat{\beta}: M \rightarrow \mathbb{R}: d\hat{\beta}(\hat{\xi}(0)) = -\hat{\lambda}(0)$$

The coercivity of J_v'' allows one to prove that there exists a function α such that

- $\Lambda_0 := \{d\alpha(x) : x \in M\} \cap \{H_1 = p_0\}$ and $M_0 := \pi\Lambda_0 = \{x \in M : h_1 \cdot \alpha(x) = p_0\}$ are $(n-1)$ -dimensional manifolds
- For $v = 1, 2$ define

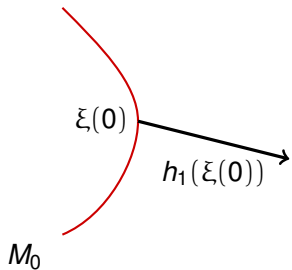
$$M_v := \left\{ \pi \mathcal{F}_{\tau_v(\ell)}^{\max}(\ell) : \ell \in \Lambda_0, \quad \tau_v(\ell) = \theta_1(\ell) \right\},$$

$$M_v^2 := \left\{ \pi \mathcal{F}_{\tau_v^2(\ell)}^{\max}(\ell) : \ell \in \Lambda_0, \quad \tau_v(\ell) = \theta_1(\ell) \right\}.$$

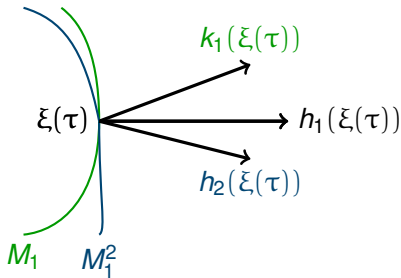
Then these four half-manifolds separate the images of the five smooth pieces of the projected maximised Hamiltonian flow.

Picture for $\nu = 1$

$$M_1 := \left\{ \pi \mathcal{F}_{\tau_1}^{\max}(\ell) : \ell \in \Lambda_0 \right\},$$



$$M_1^2 := \left\{ \pi \mathcal{F}_{\tau_1^2}^{\max}(\ell) : \ell \in \Lambda_0 \right\}.$$



Invertibility of the projected maximised flow

1: The degenerate case $d\tau_1|_{\mathcal{T}_{\hat{\lambda}(0)}\Lambda_0} \equiv d\tau_2|_{\mathcal{T}_{\hat{\lambda}(0)}\Lambda_0}$

- **Clarke's inverse function theorem**

if the convex hull of the Jacobians of each of the five smooth pieces at $(\hat{\tau}, \hat{\lambda}(0))$ is made of invertible matrices only

$\implies \pi_{\mathcal{F}^{\max}}: (-\delta, \hat{\mathcal{T}} + \delta) \times \Lambda_0 \rightarrow M$ is locally one-to-one from a neighborhood of $(\hat{\tau}, \hat{\lambda}(0))$ onto a neighborhood of $\hat{\xi}(\hat{\tau})$

Invertibility of the projected maximised flow

2: The nondegenerate case $d\tau_1|_{T_{\hat{\lambda}(0)}\Lambda_0} \neq d\tau_2|_{T_{\hat{\lambda}(0)}\Lambda_0}$

- Consider the tangent cones to the five sectors where \mathcal{F}^{\max} has a different smooth expression: **the interior of each cone is non-empty**

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- Consider the tangent cones to the five sectors where \mathcal{F}^{\max} has a different smooth expression: **the interior of each cone is non-empty**
- Show that the linearisation of each of the five smooth pieces of $\pi\mathcal{F}^{\max}$ is a proper linear function with positive determinant and degree = 1: for each couple of neighbouring tangent cones take $(\delta t_1, \delta l_1)$ and $(\delta t_2, \delta l_2)$ in each of the two cones and show that their images under the linearisation of the projected maximised flow do not overlap

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- P-Spadini 2013 (based on Pang-Ralph 1996) $\implies \pi\mathcal{F}^{\max}$ is a Lipschitz homeomorphism from $\mathcal{O}(\hat{\tau}, \hat{\lambda}(0))$ on $\mathcal{O}(\hat{\xi}(\hat{\tau}))$.

The reference trajectory

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The Hamiltonian methods can be applied:

- if $p_0 = 1$ then $\widehat{\xi}$ is a state-locally optimal trajectory;
- if $p_0 = 0$ then $\widehat{\xi}$ is isolated among admissible trajectories.

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Each second order approximation J''_{ν} $\nu = 1, 2$ is coercive on the half-space of the linearised constraints of the finite-dimensional subproblem **wih zero total variation of the times** .

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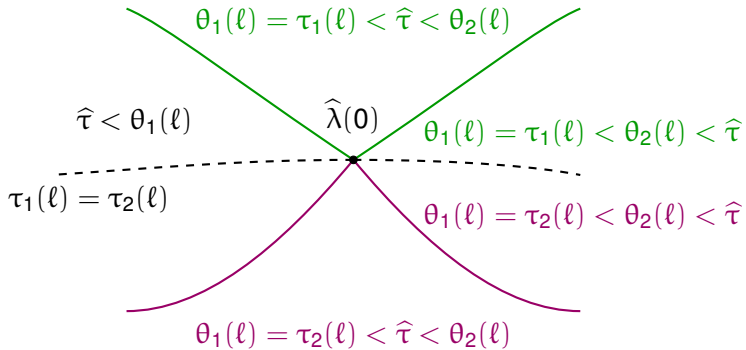
$$\text{id} \times \pi \mathcal{F}^{\max}: [0, \widehat{T}] \times \{d\alpha(x) : x \in M\} \rightarrow [0, \widehat{T}] \times M$$

is locally one-to-one at each $(t, \widehat{\lambda}(0))$, $t \in [0, \widehat{T}]$, i.e.

$$\pi \mathcal{F}_t^{\max}: \{d\alpha(x) : x \in M\} \rightarrow M$$

is locally one-to-one at $\widehat{\lambda}(0)$, for each $t \in [0, \widehat{T}]$.

Freeze at $t = \hat{\tau}$



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