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Franco Rampazzo, Maria Soledad Aronna

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LIMIT SOLUTIONS

(on estensions of input-output for unbounded controls)

FRANCO RAMPAZZO,
University of Padova,
PADOVA, ITALY
joint work with MARIA S. ARONNA (IMPA)

NETCo CONFERENCE
New trends in Optimal Control
JUNE 23-27, 2014
TOURS, FRANCE

1 HEURISTICS

2 "LIMIT" SOLUTIONS

- Existing notions of solutions
- Proposed definition of Limit Solution

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(B) **Minimum problems with end-point constraints** $x(b) \in \mathcal{S}$ as **PROPER EXTENSIONS** of regular problems ($u \in AC$)?

(Aronna's talk)

APPLICATIONS of impulsive systems:

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- Spiking models of synaptic behaviour
- Mechanical systems using some coordinates as controls
- In general, coupled fast-slow dynamics

Underlying thought:

We can accept a notion of discontinuous (=impulsive) trajectory only if it is, in some sense to be made precise, the limit of faster and faster trajectories

Outline

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A "TRIVIAL" BUT IMPORTANT CASE

$$\dot{x} = \dot{u}$$

- For this equation **one would like** to have

$$x(t) = u(t) + x(0) \quad \forall t \in [0, T] \quad (1)$$

as a solution, which is obviously true as soon as $x, u \in AC$ (i.e. absolutely continuous).

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- Another idea could be a **distributional approach**: BUT
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 - 2) it is "*wrong*" in the general nonlinear case.
- Can we take (1) as a definition for $u, x \in \mathcal{L}^1$?**

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- *the commutative case*

$$[g_{\alpha}, g_{\beta}] = 0$$

- *the non commutative case*

$$[g_{\alpha}, g_{\beta}] \neq 0$$

with the controls $u(\cdot)$ having **bounded variation**

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- 1 Due to $[g_\alpha, g_\beta] = 0$, by *multiple flow-box theorem* there exists a (global) coordinates' change

$$(x, u) \rightarrow (\xi, z) = (\xi(x, u), u)$$

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Notice: One has continuity of $u \rightarrow x$ with respect to L^1 topologies on both u and x . Actually, stronger continuity properties are verified

References include

A. Bressan and F. Rampazzo. Impulsive control systems with commutative vector fields. *J. Optim. Theory Appl.*, 71, p.67-83, (1991).

A.V. Sarychev. Nonlinear systems with impulsive and generalized function controls, vol. 9 of *Progr. Systems Control Theory*, p. 244-257, (1991).

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Namely: one bridges the jumps of u and parameterize them on s -subintervals where time $t(s)(= \varphi_0(s))$ is constant.

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single-valued version:

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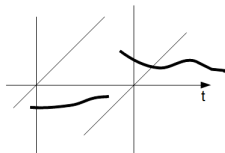
single-valued version: If $\sigma : [0, T] \rightarrow [0, 1]$ is a *Clock*, i.e. $\sigma(t) \in (\varphi_0, \varphi)^{\leftarrow}(t, u(t))$, we say that

$$t \rightarrow x := y \circ \sigma(t)$$

is a **single-valued graph-completion solution**.

$$t'(s) = \varphi_0'(s)$$

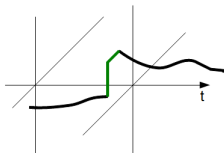
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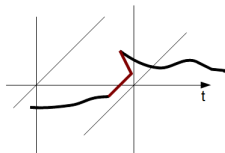
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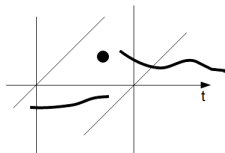
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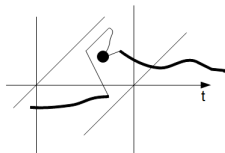
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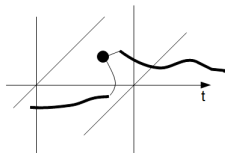
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$u \in BV$

An incomplete list of authors who have investigated this subject:

Bressan

Bressan- Rampazzo

Bressan-Mazzola

Briani-Zidani

Pereira-Vinte

Miller

Motta-Rampazzo

Camilli-Falcone

Motta-Sartori

Silva-Vinter.

Zabic-Wolenski

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- **existence** of an output (and possibly *uniqueness*) results
- former definitions of solution for impulsive systems **should be subsumed** by this extended notion

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Definition

- A map $x : [a, b] \rightarrow \mathcal{R}^n$ is a **LIMIT SOLUTION** if, for every $\tau \in [a, b]$, there exists a sequence of absolutely continuous controls (u_k^{τ}) such that

$$|(x_k^{\tau}, u_k^{\tau})(\tau) - (x, u)(\tau)| + \|(x_k^{\tau}, u_k^{\tau}) - (x, u)\|_1 \rightarrow 0,$$

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- **SIMPLE LIMIT SOLUTION**: if (u_k^{τ}) can be chosen independently of τ , i.e. $(u_k^{\tau}) = (u_k)$.
- **BV-SIMPLE LIMIT SOLUTION** if the approximating inputs u_k have **equibounded variation**.

$(x_k^{\tau}$ is the Carathéodory solution corresponding to (u_k^{τ}, v))

THE COMMUTATIVE CASE, $[g_\alpha, g_\beta] = 0$

THE COMMUTATIVE CASE, $[g_\alpha, g_\beta] = 0$ **Theorem 1.**

- **Existence and uniqueness** For every control $u \in \mathcal{L}^1$ (and every $v \in L^1$) there exists a unique limit solution of

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- **Continuous dependence:** for every $\tau \in [a, b]$ one has

$$|x_1(\tau) - x_2(\tau)| + \|x_1 - x_2\|_1 \leq M \left[|\bar{x}_1 - \bar{x}_2| + |u_1(a) - u_2(a)| + |u_1(t) - u_2(t)| + \|u_1 - u_2\|_1 \right].$$

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moreover: one has continuous dependence w.r. to the standard control $v(\cdot)$ in L^1 norm

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Remarks:

- (1) Proposition 2 was proved in Bressan-Rampazzo(1991) when f independent of v .
- (2) Proposition 2 *gives an idea* for the general definition of solution

Example

$$\dot{x} = xv + x\dot{u}, \quad x(0) = \bar{x},$$

on the interval $[0, 1]$, with $v(t) := \chi_{[0, 1/2[}$

Consider the \mathcal{L}^1 control

$$u(t) := \begin{cases} (-1)^{k+1}, & \text{for } t \in [1 - \frac{1}{k}, 1 - \frac{1}{k+1}[, \quad k \in \mathcal{N}, \\ 0, & \text{for } t = 1. \end{cases}$$

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Notice that both u and x have infinitely many discontinuities, unbounded variation, and are defined everywhere.

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*Let U have the Whitney property.
For any control pair*

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Many thanks for your patience

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The BUT... stuff

$$\dot{x} = g_1(x)\dot{u}_1 + g_2(x)\dot{u}_2, \quad x(0) = 0.$$

$$g_1(x) := (1, 0, x_2), \quad g_2(x) := (0, 1, -x_1), \quad \text{so } [g_1, g_2] = (0, 0, -2).$$

Carathéodory solution corresponding to $u \equiv (0, 0)$ is : $x_C(t) \equiv (0, 0, 0)$
 $u_k(t) := (k^{-1/2} \cos kt - 1, k^{-1/2} \sin kt)$ generates the trajectory

$$x_k(t) = (k^{-1/2} \cos kt - 1, k^{-1/2} \sin kt, -t + k^{-1} \sin kt)^t.$$

$$x_k(t) \text{ to } \hat{x}(t) := (0, 0, -t)^t$$

so \hat{x} is a (simple) limit solution. **In particular** $\hat{x} \neq x_C$

Notice that $\text{Var}(u_k) \doteq \int_0^1 |\dot{u}_k| dt \rightarrow +\infty$ **BUT...** the iterated integral

$$\int_0^1 |\dot{u}_k^2 u_k^1 - \dot{u}_k^1 u_k^2| dt$$

IS BOUNDED as k goes to ∞ .