



HAL
open science

Hamilton-Jacobi equations on networks

Fabio Camilli

► **To cite this version:**

Fabio Camilli. Hamilton-Jacobi equations on networks. NETCO 2014, 2014, Tours, France. hal-01024675

HAL Id: hal-01024675

<https://inria.hal.science/hal-01024675>

Submitted on 16 Jul 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Hamilton-Jacobi equations on networks

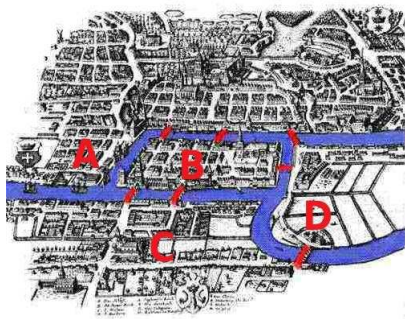
Fabio Camilli

Web page:

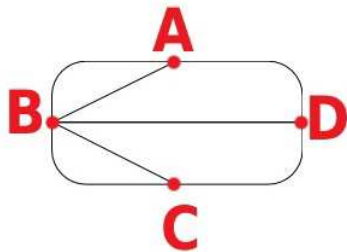
<http://www.dmmm.uniroma1.it/~fabio.camilli/>

Graph theory

Graph theory born with the Euler's solution to the famous problem concerning the crossing of the seven bridges in Königsberg in the 1735



(a) Königsberg and the Pregel river



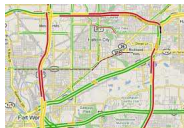
(b) Graph representation

Graph theory has been very a very successful theory in the last three hundred years and one of the main branch of Discrete Mathematics. Tools for this theory comes from Linear Algebra and Combinatorics since all the information about the graph can be encoded in a matrix representing the connection among the vertices.

Applications of Graph theory are manifold and vary from physics to biology, from social science to information systems, from quantum chemistry to sociology, etc.

Consider the following problems

- the flow of cars at different speeds on a highway
- the propagation of flames inside a pipeline
- the deformation of a spider web stirred by the wind
- the propagation of a blackout through an electric distribution network



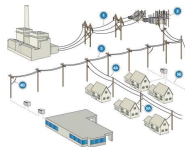
(c) highway



(d) pipeline



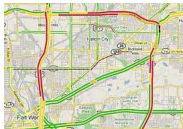
(e) web



(f) electric network

The previous models can be described by some phenomena occurring on a network, but it is not possible to give an appropriate representation by the classical Graph theory formalism since

- 1 the spatial issue cannot be neglected (length and shape of the connections)
- 2 the process is evolving in time
- 3 Inside the arcs the evolution of the physical quantities is described by some differential equations (conservation law, eikonal equation, wave equation, telegraph equation)
- 4 Important processes occur at the junctions of the network and need a proper modelization



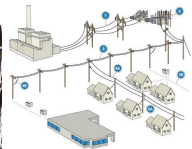
(g) highway



(h) pipeline



(i) web



(j) electric network

In the last thirty years there has been an increasing interest in the study of **differential equations on networks** and also on more general structures called **ramified spaces**.

While a theory for linear differential equations on networks is rather well developed, a corresponding theory for nonlinear differential equations is far to be complete.

A bit of history

Even if models for evolution equations on networks were already studied (for example, Beurling-Deny, Acta Math.1959), G.Lumer was one of the first mathematicians to introduce functional analytical techniques in this framework.

- G. LUMER: Espaces ramifiés, et diffusions sur les réseaux topologiques, C. R. Acad. Sci. Paris Ser. A-B 291 (1980).
- S.NICAISE: Estimées du spectre du laplacien sur un réseau topologique fini, C. R. Acad. Sc. Paris, 303, Série I,1986, p. 343-346.
- J. VON BELOW, Classical solvability of linear parabolic equations on networks. J. Diff. Equations 72 (1988), 316-337.
- S.NICAISE *Polygonal interface problems*.Methods and Procedures in Mathematical Physics, vol. 39. Verlag Peter D. Lang, Frankfurt am Main (1993)

At the same time, important contributions to the theory have been given by the Russian school

- YU. V. POKORNYI AND A. V. BOROVSKIKH, Differential equations on networks, J. Math. Sci. (N. Y.) 119 (2004), 691-718.

For an account of the current state of art, see the books

- G. BERKOLAIKO, P. KUCHMENT, *Introduction to Quantum Graphs*, Mathematical Surveys and Monographs 186; AMS (2013).
- D.MUGNOLO, *Semigroup methods for evolution equation on networks*, Understanding Complex Systems, Springer-Verlag, Berlin, 2014.

At the same time, important contributions to the theory have been given by the Russian school

- YU. V. POKORNYI AND A. V. BOROVSKIKH, Differential equations on networks, J. Math. Sci. (N. Y.) 119 (2004), 691-718.

For an account of the current state of art, see the books

- G. BERKOLAIKO, P. KUCHMENT, *Introduction to Quantum Graphs*, Mathematical Surveys and Monographs 186; AMS (2013).
- D.MUGNOLO, *Semigroup methods for evolution equation on networks*, Understanding Complex Systems, Springer-Verlag, Berlin, 2014.

Stochastic processes on networks

A theory for diffusion processes on networks with application to linear PDEs is in

- M.FREIDLIN, A.WENTZELL, Diffusion processes on graphs and the averaging principle. Ann. Probab. 21 (1993)
- M. FREIDLIN, S.J. SHEU, Diffusion processes on graphs: stochastic differential equations, large deviation principle, Probab. Theory Related Fields 116 (2000).
- V.KOSTRYKIN, J.POTTHOFF, R. SCHRADER, Brownian motions on metric graphs. J. Math. Phys. 53 (2012)

Nonlinear problems

The literature about nonlinear PDEs on networks is much more limited and confined to some specific framework. In particular

Nonlinear equations for traffic model and other problems

- M.Garavello, B.Piccoli, *Traffic flow on networks*. AIMS Series on Applied Mathematics, Springfield, 2006.
- D'Apice, Göttlich, Herty, Piccoli, *Modeling, Simulation and Optimization of Supply Chains: A Continuous Approach*, Siam book series on mathematical modeling and comp., 2009.
- Bressan's course

References on Hamilton-Jacobi equations on networks later on

Plan of the lectures:

- **Lecture 1: A short introduction to linear differential equations on networks.**
- Lecture 2: Hamilton-Jacobi equations on networks.
- Lecture 3: Vanishing viscosity, comparison among various definitions of viscosity solution on network and numerical methods
- Lecture 4: Eikonal equations on Sierpinski gasket

Plan of the lectures:

- Lecture 1: A short introduction to linear differential equations on networks.
- Lecture 2: Hamilton-Jacobi equations on networks.
- Lecture 3: Vanishing viscosity, comparison among various definitions of viscosity solution on network and numerical methods
- Lecture 4: Eikonal equations on Sierpinski gasket

Plan of the lectures:

- Lecture 1: A short introduction to linear differential equations on networks.
- Lecture 2: Hamilton-Jacobi equations on networks.
- Lecture 3: Vanishing viscosity, comparison among various definitions of viscosity solution on network and numerical methods
- Lecture 4: Eikonal equations on Sierpinski gasket

Plan of the lectures:

- Lecture 1: A short introduction to linear differential equations on networks.
- Lecture 2: Hamilton-Jacobi equations on networks.
- Lecture 3: Vanishing viscosity, comparison among various definitions of viscosity solution on network and numerical methods
- Lecture 4: Eikonal equations on Sierpinski gasket

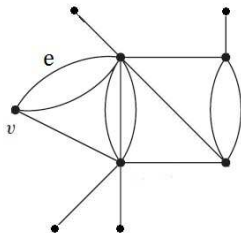
A short introduction to linear PDEs on networks

In this lecture

- Notations and functional spaces
- Transition conditions: the role of the Kirchhoff condition for the maximum principle
- Linear equations: the variational approach
- Linear equations: the probabilistic approach
- Semilinear equations: some existence and uniqueness results.

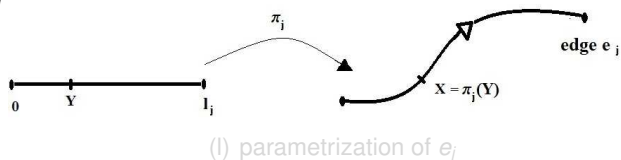
Network

A network is a connected set Γ consisting of a vertices $V := \{v_i\}_{i \in I}$ and edges $E := \{e_j\}_{j \in J}$ connecting the vertices. We assume that the network is **embedded** in \mathbb{R}^N so that any two edges can only have intersection at a vertex (cycles are not admitted).



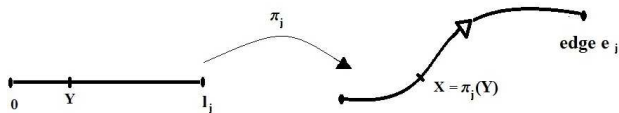
(k) Network

- A coordinate $\pi_j : [0, \ell_j] \rightarrow \mathbb{R}^N$, $j \in J$ is chosen to parametrize the edge e_j



- $Inc_i := \{j \in J : e_j \text{ incident to } v_i\}$ is the set of edges incident the vertex v_i .
- A vertex v_i is a boundary vertex if it has only one incident edge. We denote by $\partial\Gamma = \{v_i, i \in I_B\}$ the set of boundary vertices.
- A vertex v_i is a transition vertex if it has more than one incident edge. We denote by $\Gamma_T = \{v_i, i \in I_T\}$ the set of transition vertices

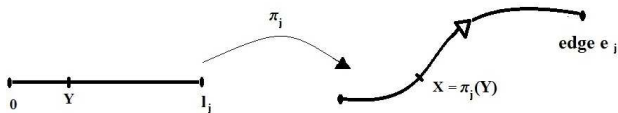
- A coordinate $\pi_j : [0, \ell_j] \rightarrow \mathbb{R}^N$, $j \in J$ is chosen to parametrize the edge e_j



(m) parametrization of e_j

- $Inc_i := \{j \in J : e_j \text{ incident to } v_i\}$ is the set of edges **incident** the vertex v_i .
- A vertex v_i is a **boundary vertex** if it has only one incident edge. We denote by $\partial\Gamma = \{v_i, i \in I_B\}$ the set of boundary vertices.
- A vertex v_i is a **transition vertex** if it has more than one incident edge. We denote by $\Gamma_T = \{v_i, i \in I_T\}$ the set of transition vertices

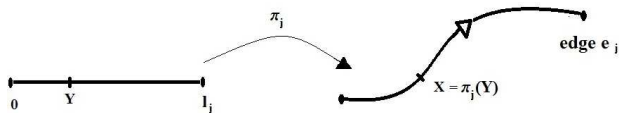
- A coordinate $\pi_j : [0, \ell_j] \rightarrow \mathbb{R}^N$, $j \in J$ is chosen to parametrize the edge e_j



(n) parametrization of e_j

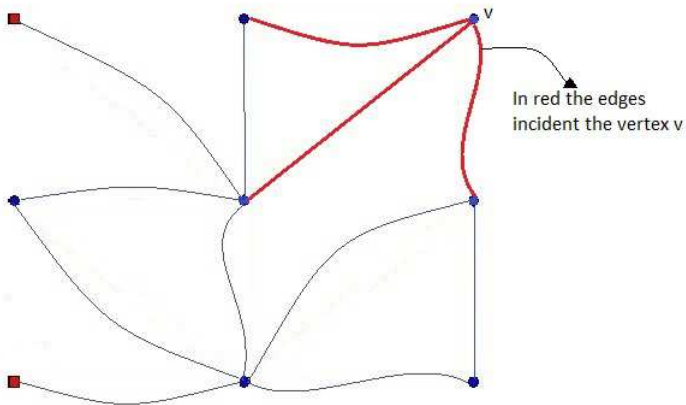
- $Inc_i := \{j \in J : e_j \text{ incident to } v_i\}$ is the set of edges **incident** the vertex v_i .
- A vertex v_i is a **boundary vertex** if it has only one incident edge. We denote by $\partial\Gamma = \{v_i, i \in I_B\}$ the set of boundary vertices.
 - A vertex v_i is a **transition vertex** if it has more than one incident edge. We denote by $\Gamma_T = \{v_i, i \in I_T\}$ the set of transition vertices

- A coordinate $\pi_j : [0, \ell_j] \rightarrow \mathbb{R}^N$, $j \in J$ is chosen to parametrize the edge e_j



(o) parametrization of e_j

- $Inc_i := \{j \in J : e_j \text{ incident to } v_i\}$ is the set of edges **incident** the vertex v_i .
- A vertex v_i is a **boundary vertex** if it has only one incident edge. We denote by $\partial\Gamma = \{v_i, i \in I_B\}$ the set of boundary vertices.
- A vertex v_i is a **transition vertex** if it has more than one incident edge. We denote by $\Gamma_T = \{v_i, i \in I_T\}$ the set of transition vertices



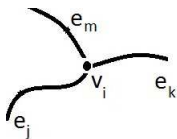
- boundary vertex
- transition vertex

(p) An example of network

Notations:

- Given $u : \Gamma \rightarrow \mathbb{R}$, $u^j : [0, l_j] \rightarrow \mathbb{R}$ denotes the restriction of u to e_j , i.e. $u^j(y) = u(\pi_j(y))$ for $y \in [0, l_j]$
- $u \in C^0(\Gamma)$ if u is continuous in the subspace topology induced by \mathbb{R}^N on Γ , i.e. $u^j \in C([0, l_j])$ for any $j \in J$ and

$$u^j(\pi_j^{-1}(v_i)) = u^k(\pi_k^{-1}(v_i)) \quad \text{for any } i \in I, j, k \in \text{Inc}_i$$

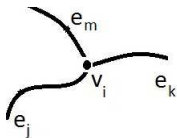


(q)

Notations:

- Given $u : \Gamma \rightarrow \mathbb{R}$, $u^j : [0, \ell_j] \rightarrow \mathbb{R}$ denotes the restriction of u to e_j , i.e. $u^j(y) = u(\pi_j(y))$ for $y \in [0, \ell_j]$
- $u \in C^0(\Gamma)$ if u is continuous in the subspace topology induced by \mathbb{R}^N on Γ , i.e. $u^j \in C([0, \ell_j])$ for any $j \in J$ and

$$u^j(\pi_j^{-1}(v_i)) = u^k(\pi_k^{-1}(v_i)) \quad \text{for any } i \in I, j, k \in \text{Inc}_i$$



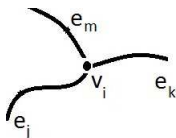
(r)

- **Differentiation** is defined with respect to the variable parametrizing the edge, i.e.

$$\partial_j u(x) := \frac{d}{dy} u^j(y), \quad \text{for } x \in e_j, y = \pi_j^{-1}(x)$$

and at a vertex v_i by

$$\partial_j u(x) := \frac{d}{dy} u^j(y) \quad \text{for } x = v_i, y = \pi_j^{-1}(x), j \in \text{Inc}_i.$$

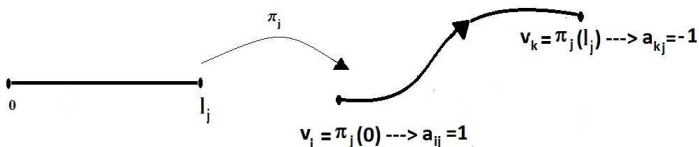


(s)

The incidence matrix

The graph is **not oriented** (in traffic flow models the network is a priori oriented), but the parametrization of the arcs e_j induces an orientation on the edges expressed by the **signed incidence matrix** $A = \{a_{ij}\}_{i \in I, j \in J}$

$$a_{ij} := \begin{cases} 1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(0) = v_i, \\ -1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(l_j) = v_i, \\ 0 & \text{otherwise.} \end{cases}$$



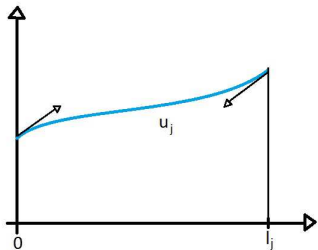
(t) Incidence matrix

The role of the incidence matrix in the differentiation

In defining the differential operators we will take into account the orientation of the segment to have the correct sign of the derivative. Suppose that $v_i = \pi_j(0)$ and $v_k = \pi_j(l_j)$. Then

$$a_{ij}\partial_j u(v_i) = a_{ij}\frac{du^j}{dy}(0) = \frac{d}{dy}u^j(0)$$

$$a_{kj}\partial_j u(v_k) = a_{kj}\frac{du^j}{dy}(l_j) = -\frac{d}{dy}u^j(l_j)$$



- $u \in L^p(\Gamma)$, $p \geq 1$ if $u^j \in L^p(0, \ell_j)$ for each $j \in J$. We set

$$\|u\|_{L^p} := \left(\sum_{j \in J} \|u^j\|_{L^p(e_j)}^p \right)^{1/p}$$

$$\|u\|_{L^\infty} := \sup_{j \in J} \|u^j\|_{L^\infty(e_j)}.$$

- $u \in W^{k,p}(\Gamma)$, for $k \in \mathbb{N}$ and $p \in (1, +\infty)$ if $u \in C^0(\Gamma)$ and $u^j \in W^{k,p}(0, \ell_j)$ for each $j \in J$. We set

$$\|u\|_{W^{k,p}} := \sum_{m=0}^k \|\partial^m u\|_{L^p}$$

- $u \in C^k(\Gamma)$, $k \in \mathbb{N}$, if $u \in C^0(\Gamma)$ and $u^j \in C^k([0, \ell_j])$ for each $j \in J$. We set

$$\|u\|_{C^k} = \max_{m \leq k} \|\partial^m u\|_{L^\infty}$$

Remark:

- The previous spaces are Banach and Hilbert space with the appropriate norms.
- It is important to observe that for a function in $W^{k,p}(\Gamma)$ is required the **continuity** at the vertices.
- No continuity condition at the vertices is prescribed for the derivatives of a function $u \in C^k(\Gamma)$.

The role of the transition condition

Maximum Principle

Assume that $w \in C^2(\Gamma)$ satisfies

$$\begin{aligned} \partial^2 w(x) &\geq 0, & x \in \Gamma, \\ \sum_{j \in \text{Inc}_i} a_{ij} \partial_j w(v_i) &\geq 0, & i \in I_T \end{aligned}$$

with $\{a_{ij}\}$ is the incidence matrix. Then w **cannot attain a maximum in $\Gamma \setminus \partial\Gamma$.**

Remark:

By $\partial^2 w \geq 0$ for $x \in \Gamma$, we intend

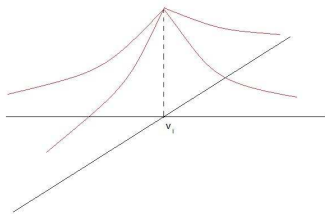
$$\partial_j^2 w(y) \geq 0 \quad \text{for } y \in (0, l_j), \forall j \in J.$$

Proof.: Set $Sw(v_i) := \sum_{j \in Inc_i} a_{ij} \partial_j w(v_i)$ and assume first

$$\partial^2 w(x) > 0 \quad \forall x \in \Gamma \quad \text{and} \quad Sw(v_i) > 0 \quad \forall i \in I_T$$

By contradiction let $x_0 \in \Gamma \setminus \partial\Gamma$ be such that w attains a maximum at x_0 .

- If $x_0 \in e_j$ for some $j \in J$, then it follows that $\partial_j w(x_0) = 0$ and $\partial_j^2 w(x_0) \leq 0$, a contradiction to $\partial_j^2 w > 0$.
- If $x_0 = v_i$ for some $i \in I_T$, then $a_{ij} \partial_j w(v_i) \leq 0$ for all $j \in Inc_i$, hence $Sw(v_i) \leq 0$, a contradiction to $Sw(v_i) > 0$.



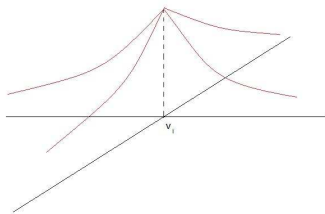
(v) Maximum at a transition vertex

Proof.: Set $Sw(v_i) := \sum_{j \in Inc_i} a_{ij} \partial_j w(v_i)$ and assume first

$$\partial^2 w(x) > 0 \quad \forall x \in \Gamma \quad \text{and} \quad Sw(v_i) > 0 \quad \forall i \in I_T$$

By contradiction let $x_0 \in \Gamma \setminus \partial\Gamma$ be such that w attains a maximum at x_0 .

- If $x_0 \in e_j$ for some $j \in J$, then it follows that $\partial_j w(x_0) = 0$ and $\partial_j^2 w(x_0) \leq 0$, a contradiction to $\partial_j^2 w > 0$.
- If $x_0 = v_i$ for some $i \in I_T$, then $a_{ij} \partial_j w(v_i) \leq 0$ for all $j \in Inc_i$, hence $Sw(v_i) \leq 0$, a contradiction to $Sw(v_i) > 0$.



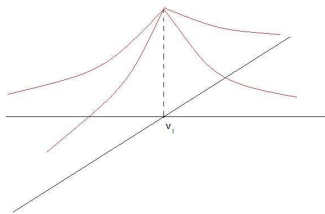
(w) Maximum at a transition vertex

Proof.: Set $Sw(v_i) := \sum_{j \in Inc_i} a_{ij} \partial_j w(v_i)$ and assume first

$$\partial^2 w(x) > 0 \quad \forall x \in \Gamma \text{ and } Sw(v_i) > 0 \quad \forall i \in I_T$$

By contradiction let $x_0 \in \Gamma \setminus \partial\Gamma$ be such that w attains a maximum at x_0 .

- If $x_0 \in e_j$ for some $j \in J$, then it follows that $\partial_j w(x_0) = 0$ and $\partial_j^2 w(x_0) \leq 0$, a contradiction to $\partial_j^2 w > 0$.
- If $x_0 = v_i$ for some $i \in I_T$, then $a_{ij} \partial_j w(v_i) \leq 0$ for all $j \in Inc_i$, hence $Sw(v_i) \leq 0$, a contradiction to $Sw(v_i) > 0$.



(x) Maximum at a transition vertex

In the general case

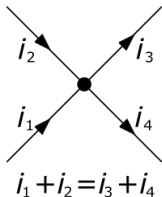
$$\partial^2 w(x) \geq 0 \quad \forall x \in \Gamma \quad \text{and} \quad Sw(v_i) \geq 0 \quad \forall i \in I_T$$

it is always possible to define $\varphi \in C^2(\Gamma)$ such that $\partial^2 \varphi(x) > 0$ for $x \in \Gamma$ and $S\varphi(v_i) > 0$ for $i \in I_T$.

Consider $w_\delta = w + \delta\varphi$, $\delta > 0$. Then w_δ cannot attain a maximum in $\Gamma \setminus \partial\Gamma$ and we conclude for $\delta \rightarrow 0$. \square

Remark:

The condition $\sum_{j \in \text{Inc}_i} a_{ij} \partial_j w(v_i) \geq 0$ rules out the maximum points at the transition vertices. This condition is called the **Kirchhoff condition** and it was introduced by Kirchhoff in electrical circuits theory (the sum of the incoming currents is null).



(y) Kirchhoff law

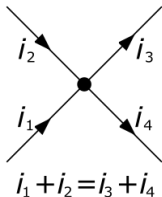
Kirchhoff condition is the simplest form of a linear transition condition. More general transition conditions are of the form

$$\sum_{j \in \text{Inc}_i} a_{ij} \beta_{ij} \partial_j w(v_i) + \gamma_i w(v_i) = 0 \quad i \in I_T$$

with $\beta_{ij} > 0$ for $i \in I, j \in J$.

Remark:

The condition $\sum_{j \in \text{Inc}_i} a_{ij} \partial_j w(v_i) \geq 0$ rules out the maximum points at the transition vertices. This condition is called the **Kirchhoff condition** and it was introduced by Kirchhoff in electrical circuits theory (the sum of the incoming currents is null).



(z) Kirchhoff law

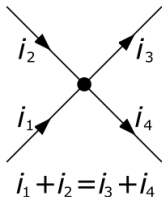
Kirchhoff condition is the simplest form of a linear transition condition. More general transition conditions are of the form

$$\sum_{j \in \text{Inc}_i} a_{ij} \beta_{ij} \partial_j w(v_i) + \gamma_i w(v_i) = 0 \quad i \in I_T$$

with $\beta_{ij} > 0$ for $i \in I, j \in J$.

Remark:

The condition $\sum_{j \in \text{Inc}_i} a_{ij} \partial_j w(v_i) \geq 0$ rules out the maximum points at the transition vertices. This condition is called the **Kirchhoff condition** and it was introduced by Kirchhoff in electrical circuits theory (the sum of the incoming currents is null).



() Kirchhoff law

Kirchhoff condition is the simplest form of a linear transition condition. More general transition conditions are of the form

$$\sum_{j \in \text{Inc}_i} a_{ij} \beta_{ij} \partial_j w(v_i) + \gamma_i w(v_i) = 0 \quad i \in I_T$$

with $\beta_{ij} > 0$ for $i \in I, j \in J$.

Linear equations on networks: the PDE approach

Consider the problem the Poisson equation

$$\begin{cases} \partial^2 u = f(x) & x \in \Gamma \\ \sum_{j \in \text{Inc}_i} a_{ij} \partial_j u(v_i) = 0 & i \in I_T \\ u(v_i) = 0 & v_i \in \partial\Gamma \end{cases} \quad (\dagger)$$

where $f \in L^2(\Gamma)$.

Definition:

- A function $u \in C^2(\Gamma)$ is said a **strong solution** of problem (\dagger) if it satisfies the problem in point-wise sense
- A function $u \in H_0^1(\Gamma)$ is said a **weak solution** of the problem (\dagger) if

$$\sum_{j \in J} \int_0^{\ell_j} \partial_j u(y) \partial_j \varphi(y) + f(y) \varphi(y) dy = 0 \quad \forall \varphi \in H_0^1(\Gamma)$$

In the weak formulation the transition conditions are **transparent** in the sense that they are included in the integral term.

Linear equations on networks: the PDE approach

Consider the problem the Poisson equation

$$\begin{cases} \partial^2 u = f(x) & x \in \Gamma \\ \sum_{j \in \text{Inc}_i} a_{ij} \partial_j u(v_i) = 0 & i \in I_T \\ u(v_i) = 0 & v_i \in \partial\Gamma \end{cases} \quad (\dagger)$$

where $f \in L^2(\Gamma)$.

Definition:

- A function $u \in C^2(\Gamma)$ is said a **strong solution** of problem (\dagger) if it satisfies the problem in point-wise sense
- A function $u \in H_0^1(\Gamma)$ is said a **weak solution** of the problem (\dagger) if

$$\sum_{j \in J} \int_0^{\ell_j} \partial_j u(y) \partial_j \varphi(y) + f(y) \varphi(y) dy = 0 \quad \forall \varphi \in H_0^1(\Gamma)$$

In the weak formulation the transition conditions are **transparent** in the sense that they are included in the integral term.

Proposition.

If $u \in C^2(\Gamma)$ is a weak solution to (\dagger) , then it is also a strong solution to (\dagger) .

Proof: Integration by parts gives

$$\begin{aligned} 0 &= \sum_{j \in J} \int_0^{\ell_j} [\partial_j u(y) \partial_j \varphi(y) + f(y) \varphi(y)] dy \\ &= - \sum_{j \in J} \int_0^{\ell_j} [\partial_j^2 u(y) - f(y)] \varphi(y) dy + \sum_{j \in J} a_{ij} [\varphi \partial_j u]_0^{\ell_j} \\ &= - \sum_{j \in J} \int_0^{\ell_j} [\partial_j^2 u(y) - f(y)] \varphi(y) dy + \sum_{v_i \in \Gamma_T} \sum_{j \in \text{Inc}_{v_i}} a_{ij} \varphi(v_i) \partial_j u(v_i) \end{aligned}$$

- Consider first a test function φ with support in e_j to get the equation inside the arc;
- Then consider a test function φ with support in $\{v_i\} \cup \cup_{j \in \text{Inc}_i} e_j$ to get the transition condition at v_i .

- **Existence and uniqueness** of a solution to the Poisson equation can be obtained by applying the classical theory of bilinear forms in $H_0^1(\Gamma)$ to the equation

$$a(u, \varphi) = (f, \varphi) \quad \forall \varphi \in H_0^1(\Gamma)$$

where

$$a(u, \varphi) = \sum_{j \in J} \int_0^{\ell_j} \partial_j u(y) \partial_j \varphi(y) dy$$

$$(f, \varphi) = \sum_{j \in J} \int_0^{\ell_j} \varphi_j(y) f_j(y) dy$$

- For the **evolutionary problem**, well posedness is obtained by means of variational or semigroup methods, see
M. KRAMAR FIJAVZ, D. MUGNOLO AND E. SIKOLYA, Variational and semigroup methods for waves and diffusion in networks. Appl. Math. Optim. 55 (2007).

The previous approach works for the linear equation

$$\begin{cases} \mathcal{L}u(x) = f & x \in \Gamma \\ \mathcal{S}u(v) = 0 & v \in \Gamma_{\mathcal{T}} \\ \mathcal{B}u(v) = 0 & v \in \partial\Gamma \end{cases}$$

where

- \mathcal{L} is given on the arc e_j by

$$\mathcal{L}^j u(y) = \partial_j \left[\sigma_j(y) \partial_j u(y) + b_j u(y) \right] + c_j \partial_j u(y) + d_j u(y), \quad y \in [0, \ell_j]$$

with $a_j, b_j, c_j, d_j : [0, \ell_j] \rightarrow \mathbb{R}$, $j \in \mathcal{J}$, measurable and bounded (no continuity conditions across the vertices is required).

- $f \in L^2(\Gamma)$
- The transition condition \mathcal{S} is given by

$$\mathcal{S}u(v_i) = \sum_{j \in \text{Inc}_i} a_{ij} \beta_j \sigma_j(v_i) \partial_j u + b_j(v_i) u(v_i) \quad i \in I_{\mathcal{T}}$$

- $\mathcal{B}u$ is a Dirichlet or a Neumann boundary condition on $\partial\Gamma$

Limitations of the previous approach

1) The structure of the transitivity condition has to be strictly linked to the one of the linear operator to perform the cancelation via integration by parts.

In particular in this approach the weights in the Kirchhoff condition can depend on the edge e_j , but not on the vertex v_i .

$$\mathcal{L}^j u(y) = \partial_j(\sigma_j(y)\partial_j u(y) + b_j u(y)) + c_j \partial_j(y)u + d_j u(y)$$

$$Su(v_i) = \sum_{j \in Inc_i} a_{ij} \beta_j \sigma_j(v_i) \partial_j u + b_j(v_i) u(v_i)$$

with the weights $\beta_j > 0$.

Hence the classical Kirchhoff condition

$$\sum_{j \in Inc_i} a_{ij} \beta_{ij} \partial_j u(v_i) = 0 \quad v_i \in \Gamma_T$$

with $\beta_{ij} > 0$ is not included.

2) **Classical solutions** are much more delicate to obtain since the transition vertices play the role of (internal) boundary points for the solution.

It is always possible to interpret a PDE on a network as **a system of 1-dimensional equations with the transition conditions as boundary data**. Hence the coupling is at the boundary.

If the system fits in the Ladyzhenskaya-Solonnikov-Ural'tseva's regularity theory, then we get existence of smooth solution. But this approach requires **very strong compatibility conditions on the initial and boundary data**, see

J. VON BELOW, Classical solvability of linear parabolic equations on networks. J. Differential Equations (1988).

A different approach: SDE on networks and Feynman-Kac formula

Consider the linear operator \mathcal{A} on $C_b(\Gamma)$ defined by

$$A_j u(x) = \frac{1}{2} \sigma_j^2(x) \partial_j^2 u + b_j(x) \partial_j^2 u \quad x \in e_j,$$

with σ_j, b_j , smooth functions and σ_j are uniformly strictly positive. The domain of \mathcal{A} is

$$D(\mathcal{A}) = \{u \in C_b^\infty(\Gamma) : \sum_{j \in \text{Inc}_i} a_{ij} \beta_{ij} \partial_j u(v_i) = 0, i \in I_T\}$$

where a_{ij} is the incidence matrix and the weights $\beta = \{\beta_{ij}\}_{i \in I_T, j \in \text{Inc}_i}$ satisfy $\beta_{ij} > 0$.

Theorem (M.FREIDLIN, A.WENTZELL, Ann. Probab. (1993))

The operator \mathcal{A} generates on Γ a Markov process $(X(t), j(t)) \in \Gamma \times \mathcal{J}$ with continuous samples defined in a unique way in the sense of distribution.

If $j(t) = j$, the first component of the Markov process is a diffusion process generated by A_j inside the arc e_j and satisfies the SDE

$$dX(t) = b_j(X(t))dt + \sigma_j(X(t))dW(t)$$

The probabilistic interpretation of the Kirchhoff's condition

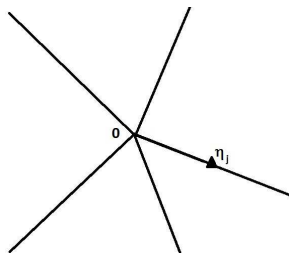
- The positivity of the coefficients β_{ij} in the Kirchhoff condition implies that the process **almost surely spends zero time at each transition vertex v_j** .
- The term $\beta_{ij}/(\sum_{j \in \text{Inc}_i} \beta_{ij})$ is the probability that the process $X(t)$ enters in the edge e_j when it is at the vertex v_j .

The probabilistic interpretation of the Kirchhoff's condition

- The positivity of the coefficients β_{ij} in the Kirchhoff condition implies that the process **almost surely spends zero time at each transition vertex v_j** .
- The term $\beta_{ij}/(\sum_{j \in \text{Inc}_i} \beta_{ij})$ is **the probability that the process $X(t)$ enters in the edge e_j when it is at the vertex v_j** .

A local description of the Markov process (Freidlin-Sheu, 2003)

Consider a star-network



() star shaped network

with only one transition vertex O and the parametrization of e_j is $\pi_j(y) = y\eta_j$, $y \in \mathbb{R}^+$ where η_j is a unit vectors in \mathbb{R}^2 .

Then there is a 1-dim Wiener process $W(t)$ and a continuous, increasing process $\ell(t)$ which are measurable with respect to the σ -field generated by the process such that

$$dX(t) = b_{j(t)}(X(t))dt + \sigma_{j(t)}(X(t))dW(t) + d\ell(t).$$

and the process $\ell(\cdot)$ increases only when $X(t) = 0$. Moreover the process $(X(t), j(t))$ satisfies the **Ito's formula**

$$f_{j(t)}(X(t)) = f_{j(0)}(X(0)) + \int_0^t \sigma_{j(s)}(X(s)) \partial_{j(s)} f(X(s)) dW_s + \int_0^t A_{j(s)} f_{j(s)}(X(s)) ds + \sum_j \beta_{0j} \partial_j f(0) \ell(t).$$

There exists a unique solution $u \in C^2(\Gamma)$ of the problem

$$\begin{cases} \mathcal{A}u + \lambda u = f(x) & x \in \Gamma \\ \sum_j \beta_{ij} a_{ij} \partial_j u(v_i) = 0 & i \in I_T \\ u(v_i) = \gamma_i & i \in I_B. \end{cases}$$

Moreover the following formula holds

$$u(x) = \mathbb{E}_x \left\{ \int_0^\tau e^{-\lambda s} f(X(s)) ds + e^{-\lambda \tau} \gamma_{i(\tau)} \right\}$$

where $X(s)$ is a Markov process defined on the graph which on each edge e_j solves the stochastic differential equation

$$dX(s) = b_j(X(s))ds + \sigma_j(X(s))dW(s),$$

$\tau = \inf\{t > 0 : X(t) \in \partial\Gamma\}$ and $i(\tau) \in I_B$ is such that $X(\tau) = v_{i(\tau)} \in \partial\Gamma$.

There exists a unique solution $u \in C^2(\Gamma)$ of the problem

$$\begin{cases} \mathcal{A}u + \lambda u = f(x) & x \in \Gamma \\ \sum_j \beta_{ij} a_{ij} \partial_j u(v_i) = 0 & i \in I_T \\ u(v_i) = \gamma_i & i \in I_B. \end{cases}$$

Moreover the following formula holds

$$u(x) = \mathbb{E}_x \left\{ \int_0^\tau e^{-\lambda s} f(X(s)) ds + e^{-\lambda \tau} \gamma_{i(\tau)} \right\}$$

where $X(s)$ is a Markov process defined on the graph which on each edge e_j solves the stochastic differential equation

$$dX(s) = b_j(X(s)) ds + \sigma_j(X(s)) dW(s),$$

$\tau = \inf\{t > 0 : X(t) \in \partial\Gamma\}$ and $i(\tau) \in I_B$ is such that $X(\tau) = v_{i(\tau)} \in \partial\Gamma$.

Some comments and remarks:

- Existence and uniqueness via probabilistic techniques also for parabolic problems.
- The advantage of the probabilistic method with respect to the variational method is to allow a general Kirchhoff condition of the type

$$\sum_{j \in \text{Inc}_i} a_{ij} \beta_{ij} \partial_j u(v_i) = 0, i \in I_T$$

with weights $\beta_{ij} > 0$ depending also on the vertex v_i besides the incident arcs e_j .

Some comments and remarks:

- Existence and uniqueness via probabilistic techniques also for parabolic problems.
- The advantage of the probabilistic method with respect to the variational method is to allow a general Kirchhoff condition of the type

$$\sum_{j \in \text{Inc}_i} a_{ij} \beta_{ij} \partial_j u(v_i) = 0, i \in I_T$$

with weights $\beta_{ij} > 0$ depending also on the vertex v_i besides the incident arcs e_j .

Some results about semi-linear equations on networks

In general semi-linear and fully nonlinear equations on networks are not well investigated. But **if the weights in the Kirchhoff condition do not depend on the vertex**, some results can be easily extended from Euclidean settings to the networks.

We give an example based on

BOCCARDO, L. ; MURAT, F. ; PUEL, J.-P. Résultats d'existence pour certains problèmes elliptiques quasilineaires. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 11 (1984), no. 2, 213-235.

Some results about semi-linear equations on networks

In general semi-linear and fully nonlinear equations on networks are not well investigated. But **if the weights in the Kirchhoff condition do not depend on the vertex**, some results can be easily extended from Euclidean settings to the networks.

We give an example based on

BOCCARDO, L. ; MURAT, F. ; PUEL, J.-P. Résultats d'existence pour certains problèmes elliptiques quasilineaires. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 11 (1984), no. 2, 213-235.

An existence result for H with quadratic growth

Consider

$$\left\{ \begin{array}{ll} -\nu \partial^2 u + H(x, u, \partial u) + \lambda u = 0, & x \in \Gamma \\ \sum_{j \in \text{Inc}_i} a_{ij} \nu_j \partial_j u(v_i) = 0 & i \in I_T \\ \partial u(x) = 0 & i \in I_B \end{array} \right. \quad (\dagger)$$

with $\nu = \{\nu_j\}_{j \in J}$, $\lambda = \{\lambda_j\}_{j \in J}$.

Definition

A weak solution of (\dagger) is a function $u \in H^1(\Gamma)$ such that

$$\sum_{j \in J} \int_{e_j} (\nu_j \partial_j u \partial_j \varphi + H^j(x, u, \partial_j u) \varphi + \lambda_j u \varphi) dx = 0 \quad \text{for any } \varphi \in H^1(\Gamma)$$

An existence result for H with quadratic growth

Consider

$$\left\{ \begin{array}{ll} -\nu \partial^2 u + H(x, u, \partial u) + \lambda u = 0, & x \in \Gamma \\ \sum_{j \in \text{Inc}_i} a_{ij} \nu_j \partial_j u(v_i) = 0 & i \in I_T \\ \partial u(x) = 0 & i \in I_B \end{array} \right. \quad (\dagger)$$

with $\nu = \{\nu_j\}_{j \in J}$, $\lambda = \{\lambda_j\}_{j \in J}$.

Definition

A weak solution of (\dagger) is a function $u \in H^1(\Gamma)$ such that

$$\sum_{j \in J} \int_{e_j} (\nu_j \partial_j u \partial_j \varphi + H^j(x, u, \partial_j u) \varphi + \lambda_j u \varphi) dx = 0 \quad \text{for any } \varphi \in H^1(\Gamma)$$

Theorem

Assume

$$\begin{cases} H^j(\cdot, r, p) \text{ is measurable in } y \in (0, \ell_j), \text{ for any } (r, p) \in \mathbb{R} \times \mathbb{R} \text{ and} \\ H^j(y, \cdot, \cdot) \text{ is continuous in } (r, p), \text{ for a.a. } y \in (0, \ell_j) \end{cases}$$

$$|H(x, r, p)| \leq C_0 + b(|s|)|p|^2 \quad \text{for a.e. } x \in \Gamma, s \in \mathbb{R}, p \in \mathbb{R}$$

$$\frac{\partial H}{\partial r}(x, r, p) \geq 0 \quad \text{for a.e. } x \in \Gamma, r \in \mathbb{R}, p \in \mathbb{R}$$

where $C_0 > 0$, $b : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function. Then there exists a weak solution to (\dagger) . Moreover

$$\|u\|_{H^1} \leq C$$

with C depending on C_0 , $\lambda_0 = \inf_{j \in J} \lambda_j$ and $\nu_0 = \inf_{j \in J} \nu_j$.

Theorem

Assume

$$\begin{cases} H^j(\cdot, r, p) \text{ is measurable in } y \in (0, \ell_j), \text{ for any } (r, p) \in \mathbb{R} \times \mathbb{R} \text{ and} \\ H^j(y, \cdot, \cdot) \text{ is continuous in } (r, p), \text{ for a.a. } y \in (0, \ell_j) \end{cases}$$

$$|H(x, r, p)| \leq C_0 + b(|s|)|p|^2 \quad \text{for a.e. } x \in \Gamma, s \in \mathbb{R}, p \in \mathbb{R}$$

$$\frac{\partial H}{\partial r}(x, r, p) \geq 0 \quad \text{for a.e. } x \in \Gamma, r \in \mathbb{R}, p \in \mathbb{R}$$

where $C_0 > 0$, $b : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function. Then there exists a weak solution to (\dagger) . Moreover

$$\|u\|_{H^1} \leq C$$

with C depending on C_0 , $\lambda_0 = \inf_{j \in J} \lambda_j$ and $\nu_0 = \inf_{j \in J} \nu_j$.

Sketch of the proof:

• **Step 1, H bounded:** Show the existence of a bounded solution via a fixed point argument applied to the map $w \mapsto u$ defined by the solution of

$$\begin{cases} -\nu \partial^2 u + \lambda u = -H(x, w, \partial w) & x \in \Gamma \\ \sum_{j \in \text{Inc}_i} a_{ij} \nu_j \partial_j u(v_i) = 0 & i \in I_T \\ \partial u(v_i) = 0 & i \in I_B \end{cases}$$

• **Step 2, H with quadratic growth:** Apply the Boccardo-Murat-Puel truncation argument to show the existence of a solution via approximation: show that the sequence $\{u_\varepsilon\}$ of the solution of the problem with bounded Hamiltonian

$$H_\varepsilon(x, r, p) = \frac{H(x, r, p)}{1 + \varepsilon |H(x, r, p)|}$$

is compact in $H_0^1(\Gamma)$ and therefore converges up to a subsequence to a solution of the limit problem. It is important to observe that in the weak formulation the transition conditions are transparent. Hence it is possible to adapt the argument in Boccardo-Murat-Puel with minor modifications.

Sketch of the proof:

• **Step 1, H bounded:** Show the existence of a bounded solution via a fixed point argument applied to the map $w \mapsto u$ defined by the solution of

$$\begin{cases} -\nu \partial^2 u + \lambda u = -H(x, w, \partial w) & x \in \Gamma \\ \sum_{j \in \text{Inc}_i} a_{ij} \nu_j \partial_j u(v_i) = 0 & i \in I_T \\ \partial u(v_i) = 0 & i \in I_B \end{cases}$$

• **Step 2, H with quadratic growth:** Apply the Boccardo-Murat-Puel truncation argument to show the existence of a solution via approximation: show that the sequence $\{u_\varepsilon\}$ of the solution of the problem with bounded Hamiltonian

$$H_\varepsilon(x, r, p) = \frac{H(x, r, p)}{1 + \varepsilon |H(x, r, p)|}$$

is compact in $H_0^1(\Gamma)$ and therefore converges up to a subsequence to a solution of the limit problem. It is important to observe that in the weak formulation the transition conditions are transparent. Hence it is possible to adapt the argument in Boccardo-Murat-Puel with minor modifications.

An example of uniqueness result¹

Theorem

Assume that $H = (H^j)_{j \in J}$ satisfies

- $H^j \in C^0([0, \ell_j] \times \mathbb{R} \times \mathbb{R})$, $j \in J$
- $H^j(y, \cdot, p)$ is strictly increasing for any $y \in [0, \ell_j]$, $p \in \mathbb{R}$, $j \in J$.

Let $w_1, w_2 \in C^2(\Gamma)$ be such that

$$\begin{cases} -\nu \partial^2 w_1 + H(x, w_1, \partial w_1) \geq -\nu \partial^2 w_2 + H(x, w_2, \partial w_2) & x \in \Gamma, \\ \sum_{j \in \text{Inc}_i} a_{ij} \nu_j \partial_j w_1(v_i) \geq \sum_{j \in \text{Inc}_i} a_{ij} \nu_j \partial_j w_2(v_i) & i \in I_T \\ w_1(v_i) \geq w_2(v_i) & i \in I_B \end{cases}$$

Then $w_1 \geq w_2$ on Γ .

¹F. CAMILLI, C. MARCHI AND D. SCHIEBORN, The vanishing viscosity limit for Hamilton-Jacobi equation on networks, J.Diff. Equations 254 (2013)

Sketch of the proof

Assume by contradiction $(w_2 - w_1)(x_0) := \max_{\Gamma}(w_2 - w_1) = \delta > 0$ with $x_0 \in \Gamma \setminus \partial\Gamma$.

• If $x_0 \in e_j$ for some $j \in J$, by the regularity of w_1 and w_2

$$w_2(x_0) = w_1(x_0) + \delta, \quad \partial_j w_2(x_0) = \partial_j w_1(x_0), \quad \partial_j^2 w_2(x_0) \leq \partial_j^2 w_1(x_0).$$

Hence

$$\begin{aligned} & -\nu \partial_j^2 w_1(x_0) + H(x_0, w_1(x_0), \partial_j w_1(x_0)) \leq \\ & -\nu \partial_j^2 w_2(x_0) + H(x_0, w_2(x_0) - \delta, \partial_j w_2(x_0)) \\ & < -\nu \partial_j^2 w_2(x_0) + H(x_0, w_2(x_0), \partial_j w_2(x_0)) \end{aligned}$$

which contradicts

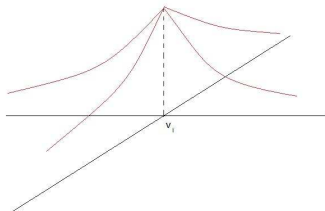
$$-\nu \partial_j^2 w_1 + H^j(x, w_1, \partial_j w_1) \geq -\nu \partial_j^2 w_2 + H^j(x, w_2, \partial_j w_2)$$

- If $x_0 = v_i \in \Gamma_T$, since $w_2 - w_1$ has a maximum point at x_0 , hence $a_{ij}\partial_j(w_2 - w_1) \leq 0$, i.e. $a_{ij}\partial_j w_2(v_i) \leq a_{ij}\partial_j w_1(v_i)$.

We claim $\partial_j w_2(v_i) = \partial_j w_1(v_i)$ for each $j \in Inc_i$. By contradiction assume $a_{ij}\partial_j w_2(v_i) < a_{ij}\partial_j w_1(v_i)$ for some $j \in Inc_i$. In this case

$$\sum_{j \in Inc_i} a_{ij} \nu_j \partial_j w_1(v_i) < \sum_{j \in Inc_i} a_{ij} \nu_j \partial_j w_2(v_i)$$

which contradicts the assumptions, hence the claim is proved.



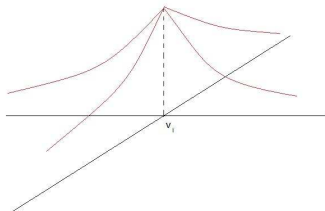
()

- If $x_0 = v_i \in \Gamma_T$, since $w_2 - w_1$ has a maximum point at x_0 , hence $a_{ij}\partial_j(w_2 - w_1) \leq 0$, i.e. $a_{ij}\partial_j w_2(v_i) \leq a_{ij}\partial_j w_1(v_i)$.

We claim $\partial_j w_2(v_i) = \partial_j w_1(v_i)$ for each $j \in Inc_i$. By contradiction assume $a_{ij}\partial_j w_2(v_i) < a_{ij}\partial_j w_1(v_i)$ for some $j \in Inc_i$. In this case

$$\sum_{j \in Inc_i} a_{ij} \nu_j \partial_j w_1(v_i) < \sum_{j \in Inc_i} a_{ij} \nu_j \partial_j w_2(v_i)$$

which contradicts the assumptions, hence the claim is proved.



()

Since $w_1(v_i) = w_2(v_i) - \delta$, $\partial_j w_2(v_i) = \partial_j w_1(v_i) \forall j$ we deduce

$$\begin{aligned} H(v_i, w_1(v_i), \partial_j w_1(v_i)) &= H(v_i, w_2(v_i) - \delta, \partial_j w_2(v_i)) \\ &< H(v_i, w_2(v_i), \partial_j w_2(v_i)) \end{aligned}$$

and by regularity of H and w_1, w_2 , the inequality holds in a neighborhood of v_i . Hence

$$\nu \partial_j^2 (w_2 - w_1)(x) \geq H(x, w_2(x), \partial_j w_2(x)) - H(x, w_1(x), \partial_j w_1(x)) > 0$$

which, together with $\partial_j w_2(v_i) = \partial_j w_1(v_i)$, contradicts that $w_2 - w_1$ attains a maximum in v_i . □

The assumption of strictly monotonicity in the zeroth-order term can be replaced either by

$$H^j(x, \cdot, \cdot) \in C^1(\mathbb{R} \times \mathbb{R}) \quad \text{for any } x \in (0, l_j), j \in J.$$

or by

$$|H^j(x, r, p) - H^j(x, s, q)| \leq K(|r - s| + |p - q|) \quad \forall r, s, p, q \in \mathbb{R}.$$

Comments and remarks

- If the structure of the Kirchhoff condition is compatible with the equation (in particular, the weights are independent of the vertex) then results for semilinear equation can be readily extended from the Euclidean case to the networks.
- The only result for semilinear equations with a general Kirchhoff condition (weights dependent on the vertex) in the following interesting paper where existence is proved via a fixed point argument.
POKORNYI, YU. V.; PENKIN, O. M.; PRYADIEV, V. L., A nonlinear boundary value problem on a graph. (Russian) Differ. Uravn. 34 (1998), no. 5, 629–637, 719; translation in Differential Equations 34 (1998), no. 5, 631-638
- All the previous results are in the semi-linear uniformly elliptic case. Is it possible to define viscosity solution for fully nonlinear second order equations on networks?

Comments and remarks

- If the structure of the Kirchhoff condition is compatible with the equation (in particular, the weights are independent of the vertex) then results for semilinear equation can be readily extended from the Euclidean case to the networks.
- The only result for semilinear equations with a general Kirchhoff condition (weights dependent on the vertex) in the following interesting paper where existence is proved via a fixed point argument.

POKORNYI, YU. V.; PENKIN, O. M.; PRYADIEV, V. L., A nonlinear boundary value problem on a graph. (Russian) Differ. Uravn. 34 (1998), no. 5, 629–637, 719; translation in Differential Equations 34 (1998), no. 5, 631-638

- All the previous results are in the semi-linear uniformly elliptic case. Is it possible to define viscosity solution for fully nonlinear second order equations on networks?

Comments and remarks

- If the structure of the Kirchhoff condition is compatible with the equation (in particular, the weights are independent of the vertex) then results for semilinear equation can be readily extended from the Euclidean case to the networks.
- The only result for semilinear equations with a general Kirchhoff condition (weights dependent on the vertex) in the following interesting paper where existence is proved via a fixed point argument.
POKORNYI, YU. V.; PENKIN, O. M.; PRYADIEV, V. L., A nonlinear boundary value problem on a graph. (Russian) Differ. Uravn. 34 (1998), no. 5, 629–637, 719; translation in Differential Equations 34 (1998), no. 5, 631-638
- All the previous results are in the **semi-linear uniformly elliptic case**. Is it possible to define **viscosity solution** for fully nonlinear second order equations on networks?

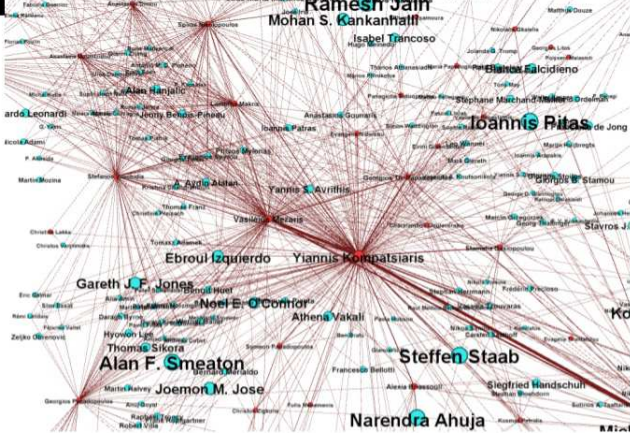


Figure: Facebook network

Thank You!

Hamilton-Jacobi equations on networks: Lecture 2

Fabio Camilli

Web page:

<http://www.dmmm.uniroma1.it/~fabio.camilli/>

In this lecture

- A definition of viscosity solution for HJ on networks
- Existence, uniqueness and stability of viscosity solutions
- A representation formula for the eikonal equation
- The evolutive problem

Viscosity solution

The theory of **viscosity solutions for 1st Hamilton-Jacobi equations** has been introduced by Crandall-Lions (Trans.AMS, '83) and Crandall-Evans-Lions (Trans AMS, '84). The main properties of viscosity solutions are

- **Uniqueness:** comparison principle and doubling of variables technique
- **Existence:** vanishing viscosity method, Perron's method and control theoretic formulas
- **Stability:** uniform convergence and Barles-Perthame's semi-relaxed limits

The basic references for 1st Hamilton-Jacobi equations are

- M. BARDI AND I. CAPUZZO DOLCETTA. Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations. Birkhäuser Boston Inc., Boston, MA, 1997.
- G. BARLES. Solutions de viscosité des équations de Hamilton-Jacobi. Springer-Verlag, Paris, 1994.

Three different paths to Hamilton-Jacobi equations

- Optimal control
- Conservation law
- Vanishing viscosity

In the Euclidean setting, under mild assumptions, the three approaches are **equivalent**, i.e. they characterize the same solution of the HJ equation, i.e. the viscosity solution.

Three different paths to Hamilton-Jacobi equations

- Optimal control
- Conservation law
- Vanishing viscosity

In the Euclidean setting, under mild assumptions, the three approaches are **equivalent**, i.e. they characterize the same solution of the HJ equation, i.e. the viscosity solution.

- **Optimal control problems constrained to network:**

Y.ACHDOU, F.CAMILLI, A.CUTRÌ, N.TCHOU, Hamilton-Jacobi equations constrained on networks, NoDEA (2013).

Y.ACHDOU, S.OUDET, N.TCHOU, Hamilton-Jacobi equations for optimal control on junctions and networks, hal-00847210, 2014

- **Hamilton-Jacobi equations and conservation laws :**

C.IMBERT, R. MONNEAU, H. ZIDANI, A Hamilton-Jacobi approach to junction problems and application to traffic flows, ESAIM: COCV (2013).

C.IMBERT, R. MONNEAU, Level-set convex Hamilton-Jacobi equations on networks, arXiv:1306.2428, 2013.

- **Vanishing viscosity and eikonal equations:**

D. SCHIEBORN, F.CAMILLI, Viscosity solutions of eikonal equations on topological networks, CVPDE (2013).

F. CAMILLI, C. MARCHI, D.SCHIEBORN, The vanishing viscosity limit for Hamilton-Jacobi equation on networks, J.Diff. Eq. (2013)

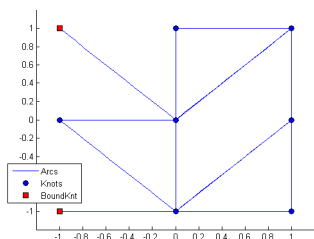
Some related papers dealing with HJ defined “non regular” geometric structures:

- A.BRESSAN, Y. HONG, Optimal control problems on stratified domains, Networks and Heterogeneous Media 2 (2007)
- C.HERMOSILLA , H.ZIDANI , Infinite Horizon Problems on Stratifiable State Constraints sets, hal-00955921, 2014
- G. BARLES, A. BRIANI, E. CHASSEIGNE, A Bellman approach for two-domains optimal control problems in \mathbb{R}^N , ESAIM Cocv(2014)
- G. BARLES, A. BRIANI, E. CHASSEIGNE, A Bellman approach for regional optimal control problems in \mathbb{R}^N , SIAM J. Control Optim. (2014)
- Z.RAO, A.SICONOLFI, H. ZIDANI , Transmission conditions on interfaces for Hamilton-Jacobi-Bellman equations, hal-00820273, 2013
- Y.GIGA, N.HAMAMUKI, A.NAKAYASU, Eikonal equations in metric spaces, Trans.Amer.Math.Soc., in press
- F.CAMILLI, D.SCHIEBORN,C.MARCHI, Eikonal equations on ramified spaces, Interfaces and Free Boundaries 15 (2013).

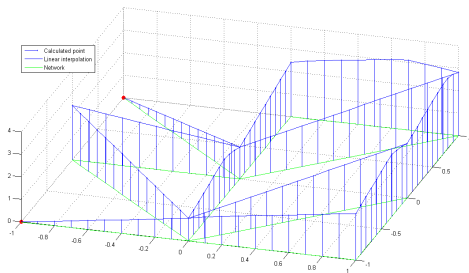
A basic request:

We aim to give a definition of viscosity solution on network which characterizes the distance from the boundary as the solution of the Eikonal equation

$$\begin{cases} |\partial u|^2 = 1 & x \in \Gamma \\ u = 0 & x \in \partial\Gamma \end{cases}$$



(a) Network



(b) Distance

The main point is to find an **appropriate transition condition at the vertices**, while inside the edges the standard definition of viscosity solution will be imposed.

For second order problems we know that the **Kirchhoff condition** (or another transition condition) is necessary for the maximum principle.

Is the Kirchhoff condition a natural condition for HJ equations?

The main point is to find an **appropriate transition condition at the vertices**, while inside the edges the standard definition of viscosity solution will be imposed.

For second order problems we know that the **Kirchhoff condition** (or another transition condition) is necessary for the maximum principle.

Is the Kirchhoff condition a natural condition for HJ equations?

The main point is to find an **appropriate transition condition at the vertices**, while inside the edges the standard definition of viscosity solution will be imposed.

For second order problems we know that the **Kirchhoff condition** (or another transition condition) is necessary for the maximum principle.

Is the Kirchhoff condition a natural condition for HJ equations?

The main point is to find an **appropriate transition condition at the vertices**, while inside the edges the standard definition of viscosity solution will be imposed.

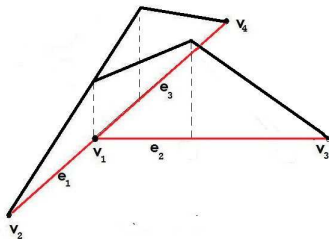
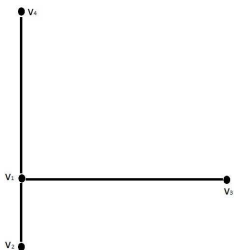
For second order problems we know that the **Kirchhoff condition** (or another transition condition) is necessary for the maximum principle.

Is the Kirchhoff condition a natural condition for HJ equations?

One could try to impose the Kirchhoff condition (HJ equations are degenerate 2nd order equation). But the Kirchhoff condition is **not necessarily satisfied by the distance function** from the boundary.

Example:

$$\begin{cases} |\partial u|^2 = 1 & x \in e_j, j = 1, 2, 3 \\ u(v_i) = 0 & i = 2, 3, 4 \\ \sum_{i=1}^3 a_{1i} \partial_i u(v_1) = 0 \end{cases}$$

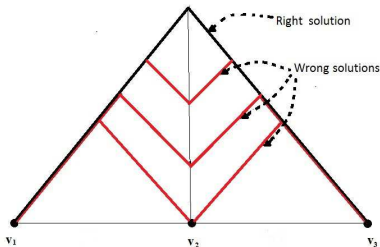
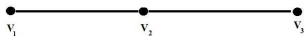


Since the Kirchhoff condition is a first order condition for a second order problem, one could try to assume only a 0-order condition, i.e. the **continuity of the solution at the transition vertices**.

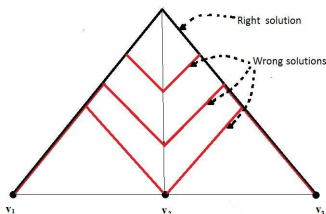
But there are infinitely many viscosity solutions inside the edges which are continuous at the transition vertices.

Example:

$$\begin{cases} |\partial u|^2 = 1 & x \in e_1 \cup e_2 \\ u(v_1) = u(v_3) = 0 \\ u \text{ continuous at } v_2 \end{cases}$$



We conclude a **Kirchhoff-type condition** does not seem to be the right **transition condition** for HJ on networks. Looking again at the previous example



we argue that for the definition of viscosity solution on the network we should **test the equation also at the transition vertices** to rule out the wrong solutions (recall that this definition has also to be consistent with the standard one).

In fact in all the approaches to viscosity solution on networks the equation is tested also at the vertices.

Some notations from the previous lecture

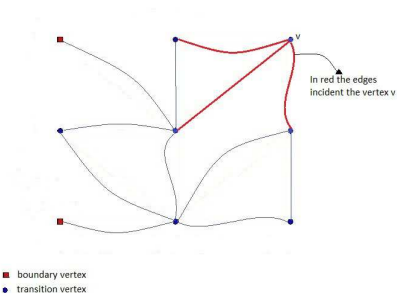
$\Gamma = (V, E)$, where $V := \{v_i\}_{i \in I}$ are the vertices and $E := \{e_j\}_{j \in J}$ are the edges.

- A coordinate $\pi_j : [0, \ell_j] \rightarrow \mathbb{R}^N$, $j \in J$ is chosen to parametrize the edge e_j
- $Inc_i := \{j \in J : e_j \text{ incident to } v_i\}$ is the set of edges **incident** the vertex v_i .
- • A vertex v_i is a **boundary vertex** if it has only one incident edge. We denote by $\partial\Gamma = \{v_i, i \in I_B\}$ the set of boundary vertices.
- • A vertex v_i is a **transition vertex** if it has more than one incident edge. We denote by $\Gamma_T = \{v_i, i \in I_T\}$ the set of transition vertices

Some notations from the previous lecture

$\Gamma = (V, E)$, where $V := \{v_i\}_{i \in I}$ are the vertices and $E := \{e_j\}_{j \in J}$ are the edges.

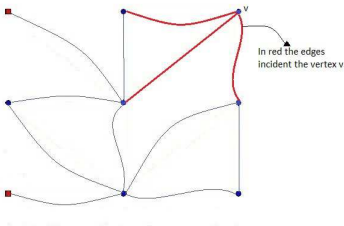
- A coordinate $\pi_j : [0, \ell_j] \rightarrow \mathbb{R}^N$, $j \in J$ is chosen to parametrize the edge e_j
- $Inc_i := \{j \in J : e_j \text{ incident to } v_i\}$ is the set of edges **incident** the vertex v_i .
- • A vertex v_i is a **boundary vertex** if it has only one incident edge. We denote by $\partial\Gamma = \{v_i, i \in I_B\}$ the set of boundary vertices.
- • A vertex v_i is a **transition vertex** if it has more than one incident edge. We denote by $\Gamma_T = \{v_i, i \in I_T\}$ the set of transition vertices



The incidence matrix

The graph is **not oriented**, but the parametrization of the arcs e_j induces an orientation on the edges expressed by the **signed incidence matrix** $A = \{a_{ij}\}_{i \in I, j \in J}$

$$a_{ij} := \begin{cases} 1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(0) = v_i, \\ -1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(\ell_j) = v_i, \\ 0 & \text{otherwise.} \end{cases}$$



- boundary vertex
- transition vertex

The incidence matrix

The graph is **not oriented**, but the parametrization of the arcs e_j induces an orientation on the edges expressed by the **signed incidence matrix** $A = \{a_{ij}\}_{i \in I, j \in J}$

$$a_{ij} := \begin{cases} 1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(0) = v_i, \\ -1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(\ell_j) = v_i, \\ 0 & \text{otherwise.} \end{cases}$$

We consider

$$(HJ) \quad \mathbf{H}(\mathbf{x}, \mathbf{u}, \mathbf{D}\mathbf{u}) = \mathbf{0} \quad \mathbf{x} \in \Gamma$$

and we will describe the approach in :

D. SCHIEBORN, F.CAMILLI, Viscosity solutions of eikonal equations on topological networks, CVPDE (2013).

Remark: In the next lecture we will compare the previous definition with the ones in

Y.ACHDOU, F.CAMILLI, A.CUTRÌ, N.TCHOU, Hamilton-Jacobi equations constrained on networks, NoDEA (2013).

C.IMBERT, R. MONNEAU, H. ZIDANI, A Hamilton-Jacobi approach to junction problems and application to traffic flows, ESAIM: COCV (2013).

The Hamiltonian H is given by a family $H^j : [0, \ell_j] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $j \in J$ satisfying

Standard assumptions inside the edges

- (H1) $H^j \in C^0([0, \ell_j] \times \mathbb{R} \times \mathbb{R})$ for $j \in J$
- (H2) $H^j(y, r, p)$ is non decreasing in r for $p \in \mathbb{R}$, $y \in [0, \ell_j]$, $j \in J$
- (H3) $|H^j(y_1, p) - H^j(y_2, q)| \leq \omega(d_\Gamma(y_1, y_2)(1 + |p|)) + \omega(|p - q|)$
 $\forall y_1, y_2 \in e_j$, $j \in J$, $p, q \in \mathbb{R}$

Specific assumptions at the vertices

- (H4) $H^j(\pi_j^{-1}(v_i), r, p) = H^j(\pi_j^{-1}(v_i), r, -p)$ for any $p \in \mathbb{R}$, $i \in I$, $j \in \text{Inc}_i$
- (H5) $H^j(\pi_j^{-1}(v_i), r, p) = H^k(\pi_k^{-1}(v_i), r, p)$ for any $p \in \mathbb{R}$, $i \in I$, $j, k \in \text{Inc}_i$

- (H4) says that H^j is independent of the orientation of the incident edge
- (H5) says that the Hamiltonian is continuous at the vertices, hence on Γ .

The Hamiltonian H is given by a family $H^j : [0, \ell_j] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $j \in J$ satisfying

Standard assumptions inside the edges

- (H1) $H^j \in C^0([0, \ell_j] \times \mathbb{R} \times \mathbb{R})$ for $j \in J$
- (H2) $H^j(y, r, p)$ is **non decreasing in r** for $p \in \mathbb{R}$, $y \in [0, \ell_j]$, $j \in J$
- (H3) $|H^j(y_1, p) - H^j(y_2, q)| \leq \omega(d_\Gamma(y_1, y_2)(1 + |p|)) + \omega(|p - q|)$
 $\forall y_1, y_2 \in e_j$, $j \in J$, $p, q \in \mathbb{R}$

Specific assumptions at the vertices

- (H4) $H^j(\pi_j^{-1}(v_i), r, p) = H^j(\pi_j^{-1}(v_i), r, -p)$ for any $p \in \mathbb{R}$, $i \in I$, $j \in \text{Inc}_i$
- (H5) $H^j(\pi_j^{-1}(v_i), r, p) = H^k(\pi_k^{-1}(v_i), r, p)$ for any $p \in \mathbb{R}$, $i \in I$, $j, k \in \text{Inc}_i$

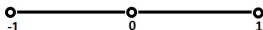
- (H4) says that H^j is **independent of the orientation** of the incident edge
- (H5) says that the Hamiltonian is **continuous at the vertices**, hence on Γ .

Remark: For the moment, neither coercivity nor convexity are assumed. These assumptions will be imposed only when necessary. Roughly speaking, while coercivity is very important for this theory, convexity is only needed for representation formula

Test Functions

A crucial point for the definition of viscosity solutions is **the space of test functions**.

- The definition of viscosity solution on network has to be consistent with the classical one in the case of the segment



- In view of the classical **doubling of variables argument**

$$\Psi(x, y) = u(x) - v(y) - d(x, y)^2/2\varepsilon$$

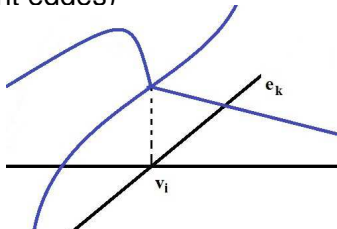
the space of test functions should contain the squared of the **geodesic distance** $d(x, y)$ on the network (i.e. the distance given by the length of the shortest path connecting two points).

Test functions

Let $\varphi \in C^0(\Gamma)$, then

- φ is a **test function at $x \in e_j$** , if $\varphi^j := \varphi \circ \pi_j : [0, l_j] \rightarrow \mathbb{R}$ is differentiable at $y = \pi_j^{-1}(x)$.
- φ is **(j,k)-test function at v_i** if $\partial_j \varphi(\pi_j^{-1}(v_i))$ and $\partial_k \varphi(\pi_k^{-1}(v_i))$ exist and $\mathbf{a}_{ij} \partial_j \varphi(\pi_j^{-1}(v_i)) + \mathbf{a}_{ik} \partial_k \varphi(\pi_k^{-1}(v_i)) = 0$, where (\mathbf{a}_{ij}) is the incidence matrix.

Remark: Condition at the vertices says that, **taking into account the orientation**, the function φ is differentiable at v_i along the direction given by the couple of edges e_j and e_k (no condition, except continuity, along the other incident edges)



Viscosity solutions

- i) If $x = \pi_j(y) \in e_j$ then a function $u \in \text{USC}(\Gamma)$ (resp., $v \in \text{LSC}(\Gamma)$) is called a subsolution (resp. supersolution) at x if for any test function φ for which $u - \varphi$ attains a local maximum (resp., a local minimum) at x , we have

$$H^j(y, u^j(y), \partial_j \varphi(y)) \leq 0, \quad (\text{resp., } H^j(y, u^j(y), \partial_j \varphi(y)) \geq 0);$$

- ii) If $x = v_i \in V$, then

- A function $u \in \text{USC}(\Gamma)$ is called a subsolution at x if **for any $j, k \in \text{Inc}_i$ and any (j, k) -test function φ** for which $u - \varphi$ attains a local maximum at x relatively to $e_j \cup e_k$, we have

$$H^j(y, u^j(y), \partial_j \varphi(y)) \leq 0.$$

- A function $u \in \text{LSC}(\Gamma)$ is called a supersolution at x if **for any $j \in \text{Inc}_i$, there exists $k \in \text{Inc}_i \setminus \{j\}$ (said feasible for j at v_i) such that for any (j, k) -test function φ** for which $u - \varphi$ attains a local minimum at x relatively to $e_j \cup e_k$, we have

$$H^j(y, u^j(y), \partial_j \varphi(y)) \geq 0.$$

Remarks:

- For $x = v_i$, since $H^j(\pi_j^{-1}(v_i), r, p) = H^k(\pi_k^{-1}(v_i), r, p)$ it is indifferent to require the sub and supersolution conditions for j or for k .
- The definitions of sub and supersolution are **asymmetric**. If we consider for supersolution a definition similar to the one for subsolution, i.e.

A function $u \in \text{LSC}(\Gamma)$ is called a supersolution at $x = v \in V$ if **for any $j, k \in \text{Inc}_i$ and any (j, k) -test function φ** for which $u - \varphi$ attains a local minimum at x relatively to $e_j \cup e_k$, we have $H^j(y, u^j(y), \partial_j \varphi(y)) \geq 0$.

the distance function from the boundary **may not satisfy the previous definition**.

Remarks:

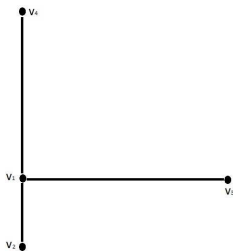
- For $x = v_i$, since $H^j(\pi_j^{-1}(v_i), r, p) = H^k(\pi_k^{-1}(v_i), r, p)$ it is indifferent to require the sub and supersolution conditions for j or for k .
- The definitions of sub and supersolution are **asymmetric**. If we consider for supersolution a definition similar to the one for subsolution, i.e.

A function $u \in \text{LSC}(\Gamma)$ is called a supersolution at $x = v \in V$ if **for any $j, k \in \text{Inc}$; and any (j, k) -test function φ** for which $u - \varphi$ attains a local minimum at x relatively to $e_j \cup e_k$, we have $H^j(y, u^j(y), \partial_j \varphi(y)) \geq 0$.

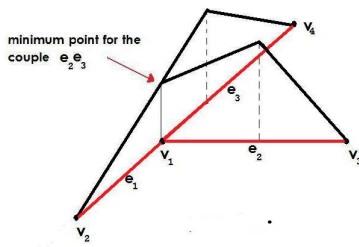
the distance function from the boundary **may not satisfy the previous definition**.

Example:

$$\begin{cases} |\partial u|^2 = 1 & x \in e_j, j = 1, 2, 3 \\ u(v_i) = 0 & i = 2, 3, 4 \end{cases}$$



(l) Network



(m) Distance

A function $v \in \text{LSC}(\Gamma)$ is called a supersolution at x if for any $j \in \text{Inc}_i$, there exists $k \in \text{Inc}_i \setminus \{j\}$ (said feasible for j at v_i) such that for any (j, k) -test function φ for which $u - \varphi$ attains a local minimum at $x = \pi_j(y)$ relatively to $e_j \cup e_k$, we have $H^j(y, u^j(y), \partial_j \varphi(y)) \geq 0$.

(note that the edge e_1 giving the **shortest path to the boundary is feasible** for edges e_2 and e_3).

Three basic results for HJ equations

Uniqueness via Comparison

Let u_1 and u_2 be a subsolution and a supersolution of (HJ) such that $\mathbf{u}_1(\mathbf{v}) \leq \mathbf{u}_2(\mathbf{v})$ for all $\mathbf{v}_i \in \partial\Gamma$. Then $\mathbf{u}_1 \leq \mathbf{u}_2$ in Γ .

Existence via Perron

Let $w \in USC(\Gamma)$ be a subsolution and $W \in LSC(\Gamma)$ a supersolution of (HJ) such that $w \leq W$ and $w_*(x) = W^*(x) = g(x)$ for $x \in \partial\Gamma$. Set $u(x) := \sup\{v(x) : v \text{ is a subsol. of (HJ) with } w \leq v \leq W \text{ on } \Gamma\}$. Then, u^* and u_* are respectively a subsolution and a supersolution to (HJ) in Γ with $u = g$ on $\partial\Gamma$.

Stability

Assume $H_n(x, r, p) \rightarrow H(x, r, p)$ locally uniformly for $n \rightarrow \infty$. Let u_n be a solution of $H_n(x, u, \partial u) = 0$ for $x \in \Gamma$. Assume $u_n \rightarrow u$ uniformly in Γ for $n \rightarrow \infty$. Then u is a solution of (HJ).

Remark: Additional assumptions will be specified where necessary

Three basic results for HJ equations

Uniqueness via Comparison

Let u_1 and u_2 be a subsolution and a supersolution of (HJ) such that $\mathbf{u}_1(\mathbf{v}) \leq \mathbf{u}_2(\mathbf{v})$ for all $\mathbf{v}_i \in \partial\Gamma$. Then $\mathbf{u}_1 \leq \mathbf{u}_2$ in Γ .

Existence via Perron

Let $w \in USC(\Gamma)$ be a subsolution and $W \in LSC(\Gamma)$ a supersolution of (HJ) such that $w \leq W$ and $w_*(x) = W^*(x) = g(x)$ for $x \in \partial\Gamma$. Set $u(x) := \sup\{v(x) : v \text{ is a subsol. of (HJ) with } w \leq v \leq W \text{ on } \Gamma\}$. Then, u^* and u_* are respectively a subsolution and a supersolution to (HJ) in Γ with $u = g$ on $\partial\Gamma$.

Stability

Assume $H_n(x, r, p) \rightarrow H(x, r, p)$ locally uniformly for $n \rightarrow \infty$. Let u_n be a solution of $H_n(x, u, \partial u) = 0$ for $x \in \Gamma$. Assume $u_n \rightarrow u$ uniformly in Γ for $n \rightarrow \infty$. Then u is a solution of (HJ).

Remark: Additional assumptions will be specified where necessary



Three basic results for HJ equations

Uniqueness via Comparison

Let u_1 and u_2 be a subsolution and a supersolution of (HJ) such that $\mathbf{u}_1(\mathbf{v}) \leq \mathbf{u}_2(\mathbf{v})$ for all $\mathbf{v}_i \in \partial\Gamma$. Then $\mathbf{u}_1 \leq \mathbf{u}_2$ in Γ .

Existence via Perron

Let $w \in USC(\Gamma)$ be a subsolution and $W \in LSC(\Gamma)$ a supersolution of (HJ) such that $w \leq W$ and $w_*(x) = W^*(x) = g(x)$ for $x \in \partial\Gamma$. Set $u(x) := \sup\{v(x) : v \text{ is a subsol. of (HJ) with } w \leq v \leq W \text{ on } \Gamma\}$. Then, u^* and u_* are respectively a subsolution and a supersolution to (HJ) in Γ with $u = g$ on $\partial\Gamma$.

Stability

Assume $H_n(x, r, p) \rightarrow H(x, r, p)$ locally uniformly for $n \rightarrow \infty$. Let u_n be a solution of $H_n(x, u, \partial u) = 0$ for $x \in \Gamma$. Assume $u_n \rightarrow u$ uniformly in Γ for $n \rightarrow \infty$. Then u is a solution of (HJ).

Remark: Additional assumptions will be specified where necessary

Proof of the comparison principle

Let u, v be a sub and supersolution of

$$\lambda u + H(x, \partial u) = 0, \quad x \in \Gamma$$

and $u \leq v$ on $\partial\Gamma$. Assume by contradiction that there exists $z \in \Gamma \setminus \partial\Gamma$ such that

$$u(z) - v(z) = \max_{\Gamma} \{u - v\} = \delta > 0.$$

For $\varepsilon > 0$ define $\Phi_\varepsilon : \Gamma \times \Gamma \rightarrow \mathbb{R}$ by

$$\Phi_\varepsilon(x, y) := u(x) - v(y) - \varepsilon^{-1} d(x, y)^2$$

and let $(x_\varepsilon, y_\varepsilon)$ be a maximum point for Φ . By standard viscosity solution arguments

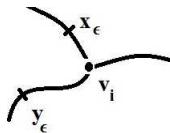
$$x_\varepsilon, y_\varepsilon \rightarrow \bar{x} \in \Gamma \setminus \partial\Gamma \quad \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} d(x_\varepsilon, y_\varepsilon)^2 = 0.$$

For ε small there exists a unique path $\xi(t)$ of length $d(x_\varepsilon, y_\varepsilon)$ in Γ connecting x_ε and y_ε which runs through at most one vertex v_i , $i \in I$. We distinguish several cases:

Case 1 $x_\varepsilon, y_\varepsilon \in e_j$, $j \in J$:

This case can be dealt with standard arguments and we skip it

Case 2: $x_\varepsilon \in e_j$, $y_\varepsilon \in e_k$, for some $j, k \in J$



The functions $\varphi(\cdot) := \varepsilon^{-1} d(x_\varepsilon, \cdot)^2$ and $\psi(\cdot) := \varepsilon^{-1} d(\cdot, y_\varepsilon)^2$ are differentiable at y_ε and x_ε with

$$\partial_j \varphi(y_\varepsilon) = \varepsilon^{-1} d(x_\varepsilon, y_\varepsilon) a_{ij}, \quad \partial_k \psi(x_\varepsilon) = \varepsilon^{-1} d(x_\varepsilon, y_\varepsilon) a_{ik}.$$

Since $u - \varphi$ has a maximum point at y_ε , $v + \psi$ has a minimum point at x_ε

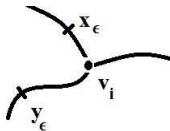
$$\begin{aligned} \lambda u(x_\varepsilon) + H^j(x_\varepsilon, \varepsilon^{-1} d(x_\varepsilon, y_\varepsilon) a_{ij}) &\leq 0 \\ \lambda v(y_\varepsilon) + H^k(y_\varepsilon, -\varepsilon^{-1} d(x_\varepsilon, y_\varepsilon) a_{ik}) &\geq 0. \end{aligned}$$

Hence

$$\begin{aligned}\lambda(u(x_\varepsilon) - v(y_\varepsilon)) &\leq H^k(y_\varepsilon, -\varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)\mathbf{a}_{ik}) - H^j(x_\varepsilon, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)\mathbf{a}_{ij}) \\ &\leq H^k(v_i, -\varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)\mathbf{a}_{ik}) - H^j(v_i, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)\mathbf{a}_{ij}) \\ &\quad + \omega_H(d(v_i, x_\varepsilon) + d(v_i, y_\varepsilon)) \\ &= H^j(v_i, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)\mathbf{a}_{ik}) - H^j(v_i, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)\mathbf{a}_{ij}) \\ &\quad + \omega_H(d(v_i, x_\varepsilon) + d(v_i, y_\varepsilon)) \\ &\leq \omega_H(\varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)) + \omega_H(d(v_i, x_\varepsilon) + d(v_i, y_\varepsilon))\end{aligned}$$

a contradiction for $\varepsilon \rightarrow 0$ since $x_\varepsilon, y_\varepsilon \rightarrow \bar{x}$ and $\lim_{\varepsilon \rightarrow 0} u(x_\varepsilon) - v(y_\varepsilon) \geq \delta > 0$.

Remark: We exploited the two assumptions at the vertices: $H^j(v_i, p) = H^k(v_i, p)$ and $H^j(v_i, -p) = H^j(v_i, p)$



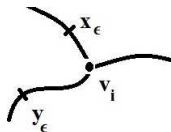
Hence

$$\begin{aligned}\lambda(u(x_\varepsilon) - v(y_\varepsilon)) &\leq H^k(y_\varepsilon, -\varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ik}) - H^j(x_\varepsilon, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ij}) \\ &\leq H^k(v_i, -\varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ik}) - H^j(v_i, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ij}) \\ &\quad + \omega_H(d(v_i, x_\varepsilon) + d(v_i, y_\varepsilon)) \\ &= H^j(v_i, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ik}) - H^j(v_i, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ij}) \\ &\quad + \omega_H(d(v_i, x_\varepsilon) + d(v_i, y_\varepsilon)) \\ &\leq \omega_H(\varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)) + \omega_H(d(v_i, x_\varepsilon) + d(v_i, y_\varepsilon))\end{aligned}$$

a contradiction for $\varepsilon \rightarrow 0$ since $x_\varepsilon, y_\varepsilon \rightarrow \bar{x}$ and $\lim_{\varepsilon \rightarrow 0} u(x_\varepsilon) - v(y_\varepsilon) \geq \delta > 0$.

Remark: We exploited the two assumptions at the vertices:

$H^j(v_i, p) = H^k(v_i, p)$ and $H^j(v_i, -p) = H^j(v_i, p)$



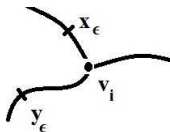
Hence

$$\begin{aligned}\lambda(u(x_\varepsilon) - v(y_\varepsilon)) &\leq H^k(y_\varepsilon, -\varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ik}) - H^j(x_\varepsilon, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ij}) \\ &\leq H^k(v_i, -\varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ik}) - H^j(v_i, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ij}) \\ &\quad + \omega_H(d(v_i, x_\varepsilon) + d(v_i, y_\varepsilon)) \\ &= H^j(v_i, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ik}) - H^j(v_i, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ij}) \\ &\quad + \omega_H(d(v_i, x_\varepsilon) + d(v_i, y_\varepsilon)) \\ &\leq \omega_H(\varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)) + \omega_H(d(v_i, x_\varepsilon) + d(v_i, y_\varepsilon))\end{aligned}$$

a contradiction for $\varepsilon \rightarrow 0$ since $x_\varepsilon, y_\varepsilon \rightarrow \bar{x}$ and $\lim_{\varepsilon \rightarrow 0} u(x_\varepsilon) - v(y_\varepsilon) \geq \delta > 0$.

Remark: We exploited the two assumptions at the vertices:

$H^j(v_i, p) = H^k(v_i, p)$ and $H^j(v_i, -p) = H^j(v_i, p)$



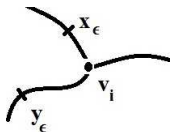
Hence

$$\begin{aligned}\lambda(u(x_\varepsilon) - v(y_\varepsilon)) &\leq H^k(y_\varepsilon, -\varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ik}) - H^j(x_\varepsilon, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ij}) \\ &\leq H^k(v_i, -\varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ik}) - H^j(v_i, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ij}) \\ &\quad + \omega_H(d(v_i, x_\varepsilon) + d(v_i, y_\varepsilon)) \\ &= H^j(v_i, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ik}) - H^j(v_i, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ij}) \\ &\quad + \omega_H(d(v_i, x_\varepsilon) + d(v_i, y_\varepsilon)) \\ &\leq \omega_H(\varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)) + \omega_H(d(v_i, x_\varepsilon) + d(v_i, y_\varepsilon))\end{aligned}$$

a contradiction for $\varepsilon \rightarrow 0$ since $x_\varepsilon, y_\varepsilon \rightarrow \bar{x}$ and $\lim_{\varepsilon \rightarrow 0} u(x_\varepsilon) - v(y_\varepsilon) \geq \delta > 0$.

Remark: We exploited the two assumptions at the vertices:

$H^j(v_i, p) = H^k(v_i, p)$ and $H^j(v_i, -p) = H^j(v_i, p)$



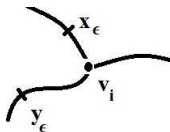
Hence

$$\begin{aligned}\lambda(u(x_\varepsilon) - v(y_\varepsilon)) &\leq H^k(y_\varepsilon, -\varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ik}) - H^j(x_\varepsilon, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ij}) \\ &\leq H^k(v_i, -\varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ik}) - H^j(v_i, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ij}) \\ &\quad + \omega_H(d(v_i, x_\varepsilon) + d(v_i, y_\varepsilon)) \\ &= H^j(v_i, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ik}) - H^j(v_i, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ij}) \\ &\quad + \omega_H(d(v_i, x_\varepsilon) + d(v_i, y_\varepsilon)) \\ &\leq \omega_H(\varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)) + \omega_H(d(v_i, x_\varepsilon) + d(v_i, y_\varepsilon))\end{aligned}$$

a contradiction for $\varepsilon \rightarrow 0$ since $x_\varepsilon, y_\varepsilon \rightarrow \bar{x}$ and $\lim_{\varepsilon \rightarrow 0} u(x_\varepsilon) - v(y_\varepsilon) \geq \delta > 0$.

Remark: We exploited the two assumptions at the vertices:

$H^j(v_i, p) = H^k(v_i, p)$ and $H^j(v_i, -p) = H^j(v_i, p)$



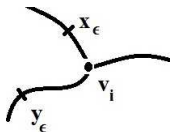
Hence

$$\begin{aligned}\lambda(u(x_\varepsilon) - v(y_\varepsilon)) &\leq H^k(y_\varepsilon, -\varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ik}) - H^j(x_\varepsilon, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ij}) \\ &\leq H^k(v_i, -\varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ik}) - H^j(v_i, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ij}) \\ &\quad + \omega_H(d(v_i, x_\varepsilon) + d(v_i, y_\varepsilon)) \\ &= H^j(v_i, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ik}) - H^j(v_i, \varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)a_{ij}) \\ &\quad + \omega_H(d(v_i, x_\varepsilon) + d(v_i, y_\varepsilon)) \\ &\leq \omega_H(\varepsilon^{-1}d(x_\varepsilon, y_\varepsilon)) + \omega_H(d(v_i, x_\varepsilon) + d(v_i, y_\varepsilon))\end{aligned}$$

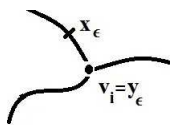
a contradiction for $\varepsilon \rightarrow 0$ since $x_\varepsilon, y_\varepsilon \rightarrow \bar{x}$ and $\lim_{\varepsilon \rightarrow 0} u(x_\varepsilon) - v(y_\varepsilon) \geq \delta > 0$.

Remark: We exploited the two assumptions at the vertices:

$$H^j(v_i, p) = H^k(v_i, p) \text{ and } H^j(v_i, -p) = H^j(v_i, p)$$



Case 3: $x_\varepsilon \in e_j$, $y_\varepsilon = v_i$, for some $j \in J$, $i \in I_T$



Define as

above $\varphi(\cdot) := \varepsilon^{-1} d(x_\varepsilon, \cdot)^2$, $\psi(\cdot) := \varepsilon^{-1} d(\cdot, y_\varepsilon)^2$. We claim that $\varphi(\cdot)$ is an admissible (j, k) -test function at $y_\varepsilon = v_i$ $\forall k \in Inc_i$ (i.e. $a_{ij}\partial_j\varphi(v_i) + a_{ik}\partial_k\varphi(v_i) = 0$). We have

$$\partial_j\varphi(y_\varepsilon) = -\varepsilon^{-1} d(x_\varepsilon, y_\varepsilon) a_{ij},$$

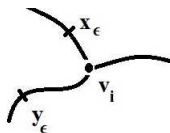
$$\partial_k\varphi(y_\varepsilon) = \varepsilon^{-1} d(x_\varepsilon, y_\varepsilon) a_{ik} \quad \text{for all } k \in Inc_i, k \neq j$$

Hence at $y_\varepsilon = v_i$

$$a_{ij}\partial_j\varphi(v_i) + a_{ik}\partial_k\varphi(v_i) = (-a_{ij}^2 + a_{ik}^2)\varepsilon^{-1} d(x_\varepsilon, y_\varepsilon) = 0$$

At this point we can proceed as in **Case 2**. The other cases are similar to the previous ones. □

Remark:



The continuity of the Hamiltonian at the vertices plays a crucial role in the proof of the previous comparison principle when $x_\epsilon \in e_j$, $y_\epsilon \in e_k$, for some $j, k \in J$.

In the other papers about HJ equations on networks the continuity of H at the vertices is not assumed, but the comparison theorem relies on [representation formulas for the value function](#) (see the next Lecture for more details)

Perron's method

Besides the assumptions for the comparison principle, we assume

(H6) $H^j(y, r, \cdot)$ is **coercive in p** for any $y \in [0, \ell_j]$, $j \in J$;

(H7) $h^i(p) := H^i(v_i, \cdot)$ is **monotone** in $|p|$ for each $i \in I$, $j \in Inc_i$

An immediate consequence of the coercivity is

Regularity of subsolutions

Let K be a compact subset of Γ and let u be a subsolution of (HJ). Then there exists a constant C_K depending only on K such that

$$|u(x) - u(y)| \leq C_K d(x, y).$$

where d is the path distance on Γ .

The proof of the Perron's method is split in several steps, as in the standard case.

Prop.1

Let u and v be two subsolutions to the Hamilton-Jacobi equation, then, $w := \max\{u, v\}$ is a subsolution. The same happens for two supersolution with “max” replaced by “min”.

The proof is more or less standard

Prop.2

Let \mathcal{V} be an arbitrary set of subsolutions of the Hamilton-Jacobi equation. Define $u := \sup_{v \in \mathcal{V}} v$ and assume that $u^* < \infty$ in Γ . Then u^* is a **subsolution** of the Hamilton-Jacobi equation.

Sketch of the proof: We only consider the case $x = v_j \in V$. By definition, $\exists x_n \rightarrow x$ such that $u^*(x) = \lim_n u(x_n)$. Fixed an admissible (j, k) -test function φ , we can assume $u^* - \varphi$ has a strict maximum at x by adding the term $-d^2(x, \cdot)$. Let $v_n \in \mathcal{V}$ be such that

$$(v_n - u)(x_n) \geq -1/n$$

and y_n be a local maximum point for $v_n - \varphi$. Up to a subsequence $y_n \rightarrow z \in \Gamma$. By

$$u(x_n) - \varphi(x_n) - \frac{1}{n} \leq v_n(x_n) - \varphi(x_n) \leq v_n(y_n) - \varphi(y_n) \leq u(y_n) - \varphi(y_n)$$

we get

$$u^*(x) - \varphi(x) \leq u^*(z) - \varphi(z)$$

hence, since x is a strict maximum point, $\lim_n y_n \stackrel{\square}{=} x$

Prop.2

Let \mathcal{V} be an arbitrary set of subsolutions of the Hamilton-Jacobi equation. Define $u := \sup_{v \in \mathcal{V}} v$ and assume that $u^* < \infty$ in Γ . Then u^* is a **subsolution** of the Hamilton-Jacobi equation.

Sketch of the proof: We only consider the case $x = v_j \in V$. By definition, $\exists x_n \rightarrow x$ such that $u^*(x) = \lim_n u(x_n)$. Fixed an admissible (j, k) -test function φ , we can assume $u^* - \varphi$ has a strict maximum at x by adding the term $-d^2(x, \cdot)$. Let $v_n \in \mathcal{V}$ be such that

$$(v_n - u)(x_n) \geq -1/n$$

and y_n be a local maximum point for $v_n - \varphi$. Up to a subsequence $y_n \rightarrow z \in \Gamma$. By

$$u(x_n) - \varphi(x_n) - \frac{1}{n} \leq v_n(x_n) - \varphi(x_n) \leq v_n(y_n) - \varphi(y_n) \leq u(y_n) - \varphi(y_n)$$

we get

$$u^*(x) - \varphi(x) \leq u^*(z) - \varphi(z)$$

hence, since x is a strict maximum point, $\lim_n y_n \overset{\square}{=} x$

Prop.2

Let \mathcal{V} be an arbitrary set of subsolutions of the Hamilton-Jacobi equation. Define $u := \sup_{v \in \mathcal{V}} v$ and assume that $u^* < \infty$ in Γ . Then u^* is a **subsolution** of the Hamilton-Jacobi equation.

Sketch of the proof: We only consider the case $x = v_j \in V$. By definition, $\exists x_n \rightarrow x$ such that $u^*(x) = \lim_n u(x_n)$. Fixed an admissible (j, k) -test function φ , we can assume $u^* - \varphi$ has a strict maximum at x by adding the term $-d^2(x, \cdot)$. Let $v_n \in \mathcal{V}$ be such that

$$(v_n - u)(x_n) \geq -1/n$$

and y_n be a local maximum point for $v_n - \varphi$. Up to a subsequence $y_n \rightarrow z \in \Gamma$. By

$$u(x_n) - \varphi(x_n) - \frac{1}{n} \leq v_n(x_n) - \varphi(x_n) \leq v_n(y_n) - \varphi(y_n) \leq u(y_n) - \varphi(y_n)$$

we get

$$u^*(x) - \varphi(x) \leq u^*(z) - \varphi(z)$$

hence, since x is a strict maximum point, $\lim_n y_n = x$

By

$$u(x_n) - \frac{1}{n} - \varphi(x_n) \leq v_n(x_n) - \varphi(x_n) \leq v_n(y_n) - \varphi(y_n)$$

we get by the Lipschitz continuity of u

$$\begin{aligned} u(x_n) - \frac{1}{n} + \varphi(y_n) - \varphi(x_n) &\leq v_n(y_n) \leq v_n(x_n) + Cd_\Gamma(x_n, y_n) \\ &\leq u(x_n) + Cd_\Gamma(x_n, y_n) \end{aligned}$$

Hence

$$y_n \rightarrow x \quad \text{and} \quad v_n(y_n) \rightarrow u^*(x).$$

By

$$u(x_n) - \frac{1}{n} - \varphi(x_n) \leq v_n(x_n) - \varphi(x_n) \leq v_n(y_n) - \varphi(y_n)$$

we get by the Lipschitz continuity of u

$$\begin{aligned} u(x_n) - \frac{1}{n} + \varphi(y_n) - \varphi(x_n) &\leq v_n(y_n) \leq v_n(x_n) + Cd_\Gamma(x_n, y_n) \\ &\leq u(x_n) + Cd_\Gamma(x_n, y_n) \end{aligned}$$

Hence

$$y_n \rightarrow x \quad \text{and} \quad v_n(y_n) \rightarrow u^*(x).$$

By

$$u(x_n) - \frac{1}{n} - \varphi(x_n) \leq v_n(x_n) - \varphi(x_n) \leq v_n(y_n) - \varphi(y_n)$$

we get by the Lipschitz continuity of u

$$\begin{aligned} u(x_n) - \frac{1}{n} + \varphi(y_n) - \varphi(x_n) &\leq v_n(y_n) \leq v_n(x_n) + Cd_{\Gamma}(x_n, y_n) \\ &\leq u(x_n) + Cd_{\Gamma}(x_n, y_n) \end{aligned}$$

Hence

$$y_n \rightarrow x \quad \text{and} \quad v_n(y_n) \rightarrow u^*(x).$$

By

$$u(x_n) - \frac{1}{n} - \varphi(x_n) \leq v_n(x_n) - \varphi(x_n) \leq v_n(y_n) - \varphi(y_n)$$

we get by the Lipschitz continuity of u

$$\begin{aligned} u(x_n) - \frac{1}{n} + \varphi(y_n) - \varphi(x_n) &\leq v_n(y_n) \leq v_n(x_n) + Cd_\Gamma(x_n, y_n) \\ &\leq u(x_n) + Cd_\Gamma(x_n, y_n) \end{aligned}$$

Hence

$$y_n \rightarrow x \quad \text{and} \quad v_n(y_n) \rightarrow u^*(x).$$

We distinguish two cases.

Case 1: For infinitely many n , $y_n \neq x$, hence $\{y_n\}_n$ is in either e_k or e_j . Wlog assume that $y_n \in e_k$. Since φ is an admissible upper test function for v_n at y_n , we get

$$v_n(y_n) + H^k(y_n, \partial_k \varphi(y_n)) \leq 0.$$

Case 2: Up to a subsequence, $y_n = x$ for any n . Since φ is a (j, k) -upper test function for v_n at v_i we get

$$v_n(y_n) + H^k(y_n, \partial_k \varphi(y_n)) \leq 0.$$

In both the case for $n \rightarrow \infty$ we get

$$u^*(x) + H^k(x, \partial_j \varphi(x)) \leq 0$$



Prop.3

Let $w \in USC(\Gamma)$ be a subsol. and $W \in LSC(\Gamma)$ a be supersol. of (HJ)
s.t. $w \leq W$. Set

$u(x) := \sup\{v(x) : v \text{ is a subsol. of (HJ) with } w \leq v \leq W \text{ on } \Gamma\}$.

Then u_* is a **supersolution** of the Hamilton-Jacobi equation.

Sketch of the proof:

In \mathbb{R}^N , the idea is the following:

Assume **u continuous**; if u is not a supersolution at x_0 , there exists a test function φ sub-tangent to u at x_0 such that

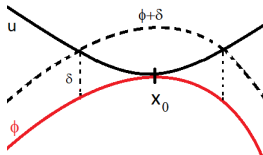
$$u(x_0) + H(x_0, D\varphi) \leq -\eta < 0$$

Then

$$\bar{u}(y) = \begin{cases} \max\{u, \varphi + \delta\}, & \text{in } B(x_0, r); \\ u, & \text{in } \mathbb{R}^N \setminus B(x_0, r). \end{cases}$$

is a subsolution such that

$\bar{u}(x_0) > u(x_0)$, a contradiction.



The idea is similar on Γ . Assume for simplicity that u is continuous and consider the case $x_0 = v_j$. If u is not a supersolution at v_j , then there exists $e_j, j \in Inc_i$ for which no other edges e_k is feasible, i.e.:

$\exists j \in Inc_i$ s.t. $\forall k \in Inc_i \setminus \{j\}$, there exists a (j, k) -lower test function φ_k for u at v_j such that

$$u(v_j) + H^j(v_j, \partial_j \varphi_k(v_j)) \leq -\eta < 0. \quad (1)$$

In this case define for $\delta > 0$, set

$$\bar{\varphi}(x) := \begin{cases} \max_{k \in K} \{\varphi_k(x)\}, & \text{if } x \in e_j, \\ \varphi_k(x), & \text{if } x \in e_k, k \in Inc_i \setminus \{j\}. \end{cases}$$

Then $\bar{\varphi}(x)$ is such that $\bar{\varphi}(v_j) = u(v_j)$ and $\bar{\varphi} \leq u$ in a neighborhood $B_r(v_j)$ of v_j . Moreover $\bar{\varphi}(x)$ is a subsolution of (HJ) in $B_r(v_j)$ (here we use the monotonicity assumption at the vertices).

We finally set

$$\bar{u}(x) := \begin{cases} \max\{u(x), \bar{\varphi} + \delta\}, & \text{if } x \in B_r(v_i), \\ u(x), & \text{if } x \in \Gamma \setminus B_r(v_i). \end{cases}$$

and we get a contradiction to the maximality of u as subsolution of the HJ equation. \square

The stability result

Recall the assumptions: $H_n \rightarrow H$ locally uniformly in $\Gamma \times \mathbb{R}$ and $u_n \rightarrow u$ uniformly.

We give an idea of the proof for $x = v_i$ in the supersolution case. Fixed $j \in \text{Inc}_i$, we have to show that there is a feasible edge e_k for u at v_i , i.e. for every (j, k) -upper test function φ for u at v_i

$$H^j(v_i, u(v_i), \partial_j \varphi(v_i)) = H^k(v_i, u(v_i), \partial_k \varphi(v_i)) \geq 0.$$

For $n \in \mathbb{N}$, let $k_n \in \text{Inc}_i$ an edge feasible for u_n at v_i . Up to a subsequence $k_n = k$ and we show that k is feasible also for u at v_i .

Let φ be a (j, k) test function such that for $u - \varphi$ has a **strict local minimum** at v_i relatively to $e_j \cup e_k$. Then there exists a sequence y_n of minimum point for $u_n - \varphi$ relatively to $e_j \cup e_k$ s.t.

$$\lim_{n \rightarrow +\infty} y_n := z = v_i, \quad \lim_{n \rightarrow +\infty} u_n(y_n) = u(v_i).$$

Case 1 $y_n \neq v_i$ for infinitely many n :

in this case y_n belongs either to e_j or to e_k , wlog say e_j . Therefore

$$H_n^j(y_n, u_n(y_n), \partial_j \varphi(y_n)) \geq 0.$$

Case 2 $x_n = v_i$ for infinitely many n :

Since $k = k_n$ is j -feasible for u_n for any n we get

$$H_n^j(v_i, u_n(v_i), \partial_j \varphi(v_i)) \geq 0.$$

Letting $n \rightarrow +\infty$ in both the last two relations we get

$$H^j(v_i, u(v_i), \partial_j \varphi(v_i)) \geq 0$$



Remark: The previous argument cannot be adapted to stability results via [semi-relaxed limits a la "Barles-Perthame"](#):

Let u_n be a sequence of supersolutions of a given HJ equation and we want to show that $u_*(x) = \liminf_{n \rightarrow \infty} u_n(x)$ is a supersolution at a vertex $x = v$.

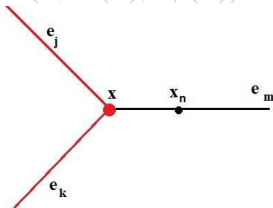
We test as before u_* by means of a (j, k) -admissible test function φ . If y_n is a local minimum point for $u_n - \varphi$ on $e_j \cup e_k$, we get

$$H(v, \lim_n u_n(y_n), \partial\varphi(v)) \geq 0$$

but if the sequence of points x_n s.t. $u_*(x) = \lim_{n \rightarrow \infty} u_n(x_n)$ belongs to a third different edge (e_m in the figure), then we could have

$\lim_n u_n(y_n) > u_*(x)$ and therefore

$$H(v, \lim_n u_n(y_n), \partial\varphi(v)) \geq H(v, u_*(v), \partial\varphi(v))$$



Remark: The previous argument cannot be adapted to stability results via [semi-relaxed limits a la "Barles-Perthame"](#):

Let u_n be a sequence of supersolutions of a given HJ equation and we want to show that $u_*(x) = \liminf_{n \rightarrow \infty} u_n(x)$ is a supersolution at a vertex $x = v$.

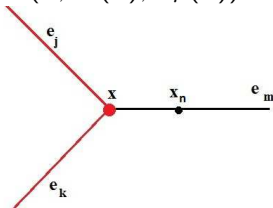
We test as before u_* by means of a (j, k) -admissible test function φ . If y_n is a local minimum point for $u_n - \varphi$ on $e_j \cup e_k$, we get

$$H(v, \lim_n u_n(y_n), \partial\varphi(v)) \geq 0$$

but if the sequence of points x_n s.t. $u_*(x) = \lim_{n \rightarrow \infty} u_n(x_n)$ belongs to a third different edge (e_m in the figure), then we could have

$\lim_n u_n(y_n) > u_*(x)$ and therefore

$$H(v, \lim_n u_n(y_n), \partial\varphi(v)) \geq H(v, u_*(v), \partial\varphi(v))$$



Remark: A similar remark applies to the Perron's method. In proving that $u := \sup_{v \in \mathcal{V}} v$ is a subsolution, we need to show that if

$$(v_n - u)(x) \geq -1/n$$

and if y_n be a local maximum point for $v_n - \varphi$, then

$$y_n \rightarrow x \quad \text{and} \quad v_n(y_n) \rightarrow u^*(x)$$

To get it we used **Lipschitz continuity of subsolutions** which is implied by coercivity

Remark: Note that for comparison principle, Perron's method and stability convexity is not necessary. For Perron we need the coercivity and the monotonicity condition in $|\rho|$ at the transition vertices.

Remark: A similar remark applies to the Perron's method. In proving that $u := \sup_{v \in \mathcal{V}} v$ is a subsolution, we need to show that if

$$(v_n - u)(x) \geq -1/n$$

and if y_n be a local maximum point for $v_n - \varphi$, then

$$y_n \rightarrow x \quad \text{and} \quad v_n(y_n) \rightarrow u^*(x)$$

To get it we used [Lipschitz continuity of subsolutions](#) which is implied by coercivity

Remark: Note that for comparison principle, Perron's method and stability convexity is not necessary. For Perron we need the coercivity and the monotonicity condition in $|\rho|$ at the transition vertices.

The Eikonal equation: a representation formula

We consider the Hamilton-Jacobi equation

$$H(x, \partial u) = 0 \quad x \in \Gamma$$

Assume that, besides the previous assumptions (continuity, coercivity), the Hamiltonian H satisfies

$$H^j(y, p) \text{ is convex in } p \in \mathbb{R} \text{ for any } y \in [0, l_j], j \in J,$$

The Lagrangian $L(x, q)$ is defined on each edge e_j by

$$L^j(x, q) = \sup_{p \in \mathbb{R}} \{p q - H^j(x, p)\} \quad x \in [0, l_j]$$

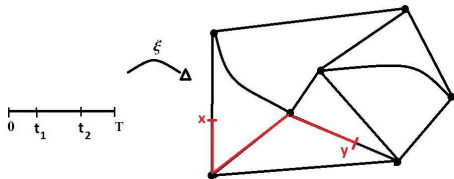
The distance function

Define $S : \Gamma \times \Gamma \rightarrow \mathbb{R}$ by

$$S(y, x) = \inf \left\{ \int_0^t L(\gamma(s), \dot{\gamma}(s)) ds : \gamma \in B_{y,x}^t \right\}$$

where $B_{y,x}^t$ is the set of paths $\gamma : [0, t] \rightarrow \Gamma$ connecting y to x and piecewise differentiable (i.e. there exist $t_0 := 0 < t_1 < \dots < t_{n+1} := t$ s.t. for any $m = 0, \dots, n$, we have $\gamma([t_m, t_{m+1}]) \subset e_{j_m}$ for some $j_m \in J$, $\pi_{j_m}^{-1} \circ \gamma \in C^1(t_m, t_{m+1})$, and $\dot{\gamma}(s) = \frac{d}{ds}(\pi_{j_m}^{-1} \circ \gamma)(s)$).

Remark: $S(y, x)$ coincides with the **path distance** $d(y, x)$ on the network for $H(x, p) = |p|^2 - 1$.



Properties of S

S is a Lipschitz continuous distance on $\Gamma \times \Gamma$. Moreover,

- i) for any $y \in \Gamma$, $S(y, \cdot)$ is a subsolution in Γ and a supersolution in $\Gamma \setminus \{y\}$,
- ii) for any $x, y \in \Gamma$

$$S(y, x) = \max\{u(x) : u \text{ is a subsolution of (HJ) s.t. } u(y) = 0\}.$$

Theorem

Assume that there exists a differentiable function ψ such that $H(x, D\psi) < 0$ in Γ . Let $g : \partial\Gamma \rightarrow \mathbb{R}$ be a continuous function satisfying

$$g(v) - g(w) \leq S(w, v), \quad \text{for any } v, w \in \partial\Gamma$$

Then

$$u(x) := \min\{g(v) + S(y, x) : v \in \partial\Gamma\}$$

is the **unique viscosity solution** of

$$\begin{cases} H(x, \partial u) = 0, & x \in \Gamma; \\ u = g, & x \in \partial\Gamma. \end{cases}$$

Remark: If $H(x, p) = |p| - f(x)$ with $f(x) \geq \eta > 0$, the $\psi \equiv 0$ satisfies the assumption of the previous theorem.

Some remarks about the evolutive problem

Assume that the graph represents an oil pipeline (or a network of computer, or the circulatory system) and at initial time a fire starts in the region Γ_0 (a virus is detected in a subnet, an embolus occurs in some vessel). The speed of propagation of the fire is known but it can depend on the state variable.

How can we find an optimal strategy to stop the fire and to minimize the burnt region?



We study the evolutive equation

$$u_t + H(x, Du) = 0 \quad (x, t) \in \Gamma \times (0, \infty)$$

where H satisfies

- $H \in C^0(\Gamma \times \mathbb{R})$;
- $H(x, \cdot)$ is convex, positive homogeneous in p for any $x \in \Gamma$;
- $\inf\{H(x, p) : |p| = 1, x \in \Gamma\} > 0$;
- $\sup_{|p|=1} (\limsup_{|x| \rightarrow \infty} H(x, p)/|x|) < +\infty$;
- $H(x, p) = H(x, -p)$ for any $x \in V, p \in \mathbb{R}$.

Remark: The Hamiltonian $H(x, p) = |p|/c(x)$ with $c(x) > 0$ on Γ satisfies the previous assumptions

Test functions

Let $\varphi \in C(\Gamma \times (0, T))$.

- i) Let $(x, t) \in e_j \times (0, t)$, $j \in J$. We say that φ is test function at (x, t) , if φ_j is differentiable at $(\pi_j^{-1}(x), t)$.
- ii) Let $(x, t) = (v_i, t)$, $i \in I_T$, $j, k \in Inc_i$, $j \neq k$, $t \in (0, t)$. We say that φ is (j, k) -test function at (x, t) , if φ_j and φ_k are differentiable at $(\pi_j^{-1}(x), t)$ and $(\pi_k^{-1}(x), t)$, respectively and

$$\begin{aligned}\partial_t \varphi_j(\pi_j^{-1}(x), t) &= \partial_t \varphi_k(\pi_k^{-1}(x), t) \\ a_{ij} \partial_j \varphi_j(\pi_j^{-1}(x), t) + a_{ik} \partial_k \varphi_k(\pi_k^{-1}(x), t) &= 0,\end{aligned}$$

where (a_{ij}) is the incidence matrix.

Viscosity solutions

- i) If $(x, t) \in e_j \times (0, T]$, $j \in J$, then a function $u \in \text{USC}(\Gamma \times (0, T])$ (resp., $v \in \text{LSC}(\Gamma \times (0, T])$) is called a subsolution (resp. supersolution) at (x, t) if for any test function φ for which $u - \varphi$ attains a local maximum (resp., a local minimum) at (x, t) , we have

$$\varphi_t(x, t) + H(x, \partial\varphi(x, t)) \leq 0 \quad (\text{resp.}, \geq 0)$$

- ii) If $(x, t) = (v_i, t)$, $i \in I_T$, $t \in (0, T]$ then

- A function $u \in \text{USC}(\Gamma \times (0, \infty))$ is called a subsolution at (x, t) if for any $j, k \in \text{Inc}_i$ and any (j, k) -test function φ for which $u - \varphi$ attains a local maximum at (x, t) relatively to $(e_j \cup e_k) \times (0, \infty)$, we have

$$\varphi_t(x, t) + H(x, \partial\varphi(x, t)) \leq 0.$$

- A function $u \in \text{LSC}(\Gamma \times (0, \infty))$ is called a supersolution at (x, t) if for any $j \in \text{Inc}_i$, there exists $k \in \text{Inc}_i$, $k \neq j$ (said i -feasible for j at (x, t)) such that for any (j, k) -test function φ for which $u - \varphi$ attains a local minimum at (x, t) relatively to $(e_j \cup e_k) \times (0, \infty)$, we have

$$\varphi_t(x, t) + H(x, \partial\varphi(x, t)) \geq 0.$$

Comparison principle

We assume for simplicity that Γ has no boundary points.

Theorem

Let $u \in USC(\Gamma \times [0, T])$, $v \in LSC(\Gamma \times [0, T])$ be a subsolution and, respectively, a supersolution of the evolutive equation such that $u(x, 0) \leq v(x, 0)$. Then $u \leq v$ in $\Gamma \times [0, T]$.

The proof is based on the doubling of variable argument

$$\Psi(x, t, \xi, s) := u(x, t) - v(\xi, s) - \delta t - \frac{d(x, \xi)^2}{\alpha} - \frac{|t - s|^2}{\alpha} - \frac{\eta}{T - t}$$

where d is the path distance on the network.

Representation formula

Theorem

Let $u_0 : \Gamma \rightarrow \mathbb{R}$ be a continuous function. Then the solution of the evolutive equation with the initial condition

$$u(x, 0) = u_0(x_0), \quad x \in \Gamma$$

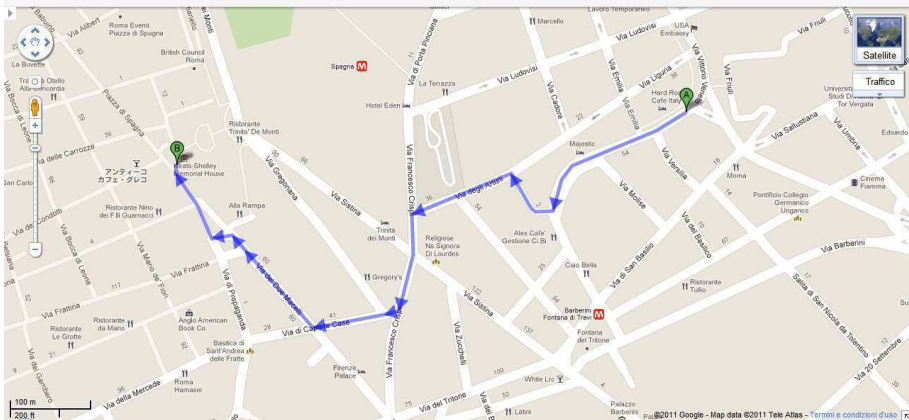
is given by

$$u(x, t) = \min\{u_0(y) : S(y, x) \leq t\}$$

where S is the distance function associated with the stationary equation $H(x, Du) = 1$ on Γ .

Google maps

Italia



Thank You!

Hamilton-Jacobi equations on networks: Lecture 3

Fabio Camilli

Web page:

<http://www.dmmm.uniroma1.it/~fabio.camilli/>

Hamilton-Jacobi on Networks: 3rd lecture

In this lecture

- The vanishing viscosity method
- Comparison with other definitions of viscosity solutions on network
- A numerical scheme

Vanishing Viscosity Approximation¹

In many problems, the vanishing viscosity method arises as a **natural selection principle** of the physical admissible solution in the class of weak solutions

Goal

Show that viscosity solution of the Hamilton-Jacobi equation

$$H(x, u, \partial u) = 0, \quad x \in \Gamma,$$

can be obtained as limit of solutions to the viscous approximation

$$-\varepsilon \partial^2 u_\varepsilon + H(x, u_\varepsilon, \partial u_\varepsilon) = 0, \quad x \in \Gamma,$$

letting the viscosity $\varepsilon \rightarrow 0^+$

¹F. CAMILLI, G. MARCHI, D. SCHIEBORN, The vanishing viscosity limit for Hamilton-Jacobi equation on networks, J.Diff. Eq. (2013)

Vanishing Viscosity Approximation¹

In many problems, the vanishing viscosity method arises as a **natural selection principle** of the physical admissible solution in the class of weak solutions

Goal

Show that viscosity solution of the Hamilton-Jacobi equation

$$H(x, u, \partial u) = 0, \quad x \in \Gamma,$$

can be obtained as limit of solutions to the viscous approximation

$$-\varepsilon \partial^2 u_\varepsilon + H(x, u_\varepsilon, \partial u_\varepsilon) = 0, \quad x \in \Gamma,$$

letting the viscosity $\varepsilon \rightarrow 0^+$

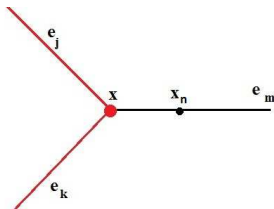
¹F. CAMILLI, C. MARCHI, D. SCHIEBORN, The vanishing viscosity limit for Hamilton-Jacobi equation on networks, J.Diff. Eq. (2013)

Since the approach via semi-relaxed limits does not work with viscosity solutions on networks (see the previous lecture), we have to resume to the classical Crandall-Evans-Lions approach:

- Existence of classical solution for the second order problem
- A-priori estimates for the solution of the second order problem
- Convergence to the solution of the HJ equation

Remark:

We test $u_*(x) = \liminf_{*n \rightarrow \infty} u_n(x) = \lim_{n \rightarrow \infty} u_n(x_n)$ by means of a (j, k) -admissible test function φ and x_n belongs to a third different edge (e_m in the figure), we cannot conclude that u_* is a supersolution



The 2nd order problem

We consider

$$\begin{cases} -\varepsilon \partial^2 u + H(x, u, \partial u) = 0 & x \in \Gamma \\ \sum_{j \in \text{Inc}_i} a_{ij} \beta_{ij} \partial_j u(v_i) = 0 & v_i \in \Gamma_T \\ u(v_i) = g(v_i) & v_i \in \partial\Gamma \end{cases}$$

where

- The Hamiltonian is continuous, convex, coercive
- a_{ij} , $i \in I$, $j \in J$ is the **incidence matrix** and the **weights** β_{ij} , $i \in I$, $j \in J$ are positive
- $g : \partial\Gamma \rightarrow \mathbb{R}$ is the **boundary datum**

Definition

A function u is a solution of the viscous HJ equation

- If $u \in C^2(\Gamma)$ (i.e. $u \in C^0(\Gamma)$ and $u^j \in C^2([0, l_j])$ for any $j \in J$)
- u satisfies the problem in **pointwise sense**

A particular case: $H(x, p) = |p|^2 - f(x)$

Let $u_\varepsilon \in C^2(\Gamma)$ be a solution of

$$\begin{cases} -\varepsilon \partial^2 u + |\partial u|^2 - f(x) = 0 & x \in \Gamma \\ \sum_{j \in \text{Inc}_i} \mathbf{a}_{ij} \beta_{ij} \partial_j u(v_i) = 0 & v_i \in \Gamma_T \\ u(v_i) = g_i & v_i \in \partial\Gamma \end{cases}$$

and set $w_\varepsilon = e^{-\frac{u_\varepsilon}{\varepsilon}} - 1$. Then $w_\varepsilon \in C^2(\Gamma)$ is a solution of the **linear problem** (the Hopf-Cole transform preserves the Kirchhoff's cond.)

$$\begin{cases} \varepsilon \partial^2 w_\varepsilon + f(x) w_\varepsilon - f(x) = 0 & x \in \Gamma \\ \sum_{j \in \text{Inc}_i} \mathbf{a}_{ij} \beta_{ij} \partial_j w_\varepsilon(v_i) = 0 & v_i \in \Gamma_T \\ w_\varepsilon(v_i) = e^{-g_i/\varepsilon} - 1 & v_i \in \partial\Gamma \end{cases}$$

By Freidlin-Wentzell '93 the linear problem admits a solution. Hence via Hopf-Cole transform we get

Proposition

For any $\varepsilon > 0$, there exists a unique solution $u_\varepsilon \in C^2(\Gamma)$ of

$$\begin{cases} -\varepsilon \partial^2 u + |\partial u|^2 - f(x) = 0 & x \in \Gamma, \\ \sum_{j \in \text{Inc}_i} a_{ij} \beta_{ij} \partial_j u(v_i) = 0 & v_i \in \Gamma_T \\ u(v_i) = g_i & v_i \in \partial\Gamma \end{cases}$$

An existence result for classical solution of PDEs with Hamiltonian satisfying quadratic growth in p in

POKORNYI, YU. V.; PENKIN, O. M.; PRYADIEV, V. L. A nonlinear boundary value problem on a graph. (Russian) Differ. Uravn. 34 (1998) (translation in Differential Equations 34 (1998))

Theorem

Assume that H is continuous and there exists h satisfying the Nagumo condition $\int_0^\infty \frac{r}{h(r)} dr = \infty$, g locally integrable and $k > 0$ s.t.

$$H^j(x, r, p) \leq g(r)h(p) \quad \text{for } j \in J, x \in [0, \ell_j], r \in \mathbb{R}, p \in \mathbb{R}$$

$$\partial_r H^j(x, r, p) \geq k \quad \text{for } j \in J, x \in [0, \ell_j], r \in \mathbb{R}, p \in \mathbb{R}$$

Then there exists a solution $u_\varepsilon \in C^2(\Gamma)$ to

$$\begin{cases} -\varepsilon \partial^2 u + H(x, u, \partial u) = 0 & x \in \Gamma, \\ \sum_{j \in \text{Inc}_i} a_{ij} \beta_{ij} \partial_j u(v_i) = 0 & v_i \in \Gamma_T \\ u(v_i) = g_i & v_i \in \partial\Gamma \end{cases}$$

Concerning the uniqueness for the 2^{nd} order problem we have

Theorem

Assume that $H(x, r, p)$ is **strictly increasing** in r . Let $w_1, w_2 \in C^2(\Gamma)$ be such that

$$\begin{cases} -\varepsilon \partial_j^2 w_1 + H(x, w_1, \partial_j w_1) \geq -\varepsilon \partial_j^2 w_2 + H(x, w_2, \partial_j w_2) & x \in \Gamma, \\ \sum_{j \in \text{Inc}_i} a_{ij} \beta_{ij} \partial_j w_1(v) \leq \sum_{j \in \text{Inc}_i} a_{ij} \beta_{ij} \partial_j w_2(v) & v_i \in \Gamma_T \\ w_1(v_i) \geq w_2(v_i) & v_i \in \partial\Gamma \end{cases}$$

Then $w_1 \geq w_2$ on Γ .

Remark:

- The monotonicity of H in u can be replaced by other assumptions on H : for example by **regularity** in $(u, \partial u)$ or by **linear growth** in $(u, \partial u)$
- The previous comparison result is also a key ingredient for the a-priori estimates for the second order problem

A priori estimates for 2nd order problem

Theorem

Assume there exists $\delta > 0$ and $\psi \in C^2(\Gamma)$ such that

$$\begin{cases} H^j(x, \psi(x), \partial_j \psi) \leq -\delta & x \in \Gamma \\ \sum_{j \in \text{Inc}_i} a_{ij} \beta_{ij} \partial_j \psi(v_i) \geq 0 & v_i \in \Gamma_T \\ \psi(v_i) = g_i & v_i \in \partial\Gamma \end{cases}$$

and let $u_\varepsilon \in C^2(\Gamma)$ be a solution of

$$\begin{cases} -\varepsilon \partial_j^2 u_\varepsilon + H(x, u_\varepsilon, \partial_j u_\varepsilon) = 0 & x \in \Gamma, \\ \sum_{j \in \text{Inc}_i} a_{ij} \beta_{ij} \partial_j u_\varepsilon(v_i) = 0 & v_i \in \Gamma_T \\ u_\varepsilon(v_i) = g_i & v_i \in \partial\Gamma \end{cases}$$

Then there is $\bar{\varepsilon}$ sufficiently small such that for any $0 < \varepsilon < \bar{\varepsilon}$, the functions u_ε are **uniformly bounded and equi-Lipschitz continuous** on Γ .

The estimates are first obtained at the vertices using the boundary and the Kirchhoff conditions, then inside the edges using the previous estimate at the vertices. The main difficulty is to prove the estimate on $|\partial u|$ at the vertices.

- For the boundary point we prove that there exists K, β independent of ε such that

$$\psi(x) \leq u_\varepsilon(x) \leq \psi(x) + Kd(x, \partial\Gamma) \quad \text{if } d_\Gamma(x, \partial\Gamma) \leq \beta$$

- For the transition vertices, we assume by contradiction that, fixed $\varepsilon > 0$, $|\partial u_\varepsilon(x)| \rightarrow \infty$ for $x \rightarrow v \in \Gamma_T$ and we get a contradiction by means of the Kirchhoff condition and the coercivity

The vanishing viscosity limit

Theorem

Let $u_\varepsilon \in C^2(\Gamma)$ be a uniformly bounded and uniformly continuous sequence of the solutions of

$$\begin{cases} -\varepsilon \partial_j^2 u_\varepsilon + H(x, u_\varepsilon, \partial_j u_\varepsilon) = 0 & x \in \Gamma, \\ \sum_{j \in \text{Inc}_i} a_{ij} \beta_{ij} \partial_j u_\varepsilon(v_i) = 0 & v_i \in \Gamma_T \\ u_\varepsilon(v_i) = g_i & v_i \in \partial\Gamma \end{cases}$$

Then, up to a subsequence, u_ε converges uniformly to $u \in C(\Gamma)$, where u is viscosity solution of

$$\begin{cases} H(x, u, \partial u) = 0 & x \in \Gamma, \\ u(v_i) = g_i & v_i \in \partial\Gamma \end{cases}$$

Remark: If the HJ equation has a unique solution u , then all the sequence u_ε converges to u .

The vanishing viscosity limit

Theorem

Let $u_\varepsilon \in C^2(\Gamma)$ be a uniformly bounded and uniformly continuous sequence of the solutions of

$$\begin{cases} -\varepsilon \partial_j^2 u_\varepsilon + H(x, u_\varepsilon, \partial_j u_\varepsilon) = 0 & x \in \Gamma, \\ \sum_{j \in \text{Inc}_i} a_{ij} \beta_{ij} \partial_j u_\varepsilon(v_i) = 0 & v_i \in \Gamma_T \\ u_\varepsilon(v_i) = g_i & v_i \in \partial\Gamma \end{cases}$$

Then, up to a subsequence, u_ε converges uniformly to $u \in C(\Gamma)$, where u is viscosity solution of

$$\begin{cases} H(x, u, \partial u) = 0 & x \in \Gamma, \\ u(v_i) = g_i & v_i \in \partial\Gamma \end{cases}$$

Remark: If the HJ equation has a unique solution u , then all the sequence u_ε converges to u .

Idea of the proof

The main point is to show that at the limit the Kirchhoff condition disappears and the limit function instead satisfies the Hamilton-Jacobi equation also at the vertices.

The Kirchhoff condition is used only in the proof of the supersolution property of the limit, while the proof of the subsolution condition is similar to the one in the classical case (recall that when restricted to the couple $e_j \cup e_k$, the notion of viscosity solution on network is equivalent to the classical one).

To prove the **supersolution condition** we have to show that given $v_i \in I_T$ and $j \in Inc_i$, there exists k feasible for j at v_i , i.e. for every (j, k) -lower test function φ for u at v_i

$$H^j(v_i, u(v_i), \partial_j \varphi(v_i)) \geq 0.$$

- The Kirchhoff condition $\sum_{j \in Inc_i} a_{ij} \beta_{ij} \partial_j u_\varepsilon(v_i) = 0$ says that we can find a subsequence u_{ε_n} and an index $k \in Inc_i$ s.t. (assume wlog that $a_{ik} = 1$)

$$\partial_k u_{\varepsilon_n}(v_i) \leq 0 \quad \forall n \in \mathbb{N}$$

- Formally, at the limit, $\partial_k u(v_i) \leq 0$, hence the function u is decreasing along the edge e_k . This means that e_k is a **minimizing edge for the “distance” from the boundary** and it can be proved that it is admissible for all the other edges $e_j, j \in Inc_i, j \neq k$. \square

To prove the **supersolution condition** we have to show that given $v_i \in I_T$ and $j \in Inc_i$, there exists k feasible for j at v_i , i.e. for every (j, k) -lower test function φ for u at v_i

$$H^j(v_i, u(v_i), \partial_j \varphi(v_i)) \geq 0.$$

- The Kirchhoff condition $\sum_{j \in Inc_i} a_{ij} \beta_{ij} \partial_j u_\varepsilon(v_i) = 0$ says that we can find a subsequence u_{ε_n} and an index $k \in Inc_i$ s.t. (assume wlog that $a_{ik} = 1$)

$$\partial_k u_{\varepsilon_n}(v_i) \leq 0 \quad \forall n \in \mathbb{N}$$

- Formally, at the limit, $\partial_k u(v_i) \leq 0$, hence the function u is decreasing along the edge e_k . This means that e_k is a **minimizing edge for the “distance” from the boundary** and it can be proved that it is admissible for all the other edges $e_j, j \in Inc_i, j \neq k$. \square

A comparison among the notions of viscosity solution for Hamilton-Jacobi equations on networks

In

F. CAMILLI E C. MARCHI, A comparison among various notions of viscosity solutions for Hamilton-Jacobi equations on networks, J. Math. Anal. Appl. 407 (2013)

we try to compare the definitions of viscosity solution on network given in

- Y.ACHDOU, F.CAMILLI, A.CUTRÌ,N.TCHOU, Hamilton-Jacobi equations constrained on networks, NoDEA Nonlinear Differential Equations Appl. 20 (2013).
- C.IMBERT, R. MONNEAU,H. ZIDANI, A Hamilton-Jacobi approach to junction problems and application to traffic flows, ESAIM: cocev 19 (2013).
- D. SCHIEBORN, F.CAMILLI: Viscosity solutions of eikonal equations on topological networks, CVPDE 46 (2013)

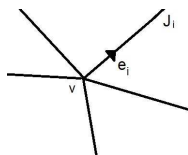
Being motivated by various applications (a control problem constrained to a network in (ACCT), traffic flow at a junction for (IMZ) and Eikonal equations and distance functions on networks for (SC)), they appear to be considerably different.

While **inside the edges** all the three definitions coincide with the standard one of viscosity solution, **at the vertices** they differ for the assumptions made on the Hamiltonians and mainly for the definitions of viscosity solution.

Nevertheless, since all the definitions give existence and uniqueness of the solution, it is worth to compare them and to show that, when restated in a common framework, the three definitions are equivalent.

Test functions at vertices

For simplicity, we consider a network composed by N semi-infinite straight edges $J_i := \mathbb{R}^+ e_i$, $i = 1, \dots, N$.



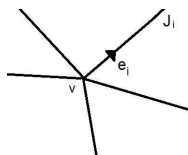
(a) network

A first crucial point is the definition of test functions

- A function $\varphi \in C(\Gamma)$ is an **{ACCT, IMZ}-admissible test function** at v_i if, for any $j \in \text{Inc}_i$, φ^j belongs to $C^1([0, l_j])$.
- A function $\varphi \in C(\Gamma)$ is said a **SC-admissible (j, k)-test function** at v_i if both $\partial_j \varphi(v_i)$ and $\partial_k \varphi(v_i)$ exist with $a_{ij} \partial_j \varphi(v_i) + a_{ik} \partial_k \varphi(v_i) = 0$.

Test functions at vertices

For simplicity, we consider a network composed by N semi-infinite straight edges $J_i := \mathbb{R}^+ e_i$, $i = 1, \dots, N$.

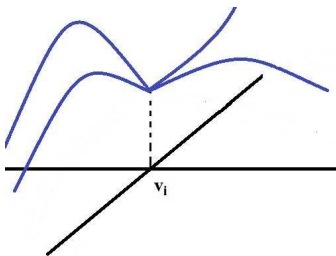


(b) network

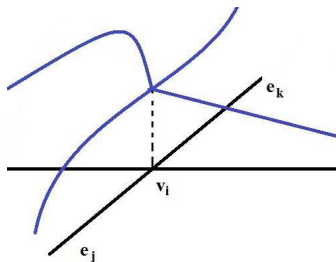
A first crucial point is the definition of test functions

- A function $\varphi \in C(\Gamma)$ is an **{ACCT, IMZ}-admissible test function** at v_i if, for any $j \in \text{Inc}_i$, φ^j belongs to $C^1([0, l_j])$.
- A function $\varphi \in C(\Gamma)$ is said a **SC-admissible (j, k)-test function** at v_i if both $\partial_j \varphi(v_i)$ and $\partial_k \varphi(v_i)$ exist with $a_{ij} \partial_j \varphi(v_i) + a_{ik} \partial_k \varphi(v_i) = 0$.

- A $\{ACCT, IMZ\}$ -test function is differentiable at v_i along each incident edge e_j , $j \in Inc_i$, but the derivatives $\partial_j \varphi(v_i)$ can assume different values.
- A (j, k) $\{CS\}$ -test function φ is required to be differentiable only along the two edges e_j and e_k , while it could be not differentiable in the other directions. Moreover, up to the orientation of the edges, $\partial_j \varphi(v_i) = \partial_k \varphi(v_i)$ and therefore φ differentiable if restricted to couple e_j, e_k



(c) $\{ACCT, IMZ\}$ -test functions



(d) SC -test functions

The ACCT-definition

Aim: Consider a control problem in \mathbb{R}^2

$$\begin{cases} \dot{y}(t) = f(y(t), a(t)) & t > 0 \\ y(0) = x \end{cases}$$

which is **constrained** to a network Γ . Find a viscosity solution characterization of the value function

$$u(x) = \inf \left\{ \int_0^\infty e^{-\lambda t} \ell(y_x(t), a(t)) dt : y_x(t) \in \Gamma \forall t > 0 \right\}, \quad x \in \Gamma.$$

Remark: Observe that known results about constrained control problems do not apply to the case of networks (in \mathbb{R}^2 a network has empty interior)

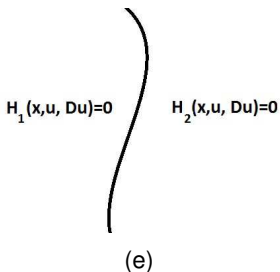
The first version of the paper has been refined and improved in

Y.ACHDOU, S.OUDET, N.TCHOU, Hamilton-Jacobi equations for optimal control on junctions and networks, 2014

using some control theoretic arguments borrowed from

G. BARLES, A. BRIANI, E. CHASSEIGNE, A Bellman approach for two-domains optimal control problems in \mathbb{R}^N , ESAIM Cocv(2014)

where a control problem discontinuous along an interface is considered

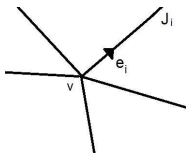


Assumptions

- $f_i : J_i \times A_i \rightarrow \mathbb{R}$ bounded and Lipschitz continuous
- $\ell_i : J_i \times A_i \rightarrow \mathbb{R}$ bounded and continuous
- $FL_i(x) \equiv \{(f_i(x, a)e_i, \ell_i(x, a)), a \in A_i\}$ convex
- Set $F_i(x) = \{f_i(x, a)e_i, a \in A_i\}$. Then $\exists \delta > 0$ s.t.

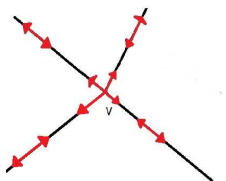
$$[-\delta e_i, \delta e_i] \subset F_i(0), \quad i = 1, \dots, N$$

Remark: The last assumption is a coercivity assumptions in a neighborhood of the vertex v : the dynamics can locally moves in all the directions.



(f) network

Observe that, even if the coefficients are assumed to be continuous on Γ , the set of admissible controls has a discontinuity when passing from inside an edge J_i to the vertex v . Hence the Hamiltonian is in any case discontinuous at the vertex v



(g) Admissible controls

Formally

$$H(x, p) = \begin{cases} \max_{a \in A_i} \{-f(x, a)p - \ell(x, a)\}, & \text{if } x \in J_i; \\ \max_{a \in \cup_i A_i} \{-f(x, a)p - \ell(x, a)\}, & \text{if } x = v. \end{cases}$$

Set

$$FL_i(x) = \{(f_i(x, a)e_i, \ell_i(x, a)), a \in A_i\} \quad x \in J_i$$

$$FL_i^+(v) = FL_i(v) \cap (J_i \times \mathbb{R}), \quad i = 1, \dots, N$$

$$FL(x) = \begin{cases} FL_i(x), & \text{if } x \in J_i \setminus \{v\}; \\ \cup_{i=1}^N FL_i^+(v), & \text{if } x = v. \end{cases}$$

Definition

A function $u \in USC(\Gamma)$ (respectively, $u \in LSC(\Gamma)$) is a subsolution (resp., supersolution) of (HJ) at x if for any admissible test function φ s.t. $u - \varphi$ attains a local maximum (resp., minimum) at x ,

$$\lambda u(x) + \max_{(\zeta, \xi) \in FL(x)} \{-\zeta \partial_i \varphi(x) - \xi\} \leq 0 \quad (\text{resp., } \geq 0).$$

In [ACCT] has been proved that

Prop.

The value function u is bounded and continuous. Moreover it is a bounded viscosity solution of

$$\lambda u(v) + \max_{(\zeta, \xi) \in FL(x)} \{-\zeta \partial_i \varphi(x) - \xi\} \leq 0$$

The proof is based on a dynamic programming argument. The coercivity assumption $[-\delta e_i, \delta e_i] \subset F_i(0)$ plays a crucial role for the continuity of u

Uniqueness

Concerning uniqueness, the proof in [ACCT] has been considerably simplified in Achdou-Oudet-Tchou using control theoretic arguments. We give an idea of their proof.

- **Outside the vertex**

Lemma 1

In $\Gamma \setminus \{v\}$, **sub- and superoptimality principles** are satisfied, i.e. if w is a supersolution (resp. a continuous subsolution), then for $x \in J_i \setminus \{v\}$

$$w(x) \geq \inf_{a, \theta} \int_0^{t \wedge \theta} e^{-\lambda s} \ell(y(s), a(s)) ds + w(y(t \wedge \theta)) e^{-\lambda t \wedge \theta}$$

$$\left(\text{resp., } w(x) \leq \inf_{a, \theta} \int_0^{t \wedge \theta} e^{-\lambda s} \ell(y(s), a(s)) ds + w(y(t \wedge \theta)) e^{-\lambda t \wedge \theta} \right)$$

where $\theta \in [\tau, \sigma]$ with τ, σ are the exit times from $J_i \setminus \{v\}$ and J_i

- At the vertex v

Lemma 2: subsolutions

There exists a neighborhood of v in which any bounded viscosity subsolution u is **Lipschitz continuous**. Moreover

$$\lambda u(v) + \max_{i=1, \dots, N} \max_{a \in A_i: f_i(v, a) = 0} \{-\ell_i(v, a)\} \leq 0$$

Lemma 3: supersolutions

Let w be a viscosity supersolution. Then either

[A] There exists $\eta > 0$, $i \in \{1, \dots, N\}$, $x_k \in J_i$ such that $\lim_k x_k = v$ and $\lim_k w(x_k) = w(v)$ and $\forall k, \exists a_k$ s.t.

$$w(x_k) \geq \int_0^\eta e^{-\lambda s} \ell(y_{x_k}(s), a_k(s)) ds + w(y_{x_k}(\eta)) e^{-\lambda \eta}$$

or

[B]

$$\lambda w(v) + \max_{i=1, \dots, N} \max_{a \in A_i: f_i(v, a) = 0} \{-\ell_i(v, a)\} \geq 0$$

Remark: The previous lemma means more or less that at v either there is a sequence of almost optimal trajectories coming from one of the incident directions or the optimal trajectory is the stationary one at the vertex.

Lemma 3: supersolutions

Let w be a viscosity supersolution. Then either

[A] There exists $\eta > 0$, $i \in \{1, \dots, N\}$, $x_k \in J_i$ such that $\lim_k x_k = v$ and $\lim_k w(x_k) = w(v)$ and $\forall k, \exists a_k$ s.t.

$$w(x_k) \geq \int_0^\eta e^{-\lambda s} \ell(y_{x_k}(s), a_k(s)) ds + w(y_{x_k}(\eta)) e^{-\lambda \eta}$$

or

[B]

$$\lambda w(v) + \max_{i=1, \dots, N} \max_{a \in A_i: f_i(v, a) = 0} \{-\ell_i(v, a)\} \geq 0$$

Remark: The previous lemma means more or less that at v either there is a sequence of almost optimal trajectories coming from one of the incident directions or the optimal trajectory is the stationary one at the vertex.

Theorem ([AOT])

Let u, w be bounded, continuous viscosity sub and supersolution of the Hamilton-Jacobi equation. Then $u \leq w$,

Idea of the proof

Set $u_\mu = \mu u + (1 - \mu)(-|x|^2 - M)$, $\mu \in (0, 1)$, which is a subsolution for M sufficiently large, and $M_\mu = \max_\Gamma(u_\mu - v) = (u_\mu - w)(x_\mu)$. Then it is sufficient to show that $M_\mu \leq 0$ for all $\mu \in (0, 1)$ and pass to the limit for $\mu \rightarrow 0$.

Theorem ([AOT])

Let u, w be bounded, continuous viscosity sub and supersolution of the Hamilton-Jacobi equation. Then $u \leq w$,

Idea of the proof

Set $u_\mu = \mu u + (1 - \mu)(-|x|^2 - M)$, $\mu \in (0, 1)$, which is a subsolution for M sufficiently large, and $M_\mu = \max_\Gamma(u_\mu - v) = (u_\mu - w)(x_\mu)$. Then it is sufficient to show that $M_\mu \leq 0$ for all $\mu \in (0, 1)$ and pass to the limit for $\mu \rightarrow 0$.

Case 1: $x_\mu \in J_i$ for some i

Standard.

Case 2: x_μ coincides with the vertex v

By Lemma 3 two possibilities:

[A]: there exists $\eta > 0$, $x_k \in J_i$ s.t. $\lim_k x_k = v$, a_k

$$w(x_k) \geq \int_0^\eta e^{-\lambda s} \ell(y_{x_k}(s), a_k(s)) ds + w(y_{x_k}(\eta)) e^{-\lambda \eta}$$

Since by Lemma 2

$$u_\mu(x_k) \leq \int_0^\eta e^{-\lambda s} \ell(y_{x_k}(s), a_k(s)) ds + u_\mu(y_{x_k}(\eta)) e^{-\lambda \eta}$$

we get $u_\mu(x_k) - w(x_k) \leq e^{-\lambda \eta} (u_\mu(y_{x_k}(\eta)) - w(y_{x_k}(\eta)))$ and therefore since $x_k \rightarrow x_\mu$ (recall that $M_\mu = (u_\mu - w)(x_\mu)$)

$$M_\mu \leq M_\mu e^{-\lambda \eta},$$

i.e. $M_\mu \leq 0$.

Case 1: $x_\mu \in J_i$ for some i

Standard.

Case 2: x_μ coincides with the vertex v

By Lemma 3 two possibilities:

[A]: there exists $\eta > 0$, $x_k \in J_i$ s.t. $\lim_k x_k = v$, a_k

$$w(x_k) \geq \int_0^\eta e^{-\lambda s} \ell(y_{x_k}(s), a_k(s)) ds + w(y_{x_k}(\eta)) e^{-\lambda \eta}$$

Since by Lemma 2

$$u_\mu(x_k) \leq \int_0^\eta e^{-\lambda s} \ell(y_{x_k}(s), a_k(s)) ds + u_\mu(y_{x_k}(\eta)) e^{-\lambda \eta}$$

we get $u_\mu(x_k) - w(x_k) \leq e^{-\lambda \eta} (u_\mu(y_{x_k}(\eta)) - w(y_{x_k}(\eta)))$ and therefore since $x_k \rightarrow x_\mu$ (recall that $M_\mu = (u_\mu - w)(x_\mu)$)

$$M_\mu \leq M_\mu e^{-\lambda \eta},$$

i.e. $M_\mu \leq 0$.

[B]: w satisfies

$$\lambda w(x_\mu) + \max_{i=1,\dots,N} \max_{a \in A_i: f_i(v,a)=0} \{-\ell_i(v, a)\} \geq 0$$

Since by Lemma 1

$$\lambda u(x_\mu) + \max_{i=1,\dots,N} \max_{a \in A_i: f_i(v,a)=0} \{-\ell_i(v, a)\} \leq 0$$

hence $\lambda u(x_\mu) \leq \lambda w(x_\mu)$ and therefore $M_\mu = (u_\mu - w)(x_\mu) \leq 0$. □

The IMZ-definition

Also in this case, the first paper has been extended in

C.IMBERT, R. MONNEAU, Level-set convex Hamilton-Jacobi equations on networks, 2013

Here the important point is to show that the definition of viscosity solution can be linked to the one of entropy solution for conservation laws on networks.

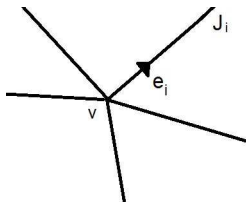
For this reason in the first paper the Hamiltonian depends only on p , while in the second paper also a dependence on x has been included. Moreover the conditions at the vertices are more in the spirit of a transition condition rather than the verification of the equation as in [ACCT] and [SC].

For simplicity I will state the assumptions and the definitions only in the x -independent case

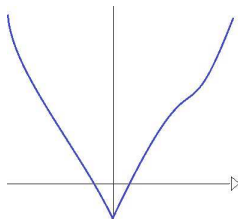
Assumptions

The network is as in [ACCT]

- **Continuity:** $H_i \in C^0(\mathbb{R})$
- **Level set convexity:** $\exists p_0^i \in \mathbb{R}$ s.t. H^i is non-increasing on $(-\infty, p_0^i)$, and non-decreasing on $(p_0^i, +\infty)$
- **Coercivity:** $\lim_{|p| \rightarrow \infty} H_i(p) = +\infty$



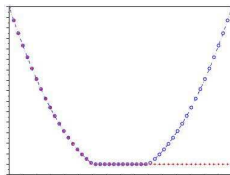
(h) network



(i) Hamiltonian H_i

Define the function H_i^- by

$$H_i^-(p) := \begin{cases} H^i(p), & \text{if } p < p_0^i \\ H^i(p_0^i), & \text{if } p \geq p_0^i. \end{cases}$$



(j) $H_i^-(p)$

Definition at the vertex

A function $u \in USC(\Gamma)$ (respectively, $u \in LSC(\Gamma)$) is a subsolution (resp., supersolution) of (HJ) at v if for any admissible φ s.t. $u - \varphi$ has a local maximum (resp., minimum) point at v , then

$$u(v) + \max_{i \in I} H_i^-(\partial_i \varphi(v)) \leq 0 \quad (\text{resp., } \geq 0)$$

Comparison between ACCT and IMZ

In Camilli-Marchi (2013) it is proved for the case of H independent of x that

ACCT subsolution \iff IMZ subsolution

ACCT supersolution \iff IMZ subsolution

and therefore

ACCT solution \iff IMZ solution

(recall that admissible test functions for the two definitions coincide).

A further insight is given in Achdou-Oudet-Tchou where it is shown that

- In the [ACCT] definition of viscosity solution at the vertex is equivalent to consider the Hamiltonian

$$\lambda u + \max_{i=1, \dots, N} \max_{a \in A_i: f_i(v, a) \geq 0} \{-f_i(x, a) \cdot p - \ell_i(v, a)\} = 0$$

in place of the equation

$$\lambda u + \max_{(\zeta, \xi) \in FL(x)} \{-\zeta \partial_i u - \xi\} = 0$$

- Moreover for the Hamiltonian $\max_{i \in I} H_i^-(p)$ considered in [IMZ], it holds

$$H_i^-(p) = \max_{a \in A_i: f_i(v, a) \geq 0} \{-f_i(v, a) \cdot p - \ell_i(v, a)\}$$

The former statement shows that at the vertex only controls pointing inside the edges play a role in the equation, while the latter one shows that at a vertex the Hamiltonians in [ACCT] and [IMZ] coincide.

In conclusion, under the assumptions of convexity and coercivity, since the definitions in [ACCT] and [IMZ] use the same test functions and the same Hamiltonians, they coincide, i.e.

ACCT subsolution \iff IMZ subsolution

ACCT supersolution \iff IMZ subsolution

Concerning [IMZ] (and [IM])

- In [IMZ], the proof of the comparison principle is based on super-optimality principles and comparison with the value function, i.e. on control theoretic arguments.
- This proof is improved in [IM] using only PDE argument (doubling of the variables).
- The proof of the comparison result in the x -independent case is extended to the x -dependent case using a perturbation argument, but the argument is rather involved.
- In [IM] **more general transmission conditions** at the vertices are considered, including the possibility of adding a running cost at the junctions

Comparison between [ACCT]-[IMZ] and [SC]

The comparison between the definitions in [ACCT] (and [IMZ]) and [SC] is more difficult since the space of test functions are different

- **Subsolution:**

Since in all the three cases the Hamiltonians are assumed to be coercive, the subsolutions are Lipschitz continuous and a.e. subsolutions of the equation. It is well known that in the convex case, a.e. subsolutions and viscosity solutions are equivalent. Hence, it is reasonable to expect that

ACCT subsolution \iff SC subsolution

and also

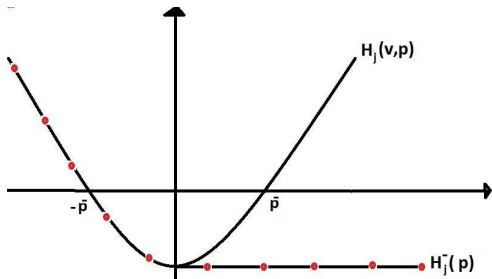
IMZ subsolution \iff SC subsolution

Wlog we assume: $u(v) = 0$ and $\pi_j(v) = 0$ for all $j \in J$, i.e. all the edges are oriented in the direction going outside of v .

Recall H is continuous at v and $H_j(v, p) = H_j(v, -p)$. Hence $\exists \bar{p}$ $H_j(v, p) > 0$ if, and only if, $p \in (-\infty, -\bar{p}) \cup (\bar{p}, +\infty)$.

It follows that

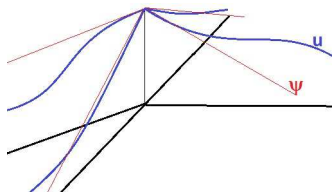
$$H_j^-(p) = \begin{cases} H_j(v, p), & \text{if } p \leq 0; \\ H_j(v, 0), & \text{if } p \geq 0. \end{cases}$$



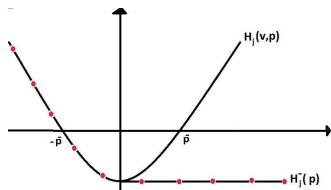
(k) $H_j^-(p)$

Since u is Lipschitz we have $\partial_j^+ u(v) =: [\rho_j, +\infty)$ for each $j \in J$. Hence the function ψ with $\partial_j \psi = \rho_j$ on J_j and $\psi_j(v) = 0$ is an (IMZ)-admissible upper test function and therefore by definition of (IMZ)-viscosity solution $H_j^-(\rho_j) \leq 0$ for each $j \in J$, or equivalently

$$\rho_j \geq -\bar{\rho} \quad \forall j \in J.$$



(l) ψ



(m) $H_j^-(\rho)$

Consider now a (SC)-admissible (j, k) -upper test function φ for u at v . We want to prove that $H^j(v, \partial_j \varphi) = H^k(v, \partial_k \varphi) \leq 0$, namely $\partial_j \varphi(v), \partial_k \varphi(v) \in [-\bar{p}, \bar{p}]$. Assume by contradiction that $\partial_k \varphi(v) > \bar{p}$. Recalling that $a_{ij} \partial_j \varphi(v) + a_{ik} \partial_k \varphi(v) = \partial_j \varphi(v) + \partial_k \varphi(v) = 0$, we have

$$-\bar{p} > -D \partial_k \varphi(v) = \partial_j \varphi(v) \geq p_j$$

which contradicts $p_j \geq -\bar{p}$. Therefore, we have $\partial_k \varphi(v) \leq \bar{p}$ and, similarly, $\partial_j \varphi(v) \leq \bar{p}$. By $\partial_k \varphi(v) = -\partial_j \varphi(v)$, it follows that $\partial_j \varphi(v), \partial_k \varphi(v) \in [-\bar{p}, \bar{p}]$.

The proof of **IMZ-ACCT subsolution** \iff **SC subsolution** is based on similar arguments. □

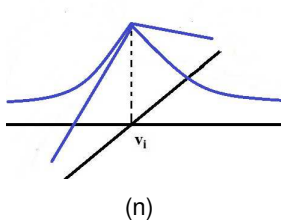
- **Supersolution:**

Concerning the supersolutions, the main problem is that the test functions in the three definitions are different:

In this case, it is possible to prove that

ACCT-IMZ supersolution \implies SC supersolution

but if it happens that at a vertex v_i there exists a (ACCT)-admissible lower test function, but not a (SC)-admissible lower test function,



then the information available from the (SC)-definition of supersolution are not sufficient to obtain an equivalent property for (IMZ)-definition.

But at the level of solution it is possible to recover equivalence. In fact

ACCT-IMZ solution \iff SC solution

Idea of the proof: SC solution \implies IMZ-ACCT solution

We have only to prove that **SC supersolution \implies IMZ-ACCT supersolution.**

Let u be a (SC)-solution; then u is Lipschitz continuous and

$\partial_j^- u(v) = (-\infty, p_j]$ for some $p_j \in \mathbb{R}$.

If ψ is a (ACCT-IMZ)-admissible lower test function, then, $\partial_j \psi(v) \leq p_j$ for each $j \in J$ and we have to prove that

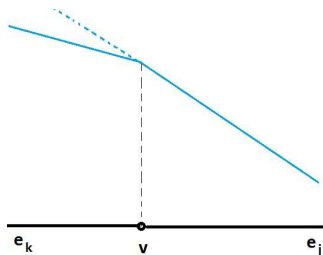
$$u(v) + H^-(\partial_j \psi) \geq 0 \quad \text{for some } j \in J.$$

We distinguish two cases according to the existence or not existence of (SC)-admissible lower test functions. I will sketch the proof only in the case of non existence.

Case (i): Non existence of a (SC)-lower test function.

Fix arbitrarily an edge e_j and let e_k be the feasible edge in the definition of (SC)-supersolution.

Observe that there exists an admissible (j, k) -lower test function if, and only if, there holds $p_j \geq -p_k$, hence $p_j < -p_k$. Moreover either p_j or p_k is negative and wlog we assume $p_j < 0$.



(o) $p_j < -p_k$

On the the segment $e_j \cup e_k$, the definition of (CS)-solution coincides with the classical definition of solution in viscosity sense. By a result of Jensen-Souganidis (Trans. AMS, 1987) it follows that that either u is differentiable in v (but this is not the case since $p_j \neq p_k$) or there holds

$$H_j(v, p_j) = H_k(v, -p_k) = -u(v).$$

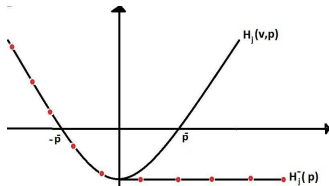
Since $p_j < 0$, we deduce $p_j = -\bar{p}$

$$u(v) + H_j^-(p_j) = 0,$$

and , by the monotonicity of H^- .

$$u(v) + H_j^-(p) \geq 0, \quad \forall p \leq p_j$$

Since $\partial_j^- u(v) = (-\infty, p_j]$, then the (ACCT-IMZ) definition of supersolution follows. □



Remark: In the next lecture, we will also show that the **Giga-Hamamuki-Nakayasu definition of viscosity solution**, which is given for general metric spaces, when recast in particular case of networks, is equivalent to the (SC)-definition of viscosity solution

Summarizing

- ACCT** : coercivity, convexity, no continuity condition at the vertices. Comparison principle and existence based about control theoretic arguments
- IMZ** : coercivity, level set convexity, no continuity condition at the vertices. Comparison principle and existence via control theoretic argument (in [IM] comparison via doubling of variables argument, but the proof is rather involved)
- SC** : coercivity, continuity at the vertices. Comparison principle and existence via PDE argument

It seems to be natural to consider the appropriate definition of viscosity solution for the specific application (control problem constrained to a network in (ACCT), traffic flow at a junction for (IMZ) and Eikonal equations and distance functions on networks for (SC)).

Summarizing

- ACCT** : coercivity, convexity, no continuity condition at the vertices. Comparison principle and existence based about control theoretic arguments
- IMZ** : coercivity, level set convexity, no continuity condition at the vertices. Comparison principle and existence via control theoretic argument (in [IM] comparison via doubling of variables argument, but the proof is rather involved)
- SC** : coercivity, continuity at the vertices. Comparison principle and existence via PDE argument

It seems to be natural to consider the appropriate definition of viscosity solution for the specific application (control problem constrained to a network in (ACCT), traffic flow at a junction for (IMZ) and Eikonal equations and distance functions on networks for (SC)).

A numerical scheme for HJ on networks¹

We consider a numerical scheme for the problem

$$\begin{cases} |Du| = f(x) & x \in \Gamma \\ u = g & x \in \partial\Gamma \end{cases}$$

where $f \in C^0(\Gamma)$, $\min_{\Gamma}\{f\} > 0$.

Grid

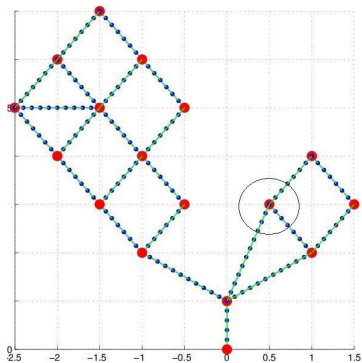
We fix an uniform partition $y_k = mh_j$, $m = 0, \dots, M_j$, of the interval $[0, l_j]$, $j \in J$ where $h_j = \frac{l_j}{M_j}$ is the **space step**. For simplicity in the following we consider the case $h_j \equiv h$, $\forall j \in J$.

We denote $\mathcal{G}_h = \{x_{j,m} = \pi_j(y_m), j \in J, m = 0, \dots, M_j\}$ the spatial grid defined on Γ

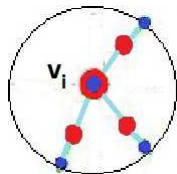
¹F.CAMILLI, A.FESTA E D.SCHIEBORN, An approximation scheme for an Hamilton-Jacobi equation defined on a network, Applied Num. Math. (2013) ▶

Definition

For $x_1, x_2 \in \mathcal{G}_h$, we say that x_1 and x_2 are **adjacent** and we write $x_1 \sim x_2$ if and only if they are the image of two adjacent grid points, i.e. $x_i = \pi_j(y_i)$, for $i = 1, 2$ and $j \in J$ with $|y_1 - y_2| = h$.



(q) grid



(r) adjacent vertices to v_i

Discrete eikonal equation

For a function $u : \mathcal{G}_h \rightarrow \mathbb{R}$, we consider the **discrete eikonal equation**

$$\begin{cases} \max_{y \sim x} \left\{ -\frac{1}{h}(u(y) - u(x)) \right\} = f(x) & \text{in } \mathcal{G}_h \setminus \partial\Gamma \\ u = g & \text{on } \partial\Gamma \end{cases}$$

Remark:

The discrete eikonal equation can be rewritten as

$$u(x) = \min_{y \sim x} \{ u(y) + hf(x) \}$$

which can be seen as a discrete dynamic programming equation

Theorem

Let $g : \partial\Gamma \rightarrow \mathbb{R}$ be such that $g(x) \leq \inf \left\{ \sum_{k=0}^N hf(x_k) + g(y) \right\}$

$\forall x, y \in \partial\Gamma$ where the minimum is over all the paths

$\{x_0 = x, x_1, \dots, x_N = y\}$ with $x_i \sim x_{i+1}$, $i = 0, \dots, N - 1$. Then, the **unique solution** of discrete eikonal equation is given by

$$u_h(x) = \min \left\{ \sum_{k=0}^N hf(x_k) + g(y) \right\}$$

where the minimum is taken over all $y \in \partial\Gamma$ and

$\{x_0 = x, x_1, \dots, x_N = y\}$ with $x_i \sim x_{i+1}$, $i = 0, \dots, N - 1$.

Moreover

$$|u_h(x)| \leq \max_{\partial\Gamma} |g| + d(x, \partial\Gamma) \max_{\Gamma} \{f\} \quad \forall x \in \mathcal{G}_h$$

$$\frac{|u_h(y) - u_h(x)|}{h} \leq \max_{\Gamma} \{f\} \quad \forall x, y \in \mathcal{G}_h, x \sim y$$

Theorem

Let $g : \partial\Gamma \rightarrow \mathbb{R}$ be such that $g(x) \leq \inf \left\{ \sum_{k=0}^N hf(x_k) + g(y) \right\}$

$\forall x, y \in \partial\Gamma$ where the minimum is over all the paths

$\{x_0 = x, x_1, \dots, x_N = y\}$ with $x_i \sim x_{i+1}$, $i = 0, \dots, N - 1$. Then, the **unique solution** of discrete eikonal equation is given by

$$u_h(x) = \min \left\{ \sum_{k=0}^N hf(x_k) + g(y) \right\}$$

where the minimum is taken over all $y \in \partial\Gamma$ and

$\{x_0 = x, x_1, \dots, x_N = y\}$ with $x_i \sim x_{i+1}$, $i = 0, \dots, N - 1$.

Moreover

$$|u_h(x)| \leq \max_{\partial\Gamma} |g| + d(x, \partial\Gamma) \max_{\Gamma} \{f\} \quad \forall x \in \mathcal{G}_h$$

$$\frac{|u_h(y) - u_h(x)|}{h} \leq \max_{\Gamma} \{f\} \quad \forall x, y \in \mathcal{G}_h, x \sim y$$

Convergence

Theorem

Let u_h and u be respectively the solutions of the discrete and of the continuous eikonal equations. Then

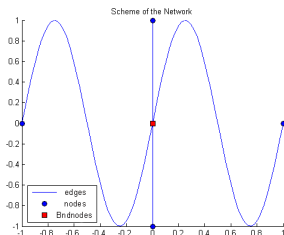
$$|u(x) - u_h(x)| \leq C\omega_f(h^{1/2}) \quad \forall x \in \mathcal{G}_h$$

where ω_f is the modulus of continuity of f .

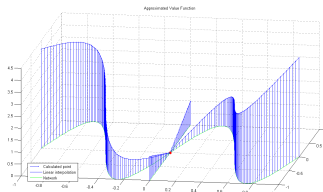
The proof is based on the classical doubling of variables argument in viscosity solution theory adapted to networks

$$\Psi(x, y) = u_h(x) - u(y) - \frac{d(x, y)^2}{2\varepsilon} \quad x \in \mathcal{G}_h, y \in \Gamma.$$

Test1: $f \equiv 1$



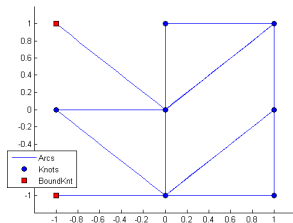
(s) Network



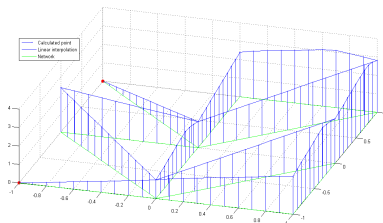
(t) Solution

h	$\ \cdot\ _\infty$	$Ord(L_\infty)$	$\ \cdot\ _2$	$Ord(L_2)$
0.2	0.1468		0.1007	
0.1	0.0901	0.7043	0.0639	0.6562
0.05	0.0630	0.5162	0.0491	0.3801
0.025	0.0450	0.4854	0.0402	0.2885
0.0125	0.0321	0.4874	0.029	0.4711

Test 2: $f \equiv 1$

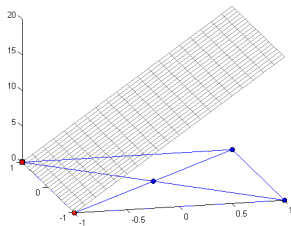


(u) Network

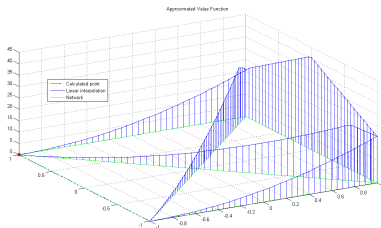


(v) Solution

Test 3: f non constant



(w) Network and f



(x) Solution
HJ on networks

Comments and remarks

- Is the coercivity really necessary for HJ on networks?
- Is it possible to give a PDE proof of comparison principle if H is discontinuous at the vertices (Imbert-Monneau?). This would be interesting also in the Euclidean setting.
- Semi-relaxed limits for stability of viscosity solutions on networks.
- Is it possible to consider transition conditions at the vertices (as in Imbert-Monneau)?
- Nonconvex Hamiltonians and Differential games

Comments and remarks

- Is the coercivity really necessary for HJ on networks?
- Is it possible to give a PDE proof of comparison principle if H is discontinuous at the vertices (Imbert-Monneau?). This would be interesting also in the Euclidean setting.
- Semi-relaxed limits for stability of viscosity solutions on networks.
- Is it possible to consider transition conditions at the vertices (as in Imbert-Monneau)?
- Nonconvex Hamiltonians and Differential games

Comments and remarks

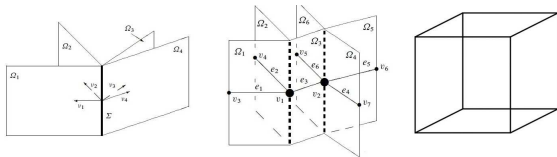
- Is the coercivity really necessary for HJ on networks?
- Is it possible to give a PDE proof of comparison principle if H is discontinuous at the vertices (Imbert-Monneau?). This would be interesting also in the Euclidean setting.
- Semi-relaxed limits for stability of viscosity solutions on networks.
- Is it possible to consider transition conditions at the vertices (as in Imbert-Monneau)?
- Nonconvex Hamiltonians and Differential games

Comments and remarks

- Is the coercivity really necessary for HJ on networks?
- Is it possible to give a PDE proof of comparison principle if H is discontinuous at the vertices (Imbert-Monneau?). This would be interesting also in the Euclidean setting.
- Semi-relaxed limits for stability of viscosity solutions on networks.
- Is it possible to consider transition conditions at the vertices (as in Imbert-Monneau)?
- Nonconvex Hamiltonians and Differential games

- Future developments:

- ▶ Applications: homogenization, large time behavior and weak KAM theory, singularly perturbed and multi-scale methods problem, etc.
- ▶ More general ramified space (for example polygonal ramified space, i.e. subsets of hyperplanes in \mathbb{R}^n which are glued together along certain edges in \mathbb{R}^{n-1}).

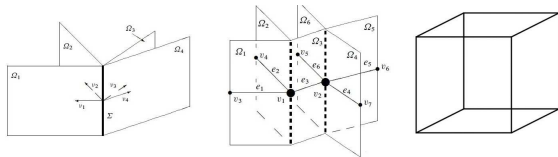


(y) Ramified spaces

- Viscosity solutions for 2^{nd} order equation

- Future developments:

- ▶ Applications: homogenization, large time behavior and weak KAM theory, singularly perturbed and multi-scale methods problem, etc.
- ▶ More general ramified space (for example polygonal ramified space, i.e. subsets of hyperplanes in \mathbb{R}^n which are glued together along certain edges in \mathbb{R}^{n-1}).

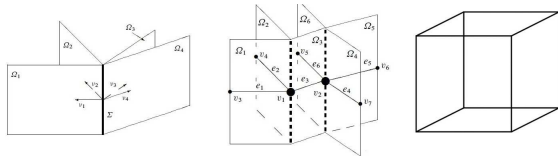


(z) Ramified spaces

- Viscosity solutions for 2^{nd} order equation

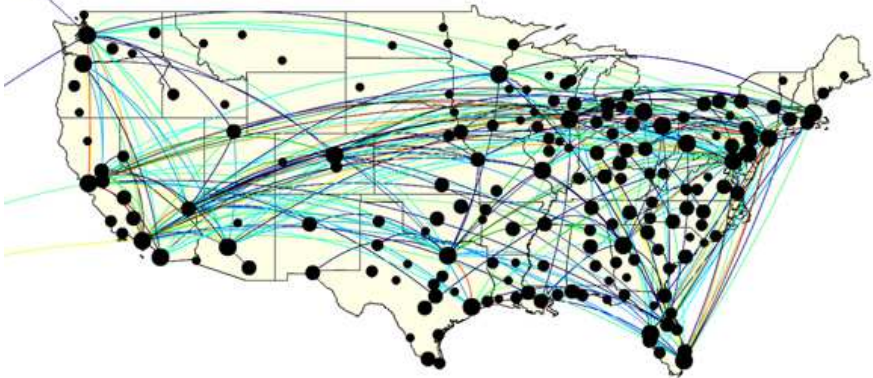
- Future developments:

- ▶ Applications: homogenization, large time behavior and weak KAM theory, singularly perturbed and multi-scale methods problem, etc.
- ▶ More general ramified space (for example polygonal ramified space, i.e. subsets of hyperplanes in \mathbb{R}^n which are glued together along certain edges in \mathbb{R}^{n-1}).



() Ramified spaces

- Viscosity solutions for 2^{nd} order equation



passengers per day (in hundreds)



Thank You!

Hamilton-Jacobi equations on networks: Lecture 4

Fabio Camilli

Web page:

<http://www.dmmm.uniroma1.it/~fabio.camilli/>

Hamilton-Jacobi equations and viscosity solutions in a “non regular” setting:

- Non regular geometric structures: networks, stratified sets, ramified spaces
 - ▶ See the references in the previous lectures
- Spaces of probability measures and metric spaces
 - ▶ Ambrosio-Feng, On a class of first order Hamilton-Jacobi equations in metric spaces, JDE (2014)
 - ▶ Gangbo-Swiech, Metric viscosity solutions of hamilton-jacobi equations, preprint (2014)
 - ▶ Giga-Hamamuki-Nakayasu, Eikonal equations in metric space, Trans.AMS, to appear.

Fractal sets are an intermediate case since:

- Fractals can be seen as the limit case of nonregular geometric sets (prefractals)
- As for metric spaces, there is no natural notion of gradient on fractals.

Hamilton-Jacobi equations and viscosity solutions in a “non regular” setting:

- Non regular geometric structures: networks, stratified sets, ramified spaces
 - ▶ See the references in the previous lectures
- Spaces of probability measures and metric spaces
 - ▶ Ambrosio-Feng, On a class of first order Hamilton-Jacobi equations in metric spaces, JDE (2014)
 - ▶ Gangbo-Swiech, Metric viscosity solutions of hamilton-jacobi equations, preprint (2014)
 - ▶ Giga-Hamamuki-Nakayasu, Eikonal equations in metric space, Trans.AMS, to appear.

Fractal sets are an intermediate case since:

- Fractals can be seen as the limit case of nonregular geometric sets (prefractals)
- As for metric spaces, there is no natural notion of gradient on fractals.

Hamilton-Jacobi equations and viscosity solutions in a “non regular” setting:

- Non regular geometric structures: networks, stratified sets, ramified spaces
 - ▶ See the references in the previous lectures
- Spaces of probability measures and metric spaces
 - ▶ Ambrosio-Feng, On a class of first order Hamilton-Jacobi equations in metric spaces, JDE (2014)
 - ▶ Gangbo-Swiech, Metric viscosity solutions of hamilton-jacobi equations, preprint (2014)
 - ▶ Giga-Hamamuki-Nakayasu, Eikonal equations in metric space, Trans.AMS, to appear.

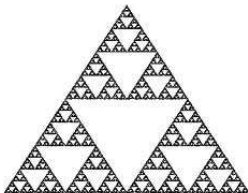
Fractal sets are an intermediate case since:

- Fractals can be seen as the limit case of nonregular geometric sets (prefractals)
- As for metric spaces, there is **no natural notion of gradient** on fractals.

Aim of this talk is to present a notion of viscosity solution for the Eikonal equation

$$|Du| = f(x) \quad x \in S$$

where S is the Sierpinski gasket (or a post critically finite fractal)¹.



(a) Sierpinski gasket

¹F. CAMILLI, R. CAPITANELLI, C. MARCHI, Eikonal equations on the Sierpinski gasket, arXiv:1404.3692, 2014

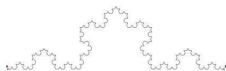
To define the eikonal equation on S we will mimic the classical construction of the [Laplacian on fractals](#):

We consider graph eikonal equations on the prefractals and we show that the solutions of these problems converge to a function which satisfies an eikonal equation on the fractal set.

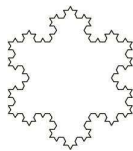
Fractals



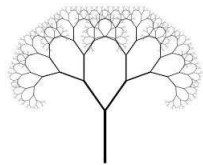
(b) Penta-gasket



(c) Koch curve



(d) Koch is-land



(e) Tree

The term **fractal** is derived from the Latin adjective *fractus*, which means **broken** in irregular fragments. The main features are

- **Self-similarity** - when broken into smaller and smaller pieces, the new pieces look exactly the same as the original
- **Dimension** - the fractals are curves that fill the space

- **K. W. T. Weierstrass** (1815-1897): Nowhere Differentiable Functions
- **G.Cantor**: Cantor set
- **Sierpinski, Julia, Koch** (beginning of '900): fractal set via the iteration of a function of a complex variable
- **P. Levy** : Fractal Random Walks (Random Fractals)
- **B. Mandelbrot** (1970s:) Mandelbrot sets and the development of a general theory on the "Fractal Geometry of Nature" (1975)

After Weierstrass invented his nowhere differentiable function, many mathematicians were alarmed at losing the property of differentiation as a constant. **Hermite** described these new functions as a "dreadful plague" and **Poincaré** wrote

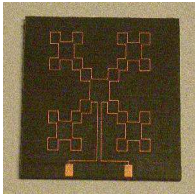
"Yesterday, if a new function was invented it was to serve some practical end; today they are specially invented only to show up the arguments of our fathers, and they will never have any other use"

- **K. W. T. Weierstrass** (1815-1897): Nowhere Differentiable Functions
- **G.Cantor**: Cantor set
- **Sierpinski, Julia, Koch** (beginning of '900): fractal set via the iteration of a function of a complex variable
- **P. Levy** : Fractal Random Walks (Random Fractals)
- **B. Mandelbrot** (1970s:) Mandelbrot sets and the development of a general theory on the "Fractal Geometry of Nature" (1975)

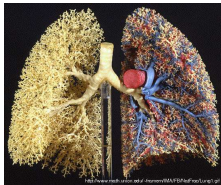
After Weierstrass invented his nowhere differentiable function, many mathematicians were alarmed at losing the property of differentiation as a constant. **Hermite** described these new functions as a "dreadful plague" and **Poincaré** wrote

*"Yesterday, if a new function was invented it was to serve some practical end; today they are specially invented only to show up the arguments of our fathers, and they **will never have any other use**".*

- **Signal Processing:** Time Series Analysis, Speech Recognition
- **Image Processing:** Fractal Compression, Fractal Dimension Segmentation
- **Simulation:** Terrain Modeling, Image Synthesis, Music, Stochastic Fields
- **Financial:** Fractal Market Analysis, Futures Markets
- **Medicine:** Histology, Monitoring, Epidemiology
- **Military:** Visual Camouflage, Covert Digital Communications



(f) Mobile antenna



(g) lung

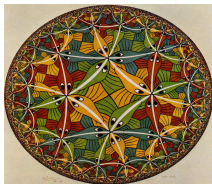


(h) camouflage

Fractals everywhere



(i) Hokusai



(j) Escher



(k) Snow flake



(l) Nautilus



(m) Cauliflower



(n) coastline

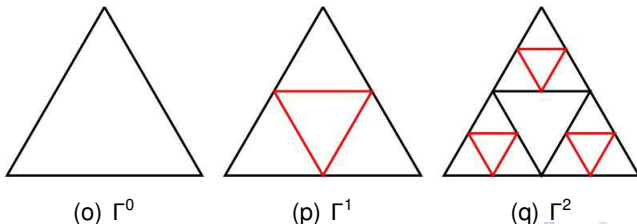
The Sierpinski gasket

Given an equilateral triangle of vertices $\Gamma^0 = \{v_1, v_2, v_3\}$, consider the 3 maps $\psi_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

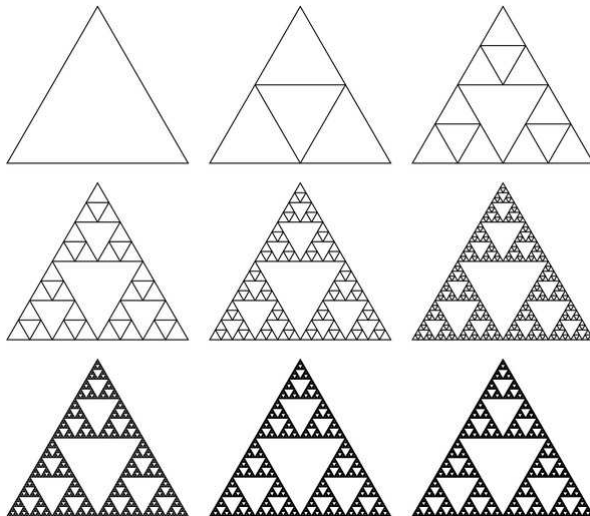
$$\psi_i(x) := v_i + \frac{1}{2}(x - v_i) \quad x \in \mathbb{R}^2, i = 1, \dots, 3.$$

Iterating the ψ_i 's, we get the set $\Gamma^\infty = \bigcup_{n=0}^{\infty} \Gamma^n$ where each prefractal Γ^n is given by the union of the images of Γ^0 under the action of the maps $\psi_{i_1} \circ \dots \circ \psi_{i_n}$ with $i_h \in \{1, \dots, 3\}$, $h = 1, \dots, n$.

The Sierpinski gasket **S** is the closure of Γ^∞ and it is the unique non empty compact set K which satisfies $K = \bigcup_{i=1}^3 \psi_i(K)$.



Sierpinski iterations



(r) Sierpinski iterations

Distance on the Sierpinski

We define the geodesic distance on the network S .

Definition (distance on the Sierpinski)

The distance between $x, y \in S$ is

$$d(x, y) := \inf \{ \ell(\gamma) : \gamma \text{ is a path joining } x \text{ to } y \} \quad (1)$$

where $\ell(\gamma)$ is the length of γ .

Proposition [Hienz-Schief,'90], [Strichartz, Trans.AMS '99]

For any $x, y \in S$, there holds

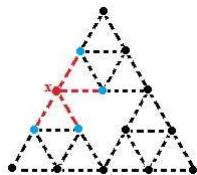
$$|x - y| \leq d(x, y) \leq 1.$$

Moreover d is continuous and the inf in (1) is a minimum.

The Laplace operator on the Sierpinski gasket

The graph (Γ^n, \sim)

On the prefractal Γ^n we consider the relation $x \sim y$ if and only if the segment connecting x and y is the image of a side of Γ under the action of some $\psi_{i_1} \circ \dots \circ \psi_{i_n}$.



(s) Vertices in \sim
with $x \in \Gamma^2$

- $\partial\Gamma^n$, the boundary of the prefractal, is given by the vertices of Γ_0
- $\Gamma_{int}^n = \Gamma^n \setminus \Gamma_0$ is the set of the internal vertices

To define the Laplacian on S , the idea is to introduce discrete operators on the (Γ^n, \sim) and to perform a limit process.

- The **probabilistic version** of this approach was introduced by Kusuoka and Lindstrøm who considered suitable **scaled random walks** on Γ^n and then passed to the limit to define a Brownian motion on S .
- An **analytical approach** was taken by Kigami who considered **scaled finite differences**

$$\Delta^n f(x) = \sum_{y \sim x} \left(\frac{5}{3}\right)^n (f(y) - f(x))^2$$

and passed to the limit to define Δf on S .

The two approaches give rise to the same **self-adjoint differential operator** on S , the Laplacian Δ on S .

The principle underlying the definition of Laplacian and harmonic functions on the Sierpinski gasket is **the minimization of the energy**.

We will try to follow an approach for the eikonal similar to the analytical one by Kigami. For the eikonal equation the principle is the **optimal control interpretation** of the solution at all the different levels (discrete and continuous on the prefractal and continuous on the fractal).

The principle underlying the definition of Laplacian and harmonic functions on the Sierpinski gasket is **the minimization of the energy**.

We will try to follow an approach for the eikonal similar to the analytical one by Kigami. For the eikonal equation the principle is the **optimal control interpretation** of the solution at all the different levels (discrete and continuous on the prefractal and continuous on the fractal).

The discrete eikonal equation on (Γ^n, \sim)

For a function $u : \Gamma^n \rightarrow \mathbb{R}$, we consider the **discrete eikonal equation**

$$\begin{cases} |Du(x)|_n = f(x) & \text{in } \Gamma^n_{int} \\ u = g & \text{on } \Gamma^0 \end{cases}$$

where

$$|Du(x)|_n := \max_{y \sim x} \left\{ -\frac{1}{h_n} (u(y) - u(x)) \right\},$$

$h_n = (1/2)^n$ and the function $f : S \rightarrow \mathbb{R}$ is continuous with

$$\lambda := \inf_S f(x) > 0.$$

Remark:

The eikonal equation can be rewritten as

$$u(x) = \inf_{y \sim x} \{u(y) + h_n f(x)\}$$

Theorem

Let $g : \Gamma^0 \rightarrow \mathbb{R}$ be such that $g(x) \leq \inf \left\{ \sum_{k=0}^N h_n f(x_k) + g(y) \right\}$

$\forall x, y \in \Gamma^0$ where the minimum is over all the paths

$\{x_0 = x, x_1, \dots, x_N = y\}$ with $x_i \sim x_{i+1}$, $i = 0, \dots, N - 1$. Then, the **unique solution** of graph eikonal equation with Dirichlet boundary condition is given by

$$u_n(x) = \min \left\{ \sum_{k=0}^N h_n f(x_k) + g(y) \right\}$$

where the minimum is taken over all $y \in \Gamma^0$ and

$\{x_0 = x, x_1, \dots, x_N = y\}$ with $x_i \sim x_{i+1}$, $i = 0, \dots, N - 1$.

Moreover

$$|u_n(x)| \leq \max_{\Gamma^0} |g| + d(x, \Gamma^0) \max_{\Gamma^n} \{f\} \quad \forall x \in \Gamma^n$$

$$\frac{|u_n(y) - u_n(x)|}{h_n} \leq \max_{\Gamma^n} \{f\} \quad \forall x, y \in \Gamma^n, x \sim y$$

Theorem

Let $g : \Gamma^0 \rightarrow \mathbb{R}$ be such that $g(x) \leq \inf \left\{ \sum_{k=0}^N h_n f(x_k) + g(y) \right\}$

$\forall x, y \in \Gamma^0$ where the minimum is over all the paths

$\{x_0 = x, x_1, \dots, x_N = y\}$ with $x_i \sim x_{i+1}$, $i = 0, \dots, N - 1$. Then, the **unique solution** of graph eikonal equation with Dirichlet boundary condition is given by

$$u_n(x) = \min \left\{ \sum_{k=0}^N h_n f(x_k) + g(y) \right\}$$

where the minimum is taken over all $y \in \Gamma^0$ and

$\{x_0 = x, x_1, \dots, x_N = y\}$ with $x_i \sim x_{i+1}$, $i = 0, \dots, N - 1$.

Moreover

$$|u_n(x)| \leq \max_{\Gamma^0} |g| + d(x, \Gamma^0) \max_{\Gamma^n} \{f\} \quad \forall x \in \Gamma^n$$

$$\frac{|u_n(y) - u_n(x)|}{h_n} \leq \max_{\Gamma^n} \{f\} \quad \forall x, y \in \Gamma^n, x \sim y$$

An immediate consequence of the previous result is that the sequence $\{u_n\}_n$ uniformly converges to a function u on the Sierpinski gasket S .

Following the the definition of Laplace equation on Sierpinski, it would be natural to say that u is the solution of the eikonal equation on S .

Still two questions

- Is it possible to characterize u by an eikonal equation defined on S ?
- Is u the value function of a control problem on S (in particular, for $f = 1$ and $g = 0$, does u coincide with the distance d on S)?

An immediate consequence of the previous result is that the sequence $\{u_n\}_n$ uniformly converges to a function u on the Sierpinski gasket S . Following the the definition of Laplace equation on Sierpinski, it would be natural to say that u is the solution of the eikonal equation on S . Still two questions

- Is it possible to characterize u by an eikonal equation defined on S ?
- Is u the value function of a control problem on S (in particular, for $f = 1$ and $g = 0$, does u coincide with the distance d on S ?)

Hence we look for a characterization of u by means of an **eikonal equation defined on S** .

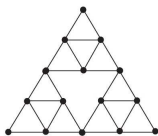
In Giga-Hamamuki-Nakayasu (Trans.AMS, to appear) is given a definition of **metric viscosity solution** for the eikonal equation in a **general metric space \mathcal{X}** . The solution of the Dirichlet problems is given by a representation formula of control type. This definition obviously applies to the Sierpinski gasket endowed with the metric induced by the path distance.

To establish the link between the limit u of the sequence $\{u_n\}$ and the definition of metric solution we proceed in the following way

- 1 We define an appropriate notion of viscosity solution for **continuous eikonal equations on the prefractal Γ^n**
- 2 We estimate uniformly in n the **distance between the solutions of the discrete and of the continuous eikonal equations on Γ^n**
- 3 We show that the solutions of the continuous eikonal equations on Γ^n are also **metric viscosity solutions** if $\mathcal{X} = \Gamma^n$.
- 4 By **stability** the sequence of the solutions of the continuous eikonal equation, and therefore also of discrete eikonal equations, on the prefractal Γ^n converges to the solution of the eikonal equation on S .

Step 1: Eikonal equations on networks

We consider the **network** S^n given by the prefractal $\Gamma^n = \{v_i\}_{i \in I}$ and by the arcs $E^n = \{e_j\}_{j \in J}$ connecting the vertices of Γ^n .



(t) Network S^2

We denote by

- $\pi_j : [0, 1] \rightarrow \mathbb{R}$, a **parametrization** of the edge e_j
- $Inc_i := \{j \in J : x_i \in e_j\}$ the set of the arcs **incident** the vertex v_i

•

$$a_{ij} := \begin{cases} 1 & \text{if } x_i \in e_j \text{ and } \pi_j(0) = x_i, \\ -1 & \text{if } x_i \in e_j \text{ and } \pi_j(\ell_j) = x_i, \\ 0 & \text{otherwise.} \end{cases}$$

the **signed incidence matrix** $A = \{a_{ij}\}_{i \in I, j \in J}$ which gives the orientation of the edge induced by π_j .

Note that the study of [Hamilton-Jacobi equation on networks](#) is not a straightforward generalization of the Euclidean setting because of the presence of the vertices.

With the aim of extending the notion of viscosity solution to networks, several different approaches have been recently proposed. Here I will consider the definition in

D. SCHIEBORN AND F. CAMILLI, *Viscosity solutions of Eikonal equations on topological network*, Calc. Var. Partial Differential Equations. 46 (2013), 671–686.

Note that the study of [Hamilton-Jacobi equation on networks](#) is not a straightforward generalization of the Euclidean setting because of the presence of the vertices.

With the aim of extending the notion of viscosity solution to networks, several different approaches have been recently proposed. Here I will consider the definition in

D. SCHIEBORN AND F. CAMILLI, *Viscosity solutions of Eikonal equations on topological network*, *Calc. Var. Partial Differential Equations*. 46 (2013), 671–686.

We give a definition of network viscosity solution for the eikonal equation

$$|Du| = f(x) \quad \text{in } S^n.$$

Definition (test function)

Let $\varphi \in C^1(S^n)$.

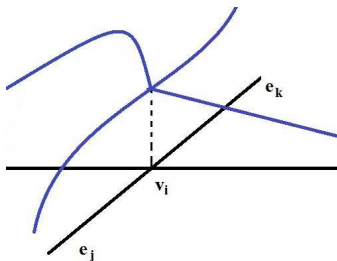
- i) Let $x \in e_j$, $j \in J$. We say that φ is test function at x , if the restriction the restriction of φ to e_j , i.e. $\varphi_j := \varphi \circ \pi_j$ is differentiable at $\pi_j^{-1}(x)$.
- ii) Let $x_i \in \Gamma_{int}^n$, $j, k \in Inc_i$, $j \neq k$. We say that φ is (j, k) -test function at x , if φ_j and φ_k are differentiable at $\pi_j^{-1}(x)$ and $\pi_k^{-1}(x)$, with

$$a_{ij} D_j \varphi(\pi_j^{-1}(x)) + a_{ik} D_k \varphi(\pi_k^{-1}(x)) = 0,$$

where (a_{ij}) is the incidence matrix

Remark:

Condition $a_{ij}D_j\varphi(\pi_j^{-1}(x)) + a_{ik}D_k\varphi(\pi_k^{-1}(x)) = 0$ says that, **taking into account the orientation**, the function φ is differentiable at x_i along the direction given by the couple of edges e_j and e_k (no condition, except continuity, along the other incident edges)



(u) test function

Definition (network viscosity solution)

- i) If $x \in e_j$, $j \in J$, then a function $u \in \text{USC}(S^n)$ (resp., $v \in \text{LSC}(S^n)$) is called a **subsolution** (resp. **supersolution**) at x if for any test function φ for which $u - \varphi$ attains a local maximum at x (resp., a local minimum), we have

$$|D_j \varphi(x)| \leq f(x) \quad (\text{resp.}, \quad |D_j \varphi(x)| \geq f(x));$$

- ii) If $x = x_i \in \Gamma_{int}^n$, then

- A function $u \in \text{USC}(S^n)$ is called a **subsolution** at x if **for any** $j, k \in \text{Inc}_i$ **and any** (j, k) -**test function** φ for which $u - \varphi$ attains a local maximum at x relatively to $e_j \cup e_k$, we have $|D_j \varphi(x)| \leq f(x)$.
- A function $v \in \text{LSC}(S^n)$ is called a **supersolution** at x if **for any** $j \in \text{Inc}_i$, **there exists** $k \in \text{Inc}_i \setminus \{j\}$ (**said feasible for** j **at** x) **such that** **for any** (j, k) -**test function** φ for which $u - \varphi$ attains a local minimum at x relatively to $e_j \cup e_k$, we have $|D_j \varphi(x)| \geq f(x)$.

Remark: Note that the definitions of sub and supersolution at the vertices **are not symmetric**.

Theorem

Let $g : \Gamma \rightarrow \mathbb{R}$ be such that $g(x) \leq \inf \left\{ \int_0^T f(\xi(t)) dt + g(y) \right\} \forall x, y \in \Gamma^0$ where the infimum is taken over all the piecewise differentiable paths ξ such that $\xi(0) = x$, $\xi(T) = y$. Then the **unique network viscosity solution** of the eikonal equation on S^n with the boundary condition $u = g$ on Γ^0 is given by

$$u(x) = \inf \left\{ \int_0^T f(\xi(t)) dt + g(y) \right\} \quad \text{for all } x \in S^n,$$

where the infimum is taken over all $y \in \Gamma^0$ and over all the paths ξ such that $\xi(0) = x$, $\xi(T) = y$.

Moreover, u is bounded and Lipschitz continuous and

$$|u(x)| \leq \max_{\Gamma^0} |g| + \max_{S^n} |f| d(x, \Gamma^0) \quad \text{for all } x \in S^n$$

$$\|Du\|_\infty \leq \max_{S^n} |f|.$$

Theorem

Let $g : \Gamma \rightarrow \mathbb{R}$ be such that $g(x) \leq \inf \left\{ \int_0^T f(\xi(t)) dt + g(y) \right\} \forall x, y \in \Gamma^0$ where the infimum is taken over all the piecewise differentiable paths ξ such that $\xi(0) = x$, $\xi(T) = y$. Then the **unique network viscosity solution** of the eikonal equation on S^n with the boundary condition $u = g$ on Γ^0 is given by

$$u(x) = \inf \left\{ \int_0^T f(\xi(t)) dt + g(y) \right\} \quad \text{for all } x \in S^n,$$

where the infimum is taken over all $y \in \Gamma^0$ and over all the paths ξ such that $\xi(0) = x$, $\xi(T) = y$.

Moreover, u is bounded and Lipschitz continuous and

$$|u(x)| \leq \max_{\Gamma^0} |g| + \max_{S^n} |f| d(x, \Gamma^0) \quad \text{for all } x \in S^n$$

$$\|Du\|_\infty \leq \max_{S^n} |f|.$$

Monotonicity of u_n

Proposition

The sequence $\{u_n\}_n$ of the solution of the eikonal equation is decreasing and converges uniformly to a continuous function u on S .

Sketch of the proof: Let u_m be the solution of eikonal equation on the network S^m , with $m > n$. Since $S^n \subset S^m$, u_m is a subsolution of eikonal equation on S^n . By the comparison principle, it follows that $u_m \leq u_n$. Moreover since the sequence $\{u_n\}_n$ is equi-bounded, equi-continuous and decreasing, then it converges uniformly to a function \bar{u} on S .

Step 2: Estimate between the solutions of discrete and continuous eikonal equation on networks

Proposition

Let u_{h_n} and u_n be respectively the solutions of discrete and continuous eikonal equations on the prefractal Γ^n . Then

$$|u_n(x) - u_{h_n}(x)| \leq C\omega_f(h_n^{1/2}) \quad \forall x \in \Gamma^n$$

where ω_f is the modulus of continuity of f . Hence the sequences u_{h_n} and u_n converge to the same limit u on S .

The proof is based on the classical doubling of variables argument in viscosity solution theory adapted to networks

$$\Psi(x, y) = u_{h_n}(x) - u_n(y) - \frac{d(x, y)^2}{2\varepsilon}$$

Step 3: Giga-Hamamuki-Nakayasu metric viscosity solutions

- Given a path $\xi : I \subset \mathbb{R} \rightarrow \mathcal{X}$, define the **metric derivative** of $|\xi|$ by

$$|\xi'| (t) := \lim_{s \rightarrow t} \frac{d_{\mathcal{X}}(\xi(s), \xi(t))}{|s - t|}.$$

(note in general $\xi'(t)$ may be not well defined).

A curve ξ is said *absolutely continuous* if $|\xi'| (t)$ exists for a.e. $t \in I$, $|\xi'| \in L^1_{loc}(I)$ and $d_{\mathcal{X}}(\xi(s), \xi(t)) \leq \int_s^t |\xi'| (r) dr$. $\mathcal{A}_{\mathcal{X}}(I)$ denotes the set of absolutely continuous curves with $\xi(0) = x$.

- A function u is said **arcwise** upper (resp., lower) semicontinuous if for each $\xi \in \mathcal{A}(I)$ the function $u \circ \xi$ is upper (lower) semicontinuous in I .
- For a function $w : \mathbb{R} \rightarrow \mathbb{R}$, denote by $D^{\pm} w(t)$ respectively its super- and subdifferential at the point t .

For $\xi \in \mathcal{A}_x(I)$ and $\Omega \subset \mathcal{X}$ define the exit and entrance time in Ω by

$$T_{\Omega}^{+}[\xi] := \inf\{t \in [0, +\infty) : \xi(t) \in \partial\Omega\} \in [0, +\infty]$$

$$T_{\Omega}^{-}[\xi] := \sup\{t \in (-\infty, 0] : \xi(t) \in \partial\Omega\} \in [-\infty, 0].$$

Definition (metric viscosity solution)

- i) A function $u \in \text{USC}_a(\Omega)$ is said a **subsolution** if for each $x \in \Omega$ and for all $\xi \in \mathcal{A}_x(I, \Omega)$, the function $w : I \rightarrow \mathbb{R}$, with $w(t) := u(\xi(t))$, satisfies $|p| \leq f(x) \quad \forall p \in D^{+}w(0)$.
- ii) A function $v \in \text{LSC}_a(\Omega)$ is said a **supersolution** if for each $x \in \Omega$ and $\varepsilon > 0$, there exists $\xi \in \mathcal{A}_x(\mathbb{R}, \Omega)$ and $w \in \text{LSC}(\mathbb{R})$ such that

$$\left\{ \begin{array}{l} T^{\pm} := T_{\Omega}^{\pm}[\xi] \text{ are both finite,} \\ w(0) = v(x), \quad w(t) \geq v(\xi(t)) - \varepsilon \quad \forall t \in (T^{-}, T^{+}), \\ |p| \geq f(x) - \varepsilon \quad \forall p \in D^{-}w(t), t \in (T^{-}, T^{+}). \end{array} \right.$$

Remark:

• In a Euclidean space $|Du(x)| = \sup_{\xi} |(u \circ \xi)'(0)|/|\xi'(0)|$. Hence reflecting this property u is a **subsolution** if $|u \circ \xi)'(0)| \leq f(x)$ in **viscosity sense** for any ξ such that $\xi(0) = x$ and $|\xi'| \leq 1$.

• The definition of a supersolution is more involved. Roughly speaking, u is a **supersolution** if for each x there is a curve ξ with $\xi(0) = x$ and $|\xi'| \leq 1$ such that $|w'(t)| \gtrsim f(\xi(t))$ for all t until ξ hits the boundary where w is an upper approximation of $u \circ \xi$ with $w(0) = u(x)$.

Note that:

- The main point is that the previous definitions are **1-dimensional** and they do not involve any definition of gradient on \mathcal{X} .
- The definition of metric viscosity solution is **consistent with the classical one** in the Euclidean space.
- The definition of subsolution and supersolution are **not symmetric**
- The definition of supersolution is **not local**

Remark:

- In a Euclidean space $|Du(x)| = \sup_{\xi} |(u \circ \xi)'(0)|/|\xi'(0)|$. Hence reflecting this property u is a **subsolution** if $|u \circ \xi)'(0)| \leq f(x)$ in **viscosity sense** for any ξ such that $\xi(0) = x$ and $|\xi'| \leq 1$.
- The definition of a supersolution is more involved. Roughly speaking, u is a **supersolution** if for each x there is a curve ξ with $\xi(0) = x$ and $|\xi'| \leq 1$ such that $|w'(t)| \gtrsim f(\xi(t))$ for all t until ξ hits the boundary where w is an upper approximation of $u \circ \xi$ with $w(0) = u(x)$.

Note that:

- The main point is that the previous definitions are **1-dimensional** and they do not involve any definition of gradient on \mathcal{X} .
- The definition of metric viscosity solution is **consistent with the classical one** in the Euclidean space.
- The definition of subsolution and supersolution are **not symmetric**
- The definition of supersolution is **not local**

Remark:

- In a Euclidean space $|Du(x)| = \sup_{\xi} |(u \circ \xi)'(0)|/|\xi'(0)|$. Hence reflecting this property u is a **subsolution** if $|(u \circ \xi)'(0)| \leq f(x)$ in **viscosity sense** for any ξ such that $\xi(0) = x$ and $|\xi'| \leq 1$.
- The definition of a supersolution is more involved. Roughly speaking, u is a **supersolution** if for each x there is a curve ξ with $\xi(0) = x$ and $|\xi'| \leq 1$ such that $|w'(t)| \gtrsim f(\xi(t))$ for all t until ξ hits the boundary where w is an upper approximation of $u \circ \xi$ with $w(0) = u(x)$.

Note that:

- The main point is that the previous definitions are **1-dimensional** and they do not involve any definition of gradient on \mathcal{X} .
- The definition of metric viscosity solution is **consistent with the classical one** in the Euclidean space.
- The definition of subsolution and supersolution are **not symmetric**
- The definition of supersolution is **not local**

Prop. (existence/uniqueness on S)

Let S the Sierpinski gasket and let $g : \Gamma \rightarrow \mathbb{R}$ be such that $g(x) \leq \inf \left\{ \int_0^T f(\xi(t)) dt + g(y) \right\}$ for all $x, y \in \Gamma^0$ where the infimum is taken over all the paths $\xi \in \mathcal{A}((0, T), S)$ such that $\xi(0) = x, \xi(T) = y$. Then the **unique metric viscosity solution** of

$$\begin{cases} |Du| = f(x) & x \in S \\ u = g & x \in \Gamma^0 \end{cases}$$

is given by $u(x) = \inf \left\{ \int_0^T f(\xi(t)) dt + g(y) \right\}$ for all $x \in S$ where the infimum is taken over all $y \in \Gamma$ and over all the paths $\xi \in \mathcal{A}((0, T), S)$ such that $\xi(0) = x, \xi(T) = y$.

Moreover u is continuous and bounded.

Step 4: Passage to the limit

Equivalence on the prefractal S^n

A function $u \in \text{USC}(S^n)$ (resp. $v \in \text{LSC}(S^n)$) is a **metric viscosity subsolution** (resp. **supersolution**) if and only if it is a **network viscosity subsolution** (resp. **supersolution**)

Remark:

- On a network arcwise continuity and continuity coincide.
- The main difficulty is to show the equivalence between the (nonlocal) definition of viscosity supersolution and the one of network supersolution.

Theorem (Convergence)

Let u_n be the sequence of the network viscosity solutions of eikonal equation on the network S^n . Then, as $n \rightarrow +\infty$, u_n tends to u uniformly in S where u is the metric viscosity solution of

$$\begin{cases} |Du| = f(x) & x \in S \\ u = g & x \in \Gamma^0 \end{cases}$$

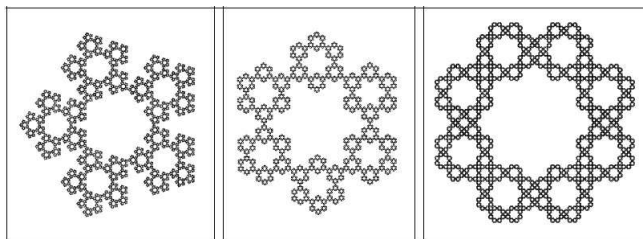
Hence, as $n \rightarrow +\infty$, the sequence $\{u_{h_n}\}_n$ of the solutions of graph eikonal equation also uniformly converges to u .

Remark:

The proof is based on stability properties of the metric viscosity solutions and the equivalence on Γ^n of metric and network viscosity solutions.

Extensions

- **More general fractals:** The interior approximation method can be extended to a class of **post-critically finite self similar sets** (generally speaking, these sets are obtained by subdividing the initial cell into cells of smaller and smaller size and the cells must intersect at isolated points) for which the corresponding prefractals are expanding, i.e. $\Gamma_n \subset \Gamma_{n+1} \forall n \in \mathbb{N}$.



(v) post critically finite fractals

- **More general Hamiltonians:** Whereas the notion of metric viscosity solution is restricted to the case of the eikonal equation $|Du| = f(x)$, the results on prefractals can be extended to the Hamilton-Jacobi equation $H(x, Du) = 0$ where the Hamiltonian $H(x, p)$ is convex and coercive with respect to p . Hence we obtain the sequence of the solutions of the Hamilton-Jacobi equation on the prefractal converges uniformly to a function u defined on the Sierpinski gasket.

At present since there is no definition of viscosity solution for the equation $H(x, Du) = 0$ on S for a general Hamiltonian, this can be seen as a constructive way to define Hamilton-Jacobi equations on the Sierpinski gasket.

- **Boundary conditions:** Following the classical definition of Laplacian on the Sierpinski gasket, we imposed the boundary conditions on the vertices of Γ^0 . Nevertheless, by easy modifications, it is possible to consider the problem in a connected subdomain Ω of S imposing the boundary condition on any finite subset of vertices contained in Ω .

- **More general Hamiltonians:** Whereas the notion of metric viscosity solution is restricted to the case of the eikonal equation $|Du| = f(x)$, the results on prefractals can be extended to the Hamilton-Jacobi equation $H(x, Du) = 0$ where the Hamiltonian $H(x, p)$ is convex and coercive with respect to p . Hence we obtain the sequence of the solutions of the Hamilton-Jacobi equation on the prefractal converges uniformly to a function u defined on the Sierpinski gasket.

At present since there is no definition of viscosity solution for the equation $H(x, Du) = 0$ on S for a general Hamiltonian, this can be seen as a constructive way to define Hamilton-Jacobi equations on the Sierpinski gasket.

- **Boundary conditions:** Following the classical definition of Laplacian on the Sierpinski gasket, we imposed the boundary conditions on the vertices of Γ^0 . Nevertheless, by easy modifications, it is possible to consider the problem in a connected subdomain Ω of S imposing the boundary condition on any finite subset of vertices contained in Ω .



Thank You!