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► **To cite this version:**

Axel Kröner. Numerical methods for optimal control of the wave equation. NETCO 2014, 2014, Tours, France. hal-01024666

HAL Id: hal-01024666

<https://inria.hal.science/hal-01024666>

Submitted on 17 Jul 2014

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Numerical methods for optimal control of the wave equation

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Tours, 25.06.2014

- Open-loop (nonsmooth) optimal control of the wave equation
 - Dirichlet boundary control problem
 - Minimum effort problem
- Closed-loop optimal control of the wave equation
- Summary

Dirichlet boundary control

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ bounded domain, $\partial\Omega \in C^2$, $T > 0$, $Q = (0, T) \times \Omega$, $\Sigma = (0, T) \times \partial\Omega$.

$$\left\{ \begin{array}{l} \min_{u \in U_{\text{ad}}} \quad J(u) = \int_0^T \int_{\Omega} (y - y_d)^2 dx dt + \frac{\alpha}{2} \int_0^T \int_{\partial\Omega} u^2 dx dt, \text{ s.t.} \\ y_{tt} - \Delta y = f \quad \text{in } Q, \\ y(0) = y_0 \quad \text{in } \Omega, \\ y_t(0) = y_1 \quad \text{in } \Omega, \\ y = u \quad \text{on } \Sigma \end{array} \right.$$

for the set of admissible controls

$$U_{\text{ad}} = \{u \in L^2(\Sigma) \mid u_a \leq u \leq u_b \text{ a.e.}\}$$

and with

$$y_0 \in L^2(\Omega), y_1 \in H^{-1}(\Omega), y \in Y = C(L^2(\Omega)) \cap C^1(H^{-1}(\Omega)), f, y_d \in L^2(Q), \\ \alpha > 0, \rho \geq 0, u_a, u_b \in L^p, p > 2.$$

Optimality condition

- There exists a **unique minimizer** $u \in U_{\text{ad}}$.
- **Necessary optimality condition:** $J'(u)(\delta u - u) \geq 0 \quad \forall \delta u \in U_{\text{ad}}$.
- Reformulation of the **optimality condition as a nonsmooth operator equation**

$$\mathcal{F}(u) = 0$$

with

$$\begin{aligned} \mathcal{F}: L^2(\Sigma) &\rightarrow L^2(\Sigma), \\ \mathcal{F}(u) &= \alpha(u - u_b) + \max(0, \alpha u_b - q(u)) + \min(0, q(u) - \alpha u_a) \end{aligned}$$

for

$$q(u) = -T^* G'(S(u))$$

and $G(y) = \frac{1}{2} \int_0^T \int_{\Omega} (y - y_d)^2 dx dt$ with control-to-state operator $S(u) = Tu + \bar{y}$, $T \in \mathcal{L}(L^2(\Sigma), Y)$, $\bar{y} \in Y$.

→ **Generalized Newton methods**

Semi-smooth Newton method

Aim: To set up a **semi-smooth Newton method**

$$G_{\mathcal{F}}(u_k)(u_{k+1} - u_k) + \mathcal{F}(u_k) = 0, \quad k = 0, 1, 2, \dots,$$

with **slant/Newton-derivative** $G_{\mathcal{F}}$ which satisfies

$$\|\mathcal{F}(x + d) - \mathcal{F}(x) - G_{\mathcal{F}}(x + h)h\|_{L_2(\Sigma)} = o(\|h\|_{L_2(\Sigma)}) \quad \text{for } \|h\|_{L_2(\Sigma)} \rightarrow 0.$$

If the **inverse generalized derivative is bounded**

$$\|G_{\mathcal{F}}^{-1}(u)\|_{L_2(\Sigma)} \leq C \|u\|_{L_2(\Sigma)} \quad \text{for } \|u - u^*\|_{L_2(\Sigma)} \text{ suff. small}$$

the method converges locally superlinearly.

[Hintermüller, Ito, Kunisch (2003), Ulbrich (2003), Chen, Nashed, Qi (2001)]

Semi-smooth Newton method

Operator $\mathcal{F}(u) = \alpha(u - u_b) + \max(0, \alpha u_b - q(u)) + \min(0, q(u) - \alpha u_a)$.

Slant/Newton differentiability

- $\max: L^p(\Sigma) \rightarrow L^2(\Sigma)$, $p > 2$, is slant/Newton-differentiable [Hintermüller et al. (2003)].
- **No smoothing property** for Dirichlet control of the wave equation:

$$\begin{aligned}y_{tt} - y_{xx} &= 0 \quad \text{in } (0, 1) \times (0, 1), \\y(t, 0) &= u(t), \quad y(t, 1) = 0, \quad y(0, x) = 0, \quad y_t(0, x) = 0.\end{aligned}$$

Here: $q(u) = -\partial_n p(u) = u \in L^2(\Sigma)$ with adjoint state p .

→ This motivates to consider the OCP for **strongly damped wave equation**:

$$y_{tt} - \Delta y - \rho \Delta y_t = f, \quad \rho > 0 \text{ small}$$

Then $q(u) = -\partial_n p(u) + \partial_n p(u)_t \in L^p(\Sigma)$, $p > 2$.

Boundedness of the inverse generalized derivative

- Follows by splitting Σ in active and inactive parts and considering them separately.



A. K., K. Kunisch, B. Vexler, SIAM Control Optim. (2011)

- **Theorem 1:** For initial state sufficiently close to the solution u^* , the Newton iteration applied to the **Dirichlet boundary control** problem for the strongly damped wave equation **converges superlinearly**.
- **Theorem 2:** In case of **distributed and Neumann boundary control** the result holds also for the undamped wave equation.
- **Theorem 3:** The analogous results of Theorem 1 and 2 hold also for the **dynamical Lamè system**.



A. K., K. Kunisch, B. Vexler, SIAM Control Optim. (2011)



A. K., Numer. Func. Anal. Optim. (2013)

Numerical example (Dirichlet boundary control)

Level	N	M	$\alpha = 10^{-4}$			$\alpha = 10^{-2}$		
			$\rho = 0$	$\rho = 0.1$	$\rho = 0.7$	$\rho = 0$	$\rho = 0.1$	$\rho = 0.7$
1	16	2	4	3	5	4	4	5
2	64	4	5	4	3	4	4	3
3	256	8	5	5	4	5	4	4
4	1024	16	6	6	6	5	7	5
5	4096	32	11	7	7	9	6	5
6	16384	64	13	9	7	10	8	5

Number of Newton steps on a sequence of uniformly refined meshes
($\rho = 0$ originally problem, $\rho > 0$ regularized problem)

Minimum effort problem

- Let $\Omega \subset \mathbb{R}^d$, ($d = 1, \dots, 4$) bounded domain, $T > 0$,

$$\left\{ \begin{array}{l} \min_{u \in \mathcal{U}} \quad J(u) = \frac{1}{2} \int_0^T \int_{\Omega} (\mathbf{C}_{\omega_o} y - z)^2 dxdt + \frac{\alpha}{2} \|u\|_{L^\infty(Q)}^2, \quad \text{s.t.} \\ \mathbf{A}y = \mathbf{B}u \quad \text{in } Q, \\ y(0) = y_0 \quad \text{in } \Omega, \\ \mathbf{C}y = 0 \quad \text{on } \Sigma. \end{array} \right.$$

- Wave operator $\mathbf{A} = \begin{pmatrix} \partial_t & -\text{id} \\ -\Delta & \partial_t \end{pmatrix}$
- Control operator $\mathbf{B} = (0, \chi_{\omega_c} \text{id}) : L_2(\Omega) \rightarrow L_2(\Omega)$, $\omega_c \subset \Omega$ open
- Control space $\mathcal{U} = L^\infty(Q)$
- Observation operator $\mathbf{C}_{\omega_o} = (\chi_{\omega_o} \text{id}, 0) : L_2(\Omega) \rightarrow L_2(\Omega)$, $\omega_o \subset \Omega$ open
- Boundary operator $\mathbf{C} = \mathbf{C}_\Omega$
- Desired state $z \in L^2(Q)$
- Elliptic case:** Clason, Ito, Kunisch (2012)

Minimum effort problem

Reformulation as a bilinear problem

$$\left\{ \begin{array}{l} \min_{(y,u,c) \in X \times U \times \mathbb{R}_0^+} J(y,c) = \frac{1}{2} \|\mathbf{C}_{\omega_o} y - z\|_{L^2(Q)}^2 + \frac{\alpha}{2} c^2, \quad \text{s.t.} \\ \mathbf{A}y = c\mathbf{B}u \quad \text{in } Q, \\ y(0) = y_0 \quad \text{in } \Omega, \\ \mathbf{C}y = 0 \quad \text{on } \Sigma, \\ \|u\|_{L^\infty(Q)} \leq 1. \end{array} \right.$$

Eliminating the control we obtain the reduced system

$$\left\{ \begin{array}{l} \mathbf{A}^* p + \mathbf{C}_{\omega_o}^* \mathbf{C}_{\omega_o} y - \mathbf{C}_{\omega_o}^* z = 0, \quad p(T) = 0, \quad \mathbf{B}p|_\Sigma = 0, \\ \alpha c - \|\mathbf{B}^* p\|_{L^1(Q)} = 0, \\ \mathbf{A}y - c\mathbf{B} \operatorname{sgn}(\mathbf{B}^* p) = 0, \quad y(0) = y_0, \quad \mathbf{C}y|_\Sigma = 0. \end{array} \right.$$

The optimality system is not (in a generalized sense) differentiable.

The regularized problem

The regularized minimum effort problem is given by

$$\left\{ \begin{array}{l} \min_{(y,u,c) \in X \times U \times \mathbb{R}_0^+} J_\beta(y, u, c) = \frac{1}{2} \|\mathbf{C}_{\omega_0} y - z\|_{L^2(Q)}^2 + \frac{\beta c}{2} \|u\|_{L^2(Q)}^2 + \frac{\alpha}{2} c^2, \\ \text{s.t.} \\ \mathbf{A}y = c\mathbf{B}u \quad \text{in } Q, \\ y(0) = y_0 \quad \text{in } \Omega, \\ \mathbf{C}y = 0 \quad \text{on } \Sigma, \\ \|u\|_{L^\infty(Q)} \leq 1 \end{array} \right.$$

for $y_0 \in Y^1$, parameters $\alpha, \beta > 0$, and $z \in L^2(Q)$.

Theorem: Let the regularized problem have a unique solution with $c^* > 0$. Then there holds

$$J(y_\beta, c_\beta) - J(y^*, c^*) = o(\beta)$$

for $\beta \rightarrow 0$.

 A. K., K. Kunisch, *Comp. Optim. Appl.* (2014)

The optimality system is given by

$$\mathbf{T}(x) = 0$$

for

$$\mathbf{T}: W \longrightarrow Z, \quad \mathbf{T}(x) = \mathbf{T}(y, p, c) = \begin{pmatrix} \mathbf{A}^* p + \mathbf{C}_{\omega_o}^* \mathbf{C}_{\omega_o} y - \mathbf{C}_{\omega_o}^* z \\ \alpha c - \|\mathbf{B}^* p\|_{L^1_\beta(Q)} \\ \mathbf{A} y - c \mathbf{B} \operatorname{sgn}_\beta(\mathbf{B}^* p) \\ y(0) - y_0 \end{pmatrix}$$

with $W = X \times Y_0^* \times \mathbb{R}^+$ and $Z = X^* \times \mathbb{R} \times Y \times Y^1$, $Y_0^* = \{p \in Y^* \mid p(T) = 0\}$.

Spaces:

$$Y^1 = H_0^1(\Omega) \times L^2(\Omega),$$

$$P^1 = L^2(\Omega) \times H_0^1(\Omega),$$

$$Y^0 = L^2(\Omega) \times H^{-1}(\Omega),$$

$$P^0 = H^{-1}(\Omega) \times L^2(\Omega),$$

$$X = L^2(Y^1) \cap H^1(Y^0),$$

$$Y = L^2(Y^0) \oplus (H^1(P^0))^*,$$

$$X^* = L^2(P^0) \oplus (H^1(Y^0))^*,$$

$$Y^* = L^2(P^1) \cap H^1(P^0).$$

Theorem: Let $x^* = (y^*, p^*, c^*)$ be a solution of the regularized problem with $c^* > 0$. Further let $x^0 \in W$ be feasible, i.e.

$$y^0(0) = y_0, \quad \partial_t y_1^0 = y_2^0, \quad \mathbf{A}^* p^0 + \mathbf{C}_{\omega_o}^* \mathbf{C}_{\omega_o} y^0 - \mathbf{C}_{\omega_o}^* z = 0.$$

Then the iterates $x^k = (y^k, p^k, c^k) \in W$ of the semi-smooth Newton method converges **locally superlinear**

$$\|x^{k+1} - x^*\|_{W^{\mathbb{R}}} \leq o(\|x^k - x^*\|_{W^{\mathbb{R}}})$$

($W^{\mathbb{R}} = X \times Y_0^* \times \mathbb{R}$).

 A. K., K. Kunisch, *Comp. Optim. Appl.*, 2014

Newton-differentiability

- Use $\text{sgn}_\beta(v) = \frac{1}{\beta}(v - \max(0, v - \beta) - \min(0, v + \beta))$.
- $\text{sgn}_\beta: L^p(Q) \rightarrow L^q(Q)$, $p > q \geq 1$ is Newton differentiable.

- The norm

$$\|\cdot\|_{L^1_\beta(Q)}: L^p(Q) \rightarrow \mathbb{R}, \quad p \geq 4$$

is Newton differentiable [Clason, Ito, and Kunisch (2012)].

- Thus we have

$$\begin{aligned} p \mapsto \mathbf{B}^* p \mapsto \|\mathbf{B}^* p\|_{L^1_\beta(Q)}, & \quad Y^* \rightarrow L^4(Q) \rightarrow \mathbb{R}, \\ p \mapsto \mathbf{B}^* p \mapsto c\mathbf{B} \text{sgn}_\beta(\mathbf{B}^* p), & \quad Y^* \rightarrow L^4(Q) \rightarrow L^2(Q) \rightarrow X^* \end{aligned}$$

for $d \leq 4$ are Newton differentiable.

Boundedness of the inverse derivative

- Follows from a priori estimates.

Numerical example (Convergence of the Newton method)

Errors in the scalar $e_c^k = |c^k - c^*|$, in the state $e_y^k = \|y^k - y^*\|_X$ and in the adjoint state $e_p^k = \|p^k - p^*\|_{L^2(P^1) \cap L^2(P^0)}$ for fixed regularization parameter β .

k	c	e_c^k	$e_c^k/e_c^{(k-1)}$	e_y^k	$e_y^k/e_y^{(k-1)}$	e_p^k	$e_p^k/e_p^{(k-1)}$	am	ap
0	10.0	8.5	-	9.6	-	0.7	-	42025	17450
1	1.215	3.0e-01	3.5e-02	8.3e+00	8.6e-01	2.3e-01	3.1e-01	32298	25740
2	1.393	1.2e-01	4.0e-01	7.5e-01	9.1e-02	3.1e-02	1.3e-01	22186	33754
3	1.489	2.1e-02	1.8e-01	3.1e-01	4.2e-01	8.5e-03	2.7e-01	26101	31073
4	1.510	4.4e-04	2.1e-02	3.3e-02	1.0e-01	1.0e-03	1.2e-01	24874	31490
5	1.510	8.2e-06	1.9e-02	1.5e-03	4.5e-02	4.4e-05	4.2e-02	25120	31450
6	1.510	9.1e-08	1.1e-02	5.5e-06	3.7e-03	9.4e-08	2.1e-03	25133	31448
7	1.510	1.6e-10	1.8e-03	3.7e-10	6.7e-05	7.0e-12	7.5e-05	25133	31448
8	1.510	0.0	0.0	0.0	0.0	0.0	0.0	25133	31448

Convergence of the semi-smooth Newton method

Path-following ($\beta \downarrow 0$)

Path-following rule

$$\beta_{n+1} = q\beta_n, \quad n \in \mathbb{N}_0, \quad q \in (0, 1), \quad \beta_0 > 0.$$

We choose

$$z = -1, \quad \alpha = 10^{-2}, \quad \beta_0 = 10^{-1}, \quad y_0(x) = (\sin(2\pi x), -4 \sin(2\pi x))$$

for $x \in \Omega = (0, 1)$ and $T = 1$. Further we set $q = 0.2$ and $\omega_c = \omega_o = \Omega$.

n	β_n	$J(u_{\beta_n}, c_{\beta_n})$	$J(u_{\beta_n}, c_{\beta_n}) - J(u^*, c^*)$	$e_{\beta_n}^J / e_{\beta_{n-1}}^J$	c_{β_n}	am	ap
0	1.0	0.25171	3.77e-02	-	2.94061	2590	0
1	2.00e-02	0.22346	9.47e-03	0.25	3.70135	6833	0
2	4.00e-03	0.21623	2.24e-03	0.24	3.96424	8720	722
3	8.00e-04	0.21444	4.53e-04	0.20	4.04065	9702	1563
4	1.60e-04	0.21407	8.57e-05	0.18	4.05790	10164	1915
5	3.20e-05	0.21401	1.47e-05	0.17	4.06136	10371	2054
6	6.40e-06	0.21400	0.00	0.00	4.06209	10520	2120

Error in the cost functional

• Theory

- Lions (1971,1985), Lasiecka & Triggiani (2000)
- Mordukhovich & Raymond (2004, 2005)
- Farahi, Rubio & Wilson (1996)
- Kowalewski, Lasiecka & Sokolowski (2010)
- strongly damped wave equation: Bucci (1992), Lasiecka, Pandolfi & Triggiani (1997)
- ...

• Numerics

- Lagnese & Leugering (2000, 2003)
- Gerdts, Greif & Pesch (2008)
- Gugat (2009), Gugat, Keimer & Leugering (2009), ...
- K. (2011, 2012), K., Kunisch & Vexler (2011), K. & Kunisch (2014)
- Kunisch & Wachsmuth (2011, 2012)
- ...
- exact controllability: Zuazua (2005), Micu & Zuazua (2005), Infante & Zuazua (1999), ...

- Optimal feedback control for a nonlinear dynamical system

$$\begin{cases} \min_{u \in \mathcal{U}_{\text{ad}}} J(x, t, u) = \frac{1}{2} \int_t^T y^T(s) M y(s) ds + \frac{\alpha}{2} \int_t^T u^T(s) u(s) ds, & \text{s.t.} \\ \frac{d}{ds} y(s) = \mathbf{F}(y(s)) + \mathbf{B}u(s), & s > t, \quad y(t) = x, \end{cases}$$

for given $\mathbf{F}: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\mathbf{B} \in \mathbb{R}^{m \times n}$, $u(s) \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $M \in \mathbb{R}^{n \times n}$ positive semi-definite, $\alpha > 0$.

- Aim: Feedback law

$$u^* = \mathbf{K}(t, y^*(t)),$$

which is robust with respect to noise (\rightarrow advantage in contrast to an open-loop optimal control),

- **Value function:** For $x \in \mathbb{R}^n$ and $t \in [0, T)$ we define

$$v(x, t) = \inf_{u \in \mathcal{U}_{\text{ad}}} J(x, t, u).$$

→ **Hamilton-Jacobi Bellman equation**

$$\begin{cases} -\partial_t v(x, t) + \mathcal{H}(x, \nabla_x v(x, t)) = 0 & (x, t) \in \mathbb{R}^n \times (0, T), \\ v(x, T) = 0 & x \in \mathbb{R}^n. \end{cases}$$

with Hamiltonian

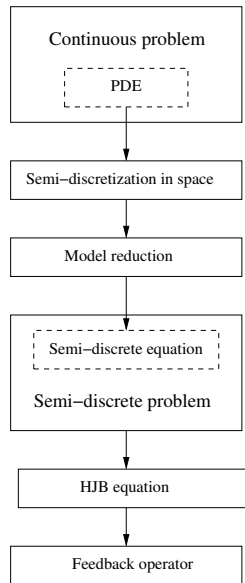
$$\mathcal{H}: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}, \quad \mathcal{H}(x, p) = \sup_{u \in U} (-(\mathbf{F}(x) + \mathbf{B}u)^T p - l(x, u)).$$

→ **Feedback law**

$$u^*(t) = P_{\mathcal{U}_{\text{ad}}} \left(-\frac{1}{\alpha} \mathbf{B}^T \nabla v(y^*(t), t) \right).$$

Application to semi-discretized PDEs

- Main challenge is the **curse of dimensionality**, since the dimension of the dynamical system is equal to the dimension of HJB equation
- Requires low dimensional discretization of the PDE
- Efficient solvers for the HJB equation
- **References for HJB-based optimal feedback control of PDEs**
 - **POD**: Kunisch, Volkwein, Xie (2004), Kunisch, Xie (2005), Alla, Falcone (2012)
 - **Spectral elements**: K., Kunisch, Zidani (2014)
 - **Balanced truncation**: K., Kalise (in preparation)
 - Gombao, Raymond (2006)



Feedback control of the wave equation

- **Control problem:** For $x \in Y^1 = H^1(\Omega) \times L^2(\Omega)$, $\Omega = (0, 1)$, $t \in [0, T]$

$$\left\{ \begin{array}{l} \min_{u \in \mathcal{U}} J(u) = \int_t^T l(y_{x,t}(s), u(s)) ds, \\ \frac{d}{ds} y_{x,t}(s) = \mathbf{F}(y_{x,t}(s), u(s)), \\ y_{x,t}(t) = x. \end{array} \right.$$

- **Dynamics:**

$$\mathbf{F}(x, u) = \mathbf{A}x + \mathbf{B}u + g(x^1)$$

with $x = (x_1, x_2) \in Y^1$ and

$$\mathbf{A} = \begin{pmatrix} 0 & I \\ c\Delta & 0 \end{pmatrix}, \quad \mathbf{B}u = (0, \tilde{u})^T, \quad \tilde{u} = \sum_{i=1}^N u_i \sin(\pi i x),$$

$u \in \mathcal{U} = L^2((t, T), [u_a, u_b]^d)$, $u_a, u_b \in \mathbb{R}$, nonlinearity g , $c > 0$, $N \in \mathbb{N}$.

- **Distributed cost:**

$$l: L^2(\Omega)^2 \times \mathbb{R}^N \rightarrow \mathbb{R}, \quad l(z, u) = \frac{1}{2} \|Cz - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|u\|^2$$

$\alpha > 0$, $C = (\text{id}, 0)$, $y_d \in L^2(\Omega)$.

Discretization with spectral elements

To obtain a low dimensional semi-discretization, we discretize the wave equation in space with **spectral elements**. The aim is to **control the lower modes**.

- **Spectral basis functions:** $\varphi_k(x) = \sin(k\pi x)$ for $k = 1, \dots, N$, $x \in \Omega = (0, 1)$.
- **Mass and stiffness matrix**

$$M_h = \text{diag} \left((1/2)_{k=1, \dots, N} \right), \quad A_h = \text{diag} \left(\left(\frac{1}{2} (k\pi)^2 \right)_{k=1, \dots, N} \right).$$

- **Discrete dynamics:**

$$\mathbf{T}_h: \mathbb{R}^{2N} \times \mathbb{R}^N \rightarrow \mathbb{R}^{2N}, \quad \mathbf{T}_h(x, u) = \mathbf{M}_h^{-1} (\mathbf{A}_h x + \mathbf{B}_h u)$$

with matrices

$$\mathbf{M}_h = \begin{pmatrix} M_h & 0 \\ 0 & M_h \end{pmatrix}, \quad \mathbf{A}_h = \begin{pmatrix} 0 & M_h \\ -cA_h & 0 \end{pmatrix}, \quad \mathbf{B}_h = \begin{pmatrix} 0 \\ M_h \end{pmatrix}.$$

The semi-discrete problem

- Discrete distributed cost

$$l_h: \mathbb{R}^{2N} \times \mathbb{R}^N \rightarrow \mathbb{R}, \quad l_h(x, u) = \frac{1}{2}(C_h x - y_d^h)^T M_h (C_h x - y_d^h) + \frac{\alpha}{2} u^T u,$$

$\alpha > 0$, $y_d^h \in \mathbb{R}^N$, with $C_h = (\text{id}_N, 0)^T$.

- The semi-discrete control problem

$$\left\{ \begin{array}{l} \min_{u^h \in \mathcal{U}} \quad J^h(u^h) = \int_t^T l_h(y_{x,t}^h(s), u^h(s)) ds, \quad \text{s.t.} \\ \frac{d}{ds} y_{x,t}^h(s) = \mathbf{T}_h(y_{x,t}^h(s), u^h(s)), \\ y_{x,t}^h(t) = x \end{array} \right.$$

for initial value $x \in \mathbb{R}^{2N}$ and $t \in [0, T)$.

→ Set up HJB equation.

Solving the HJB equation

- Obtain a **HJB equation in $2N$ dimensions**
- Discretization: Second order **ENO scheme** combined with Runge-Kutta method in time
- **Boundary condition**
 - Bounded computational domain: $(0, T) \times D$, $D \subset \mathbb{R}^{2N}$.
 - Requires an appropriate choice of the boundary condition; we choose

$$\partial_{xx}v(x, t) = 0 \quad \forall (x, t) \in \partial D \times (0, T),$$

i.e. on the discrete level

$$\begin{cases} D_{x_i}^{\pm} v(x_I) = \frac{v(x_{I^i,+}) - v(x_I)}{h_i} & \text{if } I_i = 0, \\ D_{x_i}^{\pm} v(x_I) = -\frac{v(x_{I^i,-}) - v(x_I)}{h_i} & \text{if } I_i = M_i. \end{cases}$$

- For a more sophisticated treatment using **state constraints** see [Altarovici, Bokanowski, Zidani (2012)].

Data

- Initial point: $x(z) = (1.5 \sin(\pi z) + \sin(2\pi z) + \sin(3\pi z), 0)^T$, $z \in \Omega$,
- $c = 0.025$, $T = 8$, $y_d = 0$, $\alpha = 0.1$,
- $\mathcal{K} = [-2.5, 2.5]^6$,
- Number of discretization points: $4 \times 8 \times 20 \times 4 \times 8 \times 20$.

Numerical example I

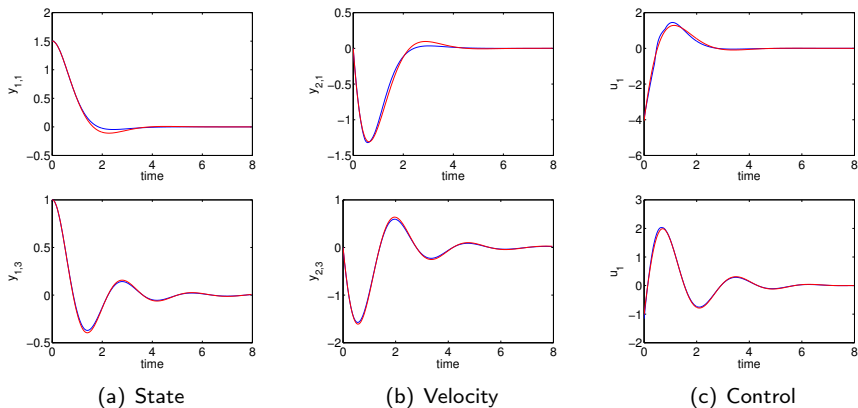


Figure : Components of the trajectory and the control of the HJB (blue curves) and Riccati (red curves) approach

Linear quadratic case: Decomposition in subproblems

We introduce N subproblems defined by

$$\left\{ \begin{array}{l} v_i(t, x) = \min_{a \in \mathcal{U}_i} \int_t^T (z_i(t)^2 + \alpha a^2) dt, \\ z_{i,t} = A_i z + B_i a, \\ z_i(t) = \hat{z}_i, \\ z_i = (z_i^1, z_i^2) \end{array} \right.$$

for $i = 1, \dots, N$ with $\mathcal{U}_i = L^2((0, T), [u_a, u_b])$ and

$$A_i = \begin{pmatrix} 0 & 1 \\ \alpha_i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $\alpha_i = (\mathbf{M}_h^{-1} \mathbf{A}_h)_{ii}$, $\hat{z}_i = (x_i, x_{i+N})$.

Theorem: For solutions v_i of the subproblems and v of the originally problem there holds

$$v(x, t) = v_1(x, t) + \dots + v_N(x, t).$$

Data

- Initial point: $x(z) = \left(\sum_{i=1}^5 0.3 \sin(i\pi z), 0 \right)^T$, $z \in \Omega$,
- $c = 0.025$, $T = 8$, $y_d = 0$, $\alpha = 0.5$,
- $\mathcal{K} = [-1.1, 1]^{10}$,
- Number of discretization points:
 $10 \times 16 \times 30 \times 50 \times 70 \times 10 \times 16 \times 30 \times 50 \times 70$.

Numerical example II

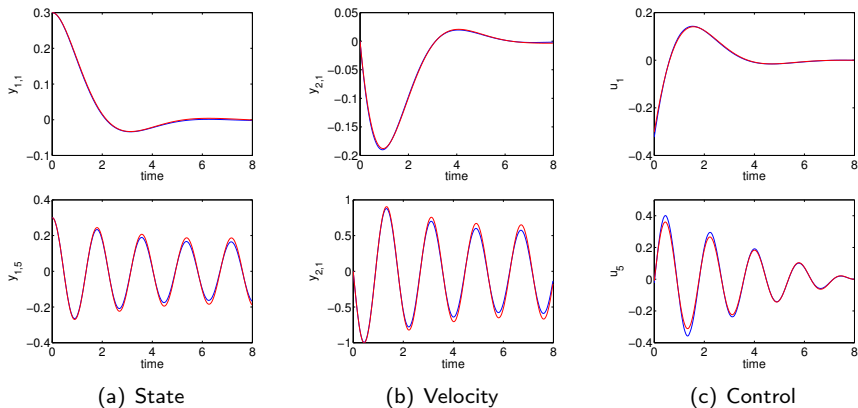
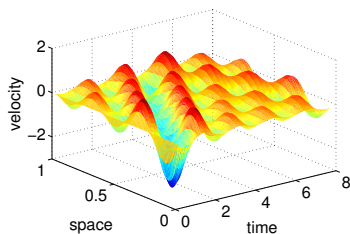
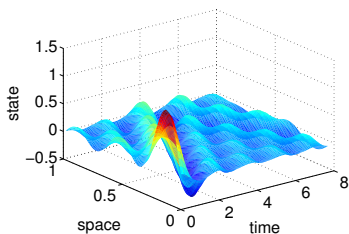
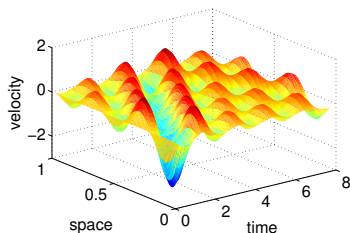
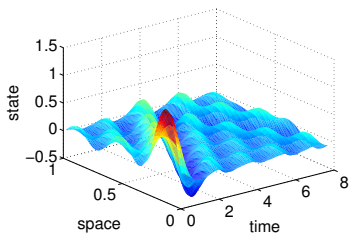


Figure : Components of the trajectory and the feedback control via HJB (blue) and Riccati (red) approach, ($\alpha = 0.5$)

Numerical example II



(a) HJB approach



(b) Riccati approach

- Perturbed dynamics

$$y_t = \mathbf{F}_h(y, u) + \mathbf{M}_h^{-1} \mathbf{B}_h \delta, \quad \delta = (\delta_1, \dots, \delta_5)^T$$

with random functions $\delta_i: [0, T] \rightarrow [-4, 4]$, $i = 1, \dots, 5$.

Numerical example III (Noise effecting the dynamics)

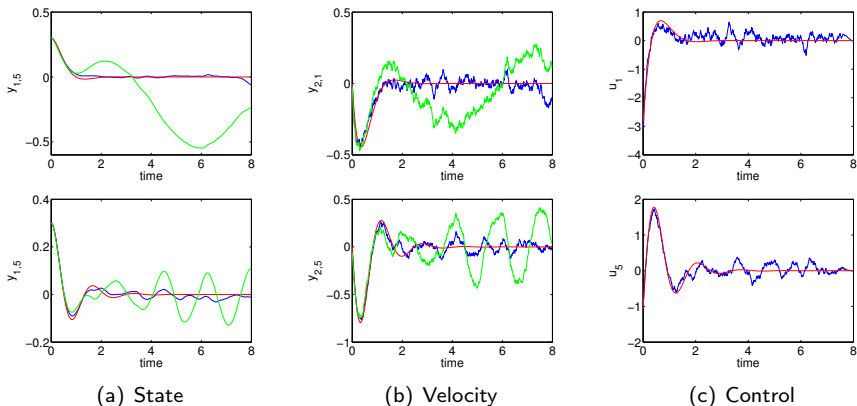
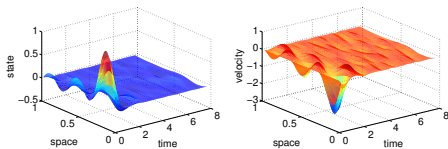
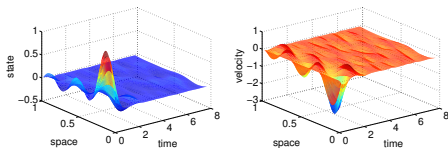


Figure : Noise in the dynamics: HJB (blue) approach for the problem with noise and via the Riccati approach (red) for the equation without noise as well as the components of the trajectory (green) when applying the optimal control of the noise free system to the perturbed system, ($\alpha = 0.01$).

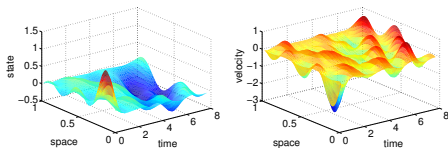
Numerical example III (Noise effecting the dynamics)



(a) HJB approach



(b) Riccati approach

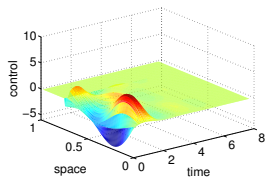
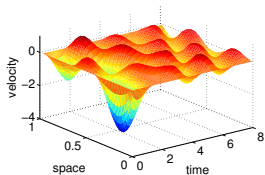
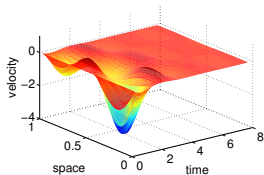
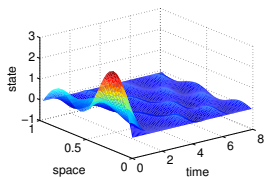
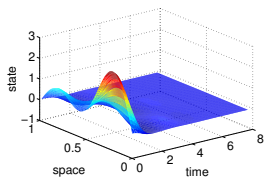


(c) Comparative solution

Numerical example IV (Local control)

Control acts locally on an open subset ω of Ω , i.e. $\mathbf{B}^\omega = \chi_\omega \mathbf{B}$, $\omega = (0, 0.75)$ and

$$\mathbf{B}_h^\omega = (0, B_2)^T, \quad B_2 = \begin{pmatrix} 0.455 & 0.075 & -0.079 \\ 0.075 & 0.375 & 0.135 \\ -0.079 & 0.135 & 0.349 \end{pmatrix}.$$



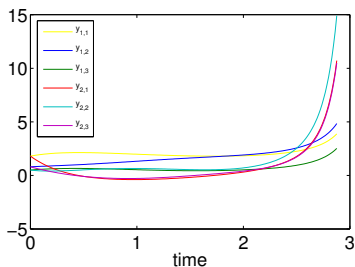
(a) Low order system

(b) High order system

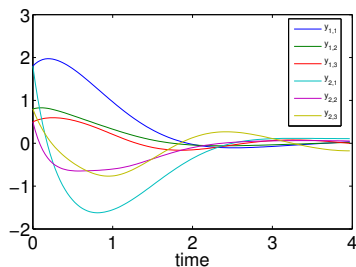
(c) Control

Numerical example V (Nonlinear dynamics)

- **Nonlinearity:** $g(x) = 0.5x^3$
- **Initial point:** $x(z) = \begin{pmatrix} 1.8 \sin(\pi z) + 0.8 \sin(2\pi z) + 0.5 \sin(3\pi z) \\ 1.8 \sin(\pi z) + 0.5 \sin(2\pi z) + 0.8 \sin(3\pi z) \end{pmatrix}$



(a) Riccati approach



(b) HJB approach

Figure : Components of the trajectory for the linearized Riccati approach and HJB approach for nonlinear dynamics

Summary

- Optimal control problems of the wave equation
- Boundary control with constraints and minimum effort problem
- Convergence of semi-smooth Newton methods
- Optimal feedback control approach for the wave equation

References

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- (5) D. Kalise, A. K., *Reduced-order minimum time control of advection-reaction-diffusion systems via dynamic programming*, Proceedings of MTNS 2014.

Thank you for your attention