# A game-theoretical model of debt and bankruptcy 

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# A game-theoretical model of debt and bankruptcy 

Alberto Bressan and Khai T. Nguyen<br>Department of Mathematics, Penn State University<br>bressan@math.psu.edu , ktn2@psu.edu

## Control system

Consider the control system of the borrower

$$
\dot{x}(t)=I(x(t)) \cdot x(t)-u(t), \quad x(0)=\bar{x}
$$

where

- $x$ : total debt, measured as a fraction of the yearly income of the borrower,
- $I(x)$ : interest rate payed on debt at a given time,
- $u \in[0,1]$ : payment rate, as a fraction of the income. This is the control variable for the borrower.


## MODELS

MODEL 1: The interest rate $I(x)$ is given a priori. This yields an optimal control problem for the borrower.


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MODEL 2: The interest rate $I(x)$ is determined by the CREDIT RATING of the borrower, which in turn depends on his global feedback strategy for repaying the debt. This corresponds to a differential game between the borrower (leading player) and a pool of risk-neutral lenders.

## Borrower may go bankrupt

Let $\rho:[0, M[\rightarrow[0, \infty[$ be instantaneous risk of bankruptcy which depends on the total debt such that $\rho$ is not decreasing and

$$
\rho(0)=0, \quad \quad \lim _{x \rightarrow M-} \rho(x)=+\infty
$$

Here the random variable $T_{b}$ denotes the bankruptcy time.

The larger the debt, the larger the risk of bankruptcy. Hence the interest rate
charged by lenders has to increase. It is natural to assume

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$$

Here, $M$ is a maximum size of the debt, so large that it immediately provokes bankruptcy. More precisely, if at time $\tau>0$ the borrower is not yet bankrupt and the total debt is $x(\tau)=y$, then

$$
\text { Prob. }\left\{T_{b} \in[\tau, \tau+\varepsilon] \mid T_{b}>\tau, x(\tau)=y\right\}=\rho(y) \cdot \varepsilon+o(\varepsilon) .
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## Distribution function of $T_{b}$

If the size of the debt is $t \mapsto x(t)$, then the distribution function

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$$

Namely,

$$
\psi(t)=\exp \left\{-\int_{0}^{t} \rho(x(s)) d s\right\}
$$

## Expected cost

Assume that

- $L(u)$ is the cost to the borrower for implementing the control $u$ such that

$$
L(0)=0, \quad L^{\prime}>0, \quad L^{\prime \prime}>0, \quad \lim _{u \rightarrow 1-} L(u)=+\infty
$$

- $r$ is the discount rate.
- $B$ is the bankruptcy cost to the borrower.

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- $B$ is the bankruptcy cost to the borrower.

The total cost

$$
\begin{align*}
J[u, \bar{x}] & \doteq E\left[\int_{0}^{T_{b}} e^{-r t} L(u(t)) d t+B e^{-r T_{b}}\right]  \tag{1}\\
& =[\text { cost of making payment }]+[\text { bankruptcy cost }] \tag{2}
\end{align*}
$$

## Optimal control problem for the borrower

One computes

$$
\begin{aligned}
& E\left[\int_{0}^{T_{b}} e^{-r t} L(u(t)) d t+B e^{-r T_{b}}\right] \\
& \quad=\int_{0}^{\infty} \exp \left\{-r t-\int_{0}^{t} \rho(x(s)) d s\right\}\{\rho(x(t)) B+L(u(t))\} d t
\end{aligned}
$$

subject to

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\end{aligned}
$$

(P) Optimal control problem with bankruptcy risk. Given the initial size $\bar{x}$ of the debt, find a control $t \mapsto u(t) \in[0,1]$ which minimizes the expected cost

$$
\begin{equation*}
J[\bar{x}, u] \doteq \int_{0}^{\infty} \exp \left\{-r t-\int_{0}^{t} \rho(x(s)) d s\right\}\{\rho(x(t)) B+L(u(t))\} d t \tag{3}
\end{equation*}
$$

subject to

$$
\begin{equation*}
\dot{x}(t)=I(x(t)) x(t)-u(t), \quad x(0)=\bar{x} \tag{4}
\end{equation*}
$$

## Hamilton-Jacobi equation

The value function

$$
V(\bar{x})=\min J[u, \bar{x}]
$$

is a viscosity solution of

$$
\begin{equation*}
V(x)=\frac{1}{r+\rho(x)} \cdot H\left(x, V^{\prime}(x)\right) \tag{5}
\end{equation*}
$$

with

$$
V(0)=0, \quad V(M)=B
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\begin{equation*}
H(x, \xi)=\min _{\omega \in[0,1]}\{L(\omega)-\xi \cdot \omega\}+\xi \cdot I(x) x+\rho(x) B \tag{6}
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The corresponding optimal feedback control is

$$
\begin{equation*}
u^{*}(\xi)=\arg \min _{\omega \in[0,1]}\{L(\omega)-\xi \cdot \omega\} . \tag{7}
\end{equation*}
$$

## A viscosity super solution

Let $W(x)$ be the cost achieved by the strategy that keeps the debt constantly equal to $x$, i.e.,

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W(x)=J[x, u] \quad \text { with } \quad u=I(x) x
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$$

Thus,

$$
W(x)=\left\{\begin{align*}
\frac{1}{r+\rho(x)} \cdot H^{\max }(x) & \text { if }  \tag{8}\\
+\infty & \text { if } x \geq x^{*}
\end{align*}\right.
$$

where

$$
H^{\max }(x) \doteq \max _{\xi} H(x, \xi)=H\left(x, \xi^{\sharp}(x)\right) .
$$

and $x^{*}$ is defined by $I\left(x^{*}\right) x^{*}=1$.

## Differential inclusion

## Hamilton-Jacobi equation

$$
\begin{equation*}
V(x)=\frac{1}{r+\rho(x)} \cdot H\left(x, V^{\prime}(x)\right) \tag{9}
\end{equation*}
$$



## Differential inclusion

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\begin{equation*}
V^{\prime} \in\left\{F^{-}(x, V), F^{+}(x, V)\right\} \tag{10}
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$$

where $V^{\prime}=F^{-}$and $V^{\prime}=F^{+}$are the two solutions of the equation (9).

## Initial data

- The value $V^{\prime}=F^{+}(x, V) \geq \xi^{\sharp}(x)$ corresponds to the choice of an optimal control such that $\dot{x}<0$.
- The value $V^{\prime}=F^{-}(x, V) \leq \xi^{\sharp}(x)$ corresponds to the choice of an optimal control such that $\dot{x}>0$.


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How to choose the appropriate initial data?

Let $t \mapsto x(t)$ be an optimal trajectory $t \mapsto x(t)$ such that

$$
\lim _{t \rightarrow \infty} x(t)=x_{0}
$$

Hence, as $t \rightarrow \infty$, the debt remains almost constant, close to $x_{0}$. We thus expect that $V(x(t)) \rightarrow W\left(x_{0}\right)$.

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Hence, as $t \rightarrow \infty$, the debt remains almost constant, close to $x_{0}$. We thus expect that $V(x(t)) \rightarrow W\left(x_{0}\right)$. Therefore, as initial data, we should choose

$$
V\left(x_{0}\right)=W\left(x_{0}\right) \quad \text { for some } x_{0}
$$

## Value function

For any $x_{0} \in\left[0, x^{*}\left[\right.\right.$, let $Z\left(\cdot, x_{0}\right)$ be the solution of

$$
Z^{\prime}(x)=\left\{\begin{array}{ll}
F^{+}(x, Z) & \text { if } x_{0}<x<x^{*},  \tag{11}\\
F^{-}(x, Z) & \text { if } 0 \leq x<x_{0},
\end{array} \quad Z\left(x_{0}\right)=W\left(x_{0}\right)\right.
$$

Moreover, let $Z(\cdot, M)$ be the solution of

$$
Z^{\prime}(x)=F^{-}(x, Z) \text { and } Z(M-)=B .
$$



The value function for the optimization problem (3)-(4) is

$$
\begin{equation*}
V(x)=\min _{x_{0} \in\left[0, x^{*}[\cup\{M\}\right.} Z\left(x ; x_{0}\right) . \tag{12}
\end{equation*}
$$

## Characterization of the value function

Theorem 1. Let the functions $I, \rho, L$ satisfy the standard assumptions. Let $W$ be the function in (8) and consider the set

$$
\begin{equation*}
\mathcal{A} \doteq\left\{x_{0} \in[0, M] ; \quad H^{\max }\left(x_{0}\right)=H\left(x_{0}, W^{\prime}\left(x_{0}\right)\right)\right\} \cup\{0, M\} . \tag{13}
\end{equation*}
$$

Then the value function for the optimization problem (3)-(4) is given by

$$
\begin{equation*}
V(x)=\min _{x_{0} \in \mathcal{A}} Z\left(x ; x_{0}\right) . \tag{14}
\end{equation*}
$$



$$
\begin{aligned}
H^{\max }\left(x_{0}\right)= & H\left(x_{0}, W^{\prime}\left(x_{0}\right)\right) \\
& \Uparrow \\
F^{+}\left(x_{0}, W\left(x_{0}\right)\right) & =F^{-}\left(x_{0}, W\left(x_{0}\right)\right) \\
& =W^{\prime}\left(x_{0}\right) .
\end{aligned}
$$

## Proof




Figure : Left: if $F^{ \pm}\left(x_{0}, W\left(x_{0}\right)\right)<W^{\prime}\left(x_{0}\right)$, then we can choose a smaller initial value $\tilde{x}_{0}<x_{0}$ and obtain a lower cost function.

Right: if $F^{ \pm}\left(x_{0}, W\left(x_{0}\right)\right)>W^{\prime}\left(x_{0}\right)$, then we can choose a bigger initial value $\tilde{x}_{0}>x_{0}$ and obtain a lower cost function.

## A game-theoretical model

## Expected pay-off for a lender

A pool-risk neutral lenders charge an interest depending on the bankruptcy risk.

- $\sigma$ : fraction of capital recovered by the lenders if bankruptcy occurs,

$$
0<\sigma<1
$$

For a unit amount of loan, the lender will receive the payoff

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$$

For a unit amount of loan, the lender will receive the payoff

$$
\begin{cases}\int_{0}^{T_{e}} l(s) e^{-r s} d s+e^{-r T_{e}} & \text { if } \quad T_{e}<T_{b}  \tag{15}\\ \int_{0}^{T_{b}} l(s) e^{-r s} d s+\sigma e^{-r T_{b}} & \text { if } \quad T_{e} \geq T_{e}\end{cases}
$$

## An ODE for the interest rate

Risk-neutral assumption implies that

$$
E\left[\int_{0}^{T_{e} \wedge T_{b}} I(x(s)) e^{-r s} d s+\left\{\begin{array}{lll}
e^{-r T_{e}} & \text { if } & T_{e}<T_{b}  \tag{16}\\
\sigma e^{-r T_{b}} & \text { if } & T_{e} \geq T_{b}
\end{array}\right]_{x(0)=\bar{x}}=1 .\right.
$$

Recalling that

$$
P(t, \bar{x})=\operatorname{Prob} .\left\{T_{b}>t ; x(0)=\bar{x}\right\}=\exp \left\{-\int_{0}^{t} \rho(x(s)) d s\right\}
$$

the interest rate is recovered by solving the ODEs

$$
\begin{equation*}
I^{\prime}(\bar{x})=\frac{\rho(\bar{x})+\lambda+r}{I(\bar{x}) \bar{x}-u(\bar{x})} \cdot(I(\bar{x})-r)+\frac{\lambda(\sigma-1) \rho(\bar{x})}{I(\bar{x}) \bar{x}-u(\bar{x})} . \tag{17}
\end{equation*}
$$

## System of equations

Let $V$ be the value function of the optimal problem for the borrower and

$$
H(x, \alpha, \xi) \doteq \min _{\omega \in[0,1]}\{L(\omega)-\omega \xi\}+\alpha x \xi++\rho(x) B
$$

The system of equations

$$
\left\{\begin{align*}
V(x) & =\frac{1}{r+\rho(x)} \cdot H\left(x, I(x), V^{\prime}(x)\right), \\
I^{\prime}(x) & =\frac{\rho(x)+\lambda+r}{I(x) x-u(x)} \cdot(I(x)-r)+\frac{\lambda(\sigma-1) \rho(x)}{I(x) x-u(x)},  \tag{18}\\
u(x) & =\arg \min _{\omega \in[0,1]}\left\{L(\omega)-V^{\prime}(x) \cdot \omega\right\},
\end{align*}\right.
$$

together with the boundary conditions

$$
\begin{equation*}
V(0)=0, \quad V(M)=B \tag{19}
\end{equation*}
$$

## Existence of admissible solution

Definition 1. A pair of functions $(V, I)$ defined for $x \in[0, M]$ is an admissible solution of (18) if the following holds

- I : $[0, M] \mapsto[0,+\infty]$ is lower semicontinuous with at most countably many points of discontinuity, and satisfies the second equation in (18) at a.e. point $x$ where $I(x)<+\infty$.
- $V$ is a continuous function, satisfying the boundary conditions (19). It provides a solution to the differential inclusion, with derivative $V^{\prime}$ having at most countably many points of jump. At each point $y$ where $V^{\prime}$ is discontinuous, the left and right limits satisfy

$$
V^{\prime}(y-) \geq V^{\prime}(y+)
$$

Theorem 2. Let the functions $\rho, L$ satisfy the standard assumptions. Then the problem (18)-(19) has an admissible solution.

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Theorem 2. Let the functions $\rho, L$ satisfy the standard assumptions. Then the problem (18)-(19) has an admissible solution.

The admissible solution constructed in Theorem 2 provides a Nash equilibrium to the debt and bankruptcy game.

## THANK YOU

