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# Dynamic programming using radial basis functions

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joint work with Alex Schreiber

# Problem

- ▶ discrete-time control system

$$x_{k+1} = f(x_k, u_k), \quad k = 0, 1, 2, \dots,$$

$f : \Omega \times U \rightarrow \Omega$  continuous

- ▶  $\Omega \subset \mathbb{R}^d$  and  $U \subset \mathbb{R}^m$  compact
- ▶ target set  $T \subset \Omega$ , compact
- ▶ **goal:** construct **feedback**  $F : S \rightarrow U$ ,  $S \subset \Omega$ , such that for the closed loop system

$$x_{k+1} = f(x_k, F(x_k)), \quad x_k \in S,$$

the target  $T$  is asymptotically stable.

# Optimal control

- ▶ cost function  $c : \Omega \times U \rightarrow [0, \infty)$  continuous,  $c(x, u) \geq \delta > 0$  for  $x \notin T$  and any  $u \in U$ .
- ▶ accumulated cost

$$J(x_0, (u_k)_k) = \sum_{k=0}^{\infty} c(x_k, u_k),$$

with trajectory  $(x_k)_k$  associated to  $x_0 \in \Omega$  and  $(u_k)_k \in U^{\mathbb{N}}$ .

- ▶ optimal value function

$$V(x) = \inf_{(u_k)_k} J(x, (u_k)_k)$$

# The Bellman equation

- ▶  $V$  fulfills the Bellman equation

$$\begin{aligned} V(x) &= \inf_{u \in U} \{c(x, u) + V(f(x, u))\} \\ &=: L[V](x) \end{aligned}$$

with boundary condition  $V(T) = 0$ .

- ▶ optimal feedback

$$F(x) = \operatorname{argmin}_{u \in U} \{c(x, u) + V(f(x, u))\}$$

(whenever the min exists)

# Numerical treatment

- ▶ assume  $V \in \mathcal{F}$
- ▶ approximation space  $\mathcal{A} \subset \mathcal{F}$ ,  $\dim(\mathcal{A}) < \infty$
- ▶ projection  $\Pi : \mathcal{F} \rightarrow \mathcal{A}$
- ▶ discretized Bellman operator

$$\Pi \circ L : \mathcal{A} \rightarrow \mathcal{A}$$

- ▶ value iteration: choose  $V^{(0)} \in \mathcal{A}$  with  $V^{(0)}(T) = 0$ ,

$$V^{(n+1)} := \Pi \circ L[V^{(n)}], \quad n = 0, 1, \dots$$

- ▶ typical  $\mathcal{A}$ : finite differences, finite elements (order  $p$ )
- ▶ problem:  $\dim(\mathcal{A}) \sim \mathcal{O}(n^d)$  for error  $\mathcal{O}(n^{-p})$

# Nonlinear approximation

## Theorem [Giroso, Anzellotti, '92]

If  $f \in H^{s,2}(\mathbb{R}^d)$ ,  $s > d/2$ , we can find

- ▶  $n$  coefficients  $c_i \in \mathbb{R}$ ,
- ▶  $n$  centers  $x_i \in \mathbb{R}^d$ ,
- ▶ and  $n$  variances  $\sigma_i > 0$  such that

$$\left\| f - \sum_{i=1}^n c_i e^{-\frac{\|x-x_i\|^2}{2\sigma_i^2}} \right\|_{\infty}^2 = \mathcal{O}(n^{-1}).$$

# Scattered data interpolation

## Problem

Given

- ▶ **sites**  $X = \{x_1, \dots, x_N\} \subset \Omega \subset \mathbb{R}^d$
- ▶ **data**  $f_1, \dots, f_N \in \mathbb{R}$ ,

find a function  $a \in \mathcal{A}$  such that

$$a(x_i) = f_i, \quad i = 1, \dots, N.$$

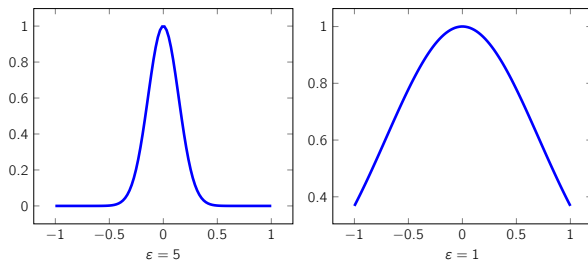
For  $\mathcal{A} = \text{span}\{a_1, \dots, a_N\}$  we get

$$Ac = f, \quad \text{with } A_{ij} = a_j(x_i).$$



# Radial basis functions

- ▶ **radial basis functions**  $a(\cdot, x_j) = \varphi(\|\cdot - x_j\|_2)$
- ▶ examples:
  - ▶ Gaussian:  $\varphi(r) = \exp(-r^2)$ ,
  - ▶ Wendland function:  $\varphi(r) = (1 - r)_+^4 \cdot (4r + 1)$
- ▶ scaling:  $a_j = a_j^\varepsilon = \varphi(\varepsilon\|\cdot - x_j\|)$



# The Kruzkov transform

- ▶ problem:  $V(x)$  increasing, but  $\varphi(x)$  decreasing as  $\|x\| \rightarrow \infty$
- ▶ Kruzkov transform:  $V \mapsto \hat{V} = e^{-V(\cdot)}$
- ▶ Kruzkov-Bellman equation

$$\hat{V}(x) = \sup_{u \in U} \{e^{-c(x,u)} \cdot \hat{V}(f(x,u))\} =: \hat{L}[V](x), \quad x \in \Omega \setminus T$$

with boundary condition  $\hat{V}(T) = 1$ .

- ▶ under the assumption  $c(x,u) \geq \delta > 0$  for  $x \notin T$ , the Kruzkov-Bellman operator  $\hat{L}$  is a contraction on  $L^\infty$ .

# Dynamic programming using radial basis functions

- ▶ approximation space

$$\mathcal{A} = \mathcal{A}_{X,\varepsilon} = \text{span}\{\varphi(\varepsilon\|\cdot - x\|_2) : x \in X\}$$

- ▶ interpolation operator on  $X$

$$\Pi : \mathcal{F} \rightarrow \mathcal{A}$$

- ▶ discretized Kruzkov-Bellman operator

$$\Pi \circ \hat{L} : \mathcal{A} \rightarrow \mathcal{A}$$

- ▶ value iteration: choose  $\hat{V}^{(0)} \in \mathcal{A}$  with  $\hat{V}^{(0)}(0) = 1$ ,

$$\hat{V}^{(n+1)} := \Pi \circ \hat{L}[\hat{V}^{(n)}], \quad n = 0, 1, \dots$$

# Weighted least squares

## Problem

Given

- ▶ **sites**  $X = \{x_1, \dots, x_N\} \subset \Omega \subset \mathbb{R}^d$ ,
- ▶ **data**  $f_1, \dots, f_N \in \mathbb{R}$ ,
- ▶ **approximation space**  $\mathcal{A} = \text{span}\{a_1, \dots, a_m\}$ ,  $m < N$ ,
- ▶ **weight function**  $w : \Omega \rightarrow \mathbb{R}$  with associated scalar product  $\langle f, g \rangle_w := \sum_{k=1}^N f(x_k)g(x_k)w(x_k)$  and induced norm

find a function  $a \in \mathcal{A}$  such that

$$\|f - a\|_w \stackrel{!}{=} \min$$

Optimal coefficient vector  $c$ :

$$Gc = f_{\mathcal{A}}$$

with **Gram matrix**  $G = (\langle a_i, a_j \rangle_w)_{ij}$  and  $f_{\mathcal{A}} = (\langle f, a_j \rangle_w)_j$ .

# Moving least squares

## Idea

In computing an approximation to the function  $f : \Omega \rightarrow \mathbb{R}$  at  $x \in \Omega$ , only the values at sites  $x_j \in X$  **close to  $x$**  should play a role.

- ▶ **moving** weight function  $w : \Omega \times \Omega \rightarrow \mathbb{R}$
- ▶  $w(x, y)$  small for  $\|x - y\|_2$  large
- ▶ inner product:  $\langle f, g \rangle_{w(\cdot, x)} := \sum_{k=1}^N f(x_k)g(x_k)w(x_k, x)$
- ▶ **moving least squares approximation**  $a$  of data  $f$  is

$$a(x) = a^x(x),$$

where  $a^x \in \mathcal{A}$  is minimizing  $\|f - a^x\|_{w(\cdot, x)}$ .

- ▶ given by solving the Gram system  $G^x c^x = f_{\mathcal{A}}^x$

# Shepard's method

D. Shepard, *A two dimensional interpolation function for irregularly spaced data*, Proc. 23rd Nat. Conf. ACM, 1968.

- ▶ simply choose  $\mathcal{A} = \text{span}\{1\}$
- ▶ Gram matrix  $G^x = \langle 1, 1 \rangle_{w(\cdot, x)} = \sum_{i=1}^N w(x_i, x)$
- ▶ right hand side  $f_{\mathcal{A}}^x = \langle f, 1 \rangle_{w(\cdot, x)} = \sum_{i=1}^N f(x_i)w(x_i, x)$
- ▶ thus we get

$$c^x = f^x / G^x = \sum_{i=1}^N f(x_i) \underbrace{\frac{w(x_i, x)}{\sum_{i=1}^N w(x_i, x)}}_{=: a_i(x)}$$

- ▶ and so the **Shepard approximant** is

$$Sf(x) = c^x \cdot 1 = \sum_{i=1}^N f(x_i) a_i(x)$$

- ▶ advantage: Shepard approximation requires **no matrix solve**

# Shepard discretization of the Bellman equation

- ▶ approximation space

$$\mathcal{A} = \text{span} \left\{ \frac{w(x_i, \cdot)}{\sum_{i=1}^N w(x_i, x)}, x_i \in X \right\}$$

- ▶ Shepard approximation operator

$$S : \mathcal{F} \rightarrow \mathcal{A}$$

- ▶ discretized Kruzkov-Bellman operator

$$S \circ \hat{L} : \mathcal{A} \rightarrow \mathcal{A}$$

- ▶ value iteration as usual

# Convergence of the value iteration

- ▶  $f \mapsto Sf$  is linear,
- ▶ for each  $x \in \Omega$ ,  $Sf(x)$  is a convex combination of the values  $f(x_1), \dots, f(x_n)$ , therefore
- ▶ the Shepard operator  $S : (L^\infty, \|\cdot\|_\infty) \rightarrow (\mathcal{A}, \|\cdot\|_\infty)$  has norm 1,

thus we get

## Lemma

*Value iteration with the discretized Kruzkov-Bellman operator  $S \circ \hat{L} : (\mathcal{A}, \|\cdot\|_\infty) \rightarrow (\mathcal{A}, \|\cdot\|_\infty)$  converges to the unique fixed point of  $S \circ \hat{L}$ .*



# Convergence for fill distance $\rightarrow 0$

fill distance of  $X \subset \Omega$

$$h = h(X, \Omega) = \sup_{x \in \Omega} \min_{x_j \in X} \|x - x_j\|_2$$

If  $f : \Omega \rightarrow \mathbb{R}$  is Lipschitz continuous with constant  $L$  then

$$\|f - Sf\|_\infty \leq CLh$$

for some constant  $C > 0$ .

## Convergence for fill distance $\rightarrow 0$

- ▶ sequence  $(X_n)_n$  of nodes sets,  $X_n \subset \Omega$ , fill distances  $h_n$ , Shepard operators  $S_n$ ,
- ▶  $K < 1$  contraction constant of  $\hat{L}$ ,
- ▶  $\hat{V}$  fixed point of  $\hat{L}$ ,  $\hat{V}_n$  fixed point of  $S_n \circ \hat{L}$

### Theorem

*If  $\hat{V}$  is Lipschitz continuous, then*

$$\|\hat{V} - \hat{V}_n\|_\infty \leq \frac{CL}{1-K}h$$

# Example 1

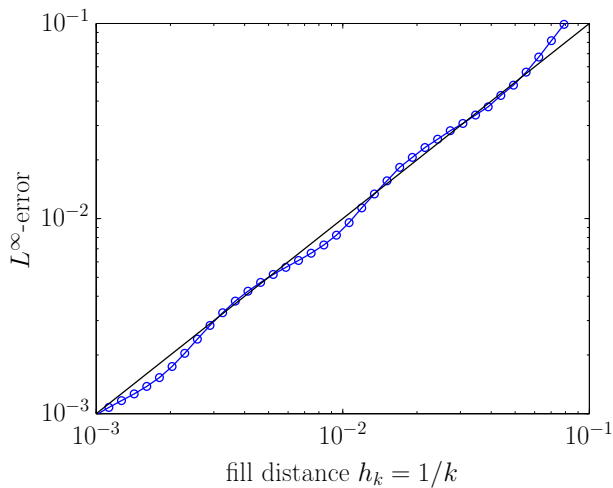
A simple 1D example

$$f(x, u) = aux, \quad c(x, u) = ax, \quad x \in [0, 1], u \in [-1, 1]$$

- ▶ optimal feedback  $u(x) = -1$
- ▶ optimal value function  $V(x) = x$
- ▶ nodes  $X_k$  equidistant with spacing  $1/k$
- ▶  $T = [0, 1/(2k)]$
- ▶  $\tilde{U} = -1 : 0.1 : 1$
- ▶  $\phi^\sigma$  : Wendland function of order 4,  $\sigma = k/5$

# Example 1

A simple 1D example



## Example 2

shortest path, geometrically complicated state constraints

We consider a boat in the mediteranian sea surrounding Greece which moves with constant speed 1, i.e.

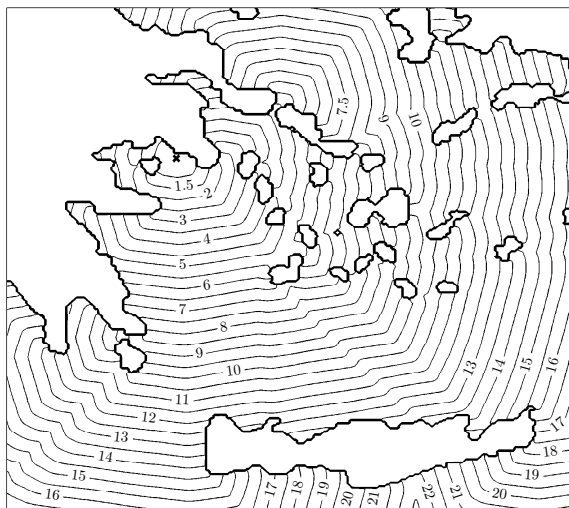
$$f(x, u) = x + hu, \quad c(x, u) \equiv 1, \quad x \in \text{neighborhood of Greece}$$

with time step  $h = 0.1$  and  $u \in \{u \in \mathbb{R}^2 : \|u\| = 1\}$

- ▶  $T$  = neighborhood of the harbour of Athens
- ▶  $X$  = equidistant in the sea on a  $275 \times 257$  grid (50301 nodes)
- ▶  $\tilde{U} = \{\exp(2\pi ij/20) : j = 0, \dots, 19\}$
- ▶  $\phi^\sigma$  : Wendland function of order 4,  $\sigma = 10$
- ▶ CPU time: 6 secs

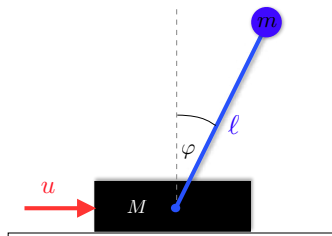
## Example 2

shortest path, geometrically complicated state constraints



## Example 3

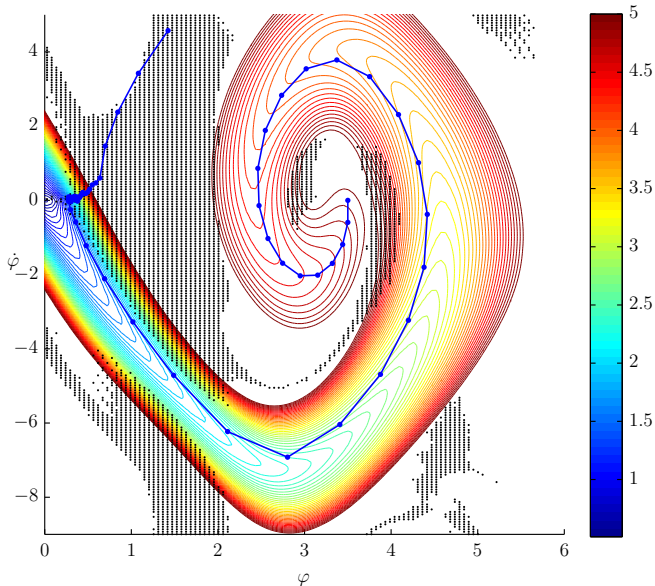
inverted pendulum, highly nonlinear dynamics



- ▶  $f$ : equations of the forced pendulum
- ▶  $c$ : quadratic deviation from the origin + quadratic in control
- ▶  $T$ : neighborhood of the origin
- ▶  $X$ : equidistant grid of  $100 \times 100$  nodes
- ▶  $\tilde{U} = -128 : 8 : 128$
- ▶  $\phi^\sigma$ : Wendland function of order 4,  $\sigma = 2.22$
- ▶ CPU time: 7 secs

# Example 3

inverted pendulum, highly nonlinear dynamics

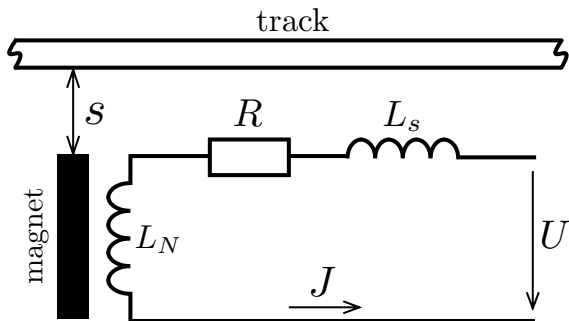






## Example 4

magnetic wheel, 3D example



## Example 4

magnetic wheel, 3D example

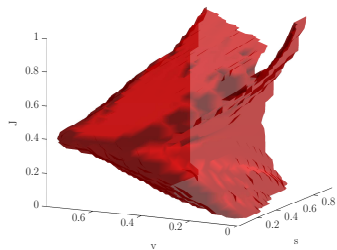
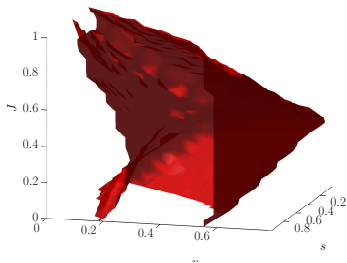
$$\begin{aligned}\dot{s} &= v, \\ \dot{v} &= \frac{CJ^2}{m_m 4s^2} - \mu g, \\ \dot{j} &= \frac{1}{L_s + \frac{C}{2s}} \left( -RJ + \frac{C}{2s^2} Jv + U \right),\end{aligned}$$

cost function  $c$  quadratic in  $s$ ,  $v$  and  $u$

- ▶  $\Omega$ : suitably chosen box
- ▶  $\tilde{U} = \{6 \cdot 10^3 u^3 \mid u \in \{-1, -0.99, \dots, 0.99, 1\}\}$
- ▶  $T$ : neighborhood of the equilibrium  $(0.01, 0, 17.155)$
- ▶  $X$ : equidistant grid of  $30 \times 30 \times 30$  nodes
- ▶  $\phi^\sigma$ : Wendland function of order 4,  $\sigma = 11.2$
- ▶ CPU time: 60 secs

# Example 4

magnetic wheel, 3D example



## Matlab code template

```
f = @(x,u) ...
c = @(x,u) ...
phi = @(r) max(spones(r)-r,0).^4.*(4*r+spones(r));
T = [0 0]; v_T = 1;
shepard = @(A) spdiags(1./sum(A')',0,size(A,1),size(A,1))*A
S = [8,10];
L = 33; U = linspace(-128,128,L)';
N = 100; X1 = linspace(-1,1,N);
[XX,YY] = meshgrid(X1*S(1),X1*S(2)); X = [XX(:) YY(:)];
ep = 1/sqrt((4*prod(S)*20/N^2)/pi);
A = shepard(phi(ep*sdistm(f(X,U),[T;X],1/ep)));
C = c(X,U);

v = zeros(N^2+1,1); v0 = ones(size(v)); TOL = 1e-12;
while norm(v-v0,inf)/norm(v,inf) > TOL
    v0 = v;
    v = [v_T; max(reshape(C.*(A*v),L,N^2))'];
end
contour(...
```

# Conclusion

## Pros

- ▶ simple convergence theory
- ▶ simple implementation, independent of state dimension
- ▶ easy to incorporate complicated state constraints

## Cons

- ▶ delicate choice of the shape parameter
- ▶ does not solve the curse of dimension ;-)