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# Complexity in control-affine systems

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We will consider affine-control systems, i.e., systems in the form

$$\dot{q}(t) = f_0(q(t)) + \sum_{i=1}^m u_i(t) f_i(q(t))$$

Here,

- the point  $q$  belongs to a smooth manifold  $M$
- the  $f_i$ 's are smooth vector fields on  $M$
- $u \in L^1([0, T], \mathbb{R}^m)$

This type of system appears in many applications

- Mechanical systems
- Quantum control
- Microswimmers (Tucsna, Alouges)
- Neuro-geometry of vision (Mumfor, Petitot)

- 1 Motion planning problem
- 2 Definitions of complexity
- 3 Asymptotic estimates in affine-control systems

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## Problem

Given  $x, y \in M$ , find an admissible trajectory steering the system from  $x$  to  $y$ , possibly under some constraints.

Possible constraints:

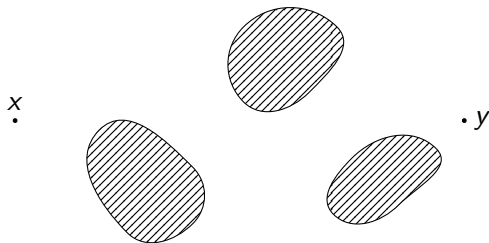
- 1 Avoiding some obstacles
- 2 Rendez-vous problem, i.e., being near certain places at certain times

## Assumption

A metric with balls  $B(q, \varepsilon)$  is fixed on  $M$ .

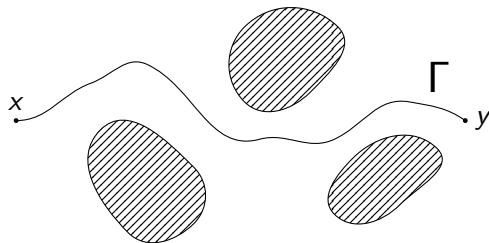
# Method

Different approaches are possible. We consider the following method:



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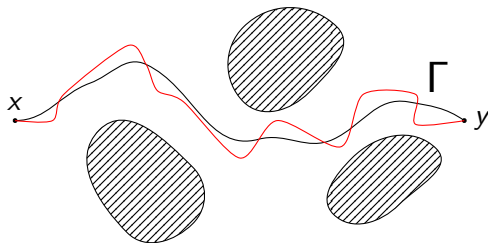
- 1 Find an (non-admissible) curve  $\Gamma \subset M$  or a path  $\gamma : [0, T] \rightarrow M$  solving the problem.





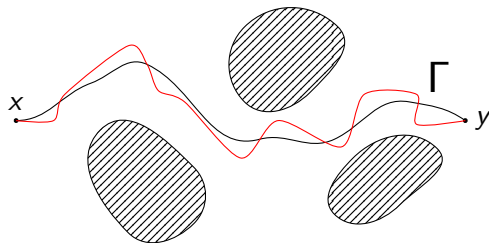
Different approaches are possible. We consider the following method:

- 1 Find an (non-admissible) curve  $\Gamma \subset M$  or a path  $\gamma : [0, T] \rightarrow M$  solving the problem.
- 2 Track  $\Gamma$  or  $\gamma$  with an admissible trajectory.



Different approaches are possible. We consider the following method:

- 1 Find an (non-admissible) curve  $\Gamma \subset M$  or a path  $\gamma : [0, T] \rightarrow M$  solving the problem.  $\rightarrow$  **global topology**
- 2 Track  $\Gamma$  or  $\gamma$  with an admissible trajectory.  $\rightarrow$  **local behavior of the control system**



We focus on quantifying the difficulty of the second step.

- 1 Motion planning problem
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Let  $J : \mathcal{U} \rightarrow [0, +\infty)$  be a cost function.

## Definition (Complexity)

A measure of the cost of approximation of a given curve/path with a certain precision

In general:

- 1 we fix a set  $\text{Adm}(\Gamma, \varepsilon)$  of admissible controls for precision  $\varepsilon$
- 2 we define complexity as

$$\sigma(\gamma, \varepsilon) = \inf_{u \in \text{Adm}(\Gamma, \varepsilon)} \frac{\text{cost of } u}{\text{cost of an } \varepsilon \text{ piece of } u} = \frac{1}{\varepsilon} \inf_{u \in \text{Adm}(\Gamma, \varepsilon)} J(u, T).$$

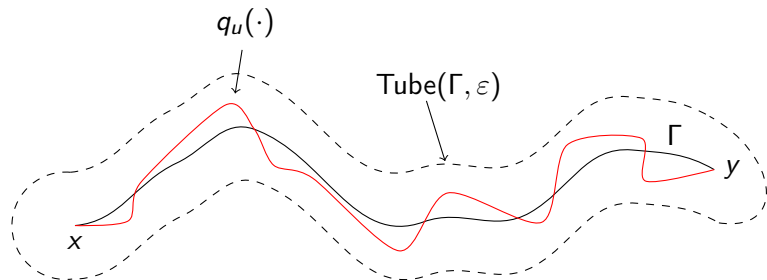
# Obstacle-avoidance problem

Let  $\Gamma \subset M$  be a curve,  $\text{Tube}(\Gamma, \varepsilon) = \bigcup_{q \in \Gamma} B(q, \varepsilon)$ , and

$$\mathcal{A}(\Gamma, \varepsilon) = \left\{ u \in L^1([0, T], \mathbb{R}^m) \mid \begin{array}{l} T > 0, q_u(T) = y, \\ q_u(\cdot) \subset \text{Tube}(\Gamma, \varepsilon) \end{array} \right\}.$$

With this set we define the *tubular approximation complexity*

$$\Sigma_a(\Gamma, \varepsilon) = \frac{1}{\varepsilon} \inf_{u \in \mathcal{A}(\Gamma, \varepsilon)} J(u, T).$$



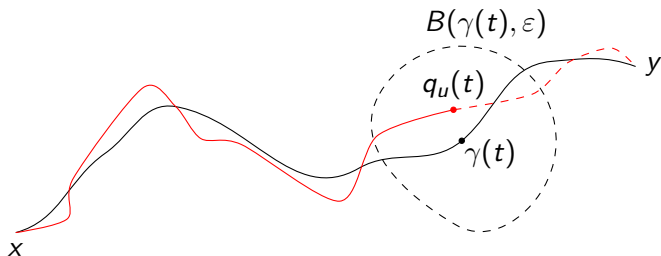
## Rendez-vous problem

Let  $\gamma : [0, T] \rightarrow M$  be a path and

$$\mathcal{N}(\gamma, \varepsilon) = \left\{ u \in L^1([0, T], \mathbb{R}^m) \mid \begin{array}{l} q_u(T) = y \text{ and } q_u(t) \in B(\gamma(t), \varepsilon) \\ \text{for any } t \in [0, T] \end{array} \right\}.$$

This set defines the *neighborhood approximation complexity*

$$\sigma_n(\gamma, \varepsilon) = \frac{1}{\varepsilon} \inf_{u \in \mathcal{N}(\gamma, \varepsilon)} J(u, T).$$



- 1 Motion planning problem
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## Particular case: nonholonomic control systems

Nonholonomic control system = control-affine system without drift

$$\dot{q}(t) = \sum_{i=1}^m u_i(t) f_i(q(t)),$$

that satisfies the Hörmander condition, i.e., such that

$$\text{Lie}_q\{f_1, \dots, f_m\} = T_q M, \quad \text{for any } q \in M.$$

- 1 The value function associated to this system w.r.t. the  $L^1$  cost is a distance, called sub-Riemannian distance.
- 2 Due to the linearity of the system, we can always reparametrize trajectories without changing their  $L^1$  cost. Hence,

Tubular approximation  
complexity



Neighborhood approximation  
complexity



- Introduced by Gromov (1996) in a different context.

- *Weak equivalence:*

$$\sigma(\Gamma, \varepsilon) \asymp g(\varepsilon) \iff C_1 \leq \frac{\sigma(\Gamma, \varepsilon)}{g(\varepsilon)} \leq C_2 \quad \text{for } \varepsilon \downarrow 0.$$

Complete results (Jean 2003).

- *Strong equivalence:*

$$\sigma(\Gamma, \varepsilon) \simeq g(\varepsilon) \iff \lim_{\varepsilon \downarrow 0} \frac{\sigma(\Gamma, \varepsilon)}{g(\varepsilon)} = 1.$$

Results in particular cases (Gauthier, Zakalyukin, et al., 2004-2013)

Recall the general form of a control-affine system

$$\dot{q}(t) = f_0(q(t)) + \sum_{i=1}^m u_i(t) f_i(q(t)).$$

We will consider:

- *strong Hörmander condition*:  $\text{Lie}_q\{f_1, \dots, f_m\} = T_q M$  for any  $q \in M$ .
- The set of controls is

$$\mathcal{U} = \bigcup_{T \in (0, T]} L^1([0, T], \mathbb{R}^m).$$

- The cost  $J$  is the  $L^1$ -norm of  $u$ .

Consequences:

- 1 Small time local controllability.
- 2 The associated driftless system ( $f_0 = 0$ ) is a nonholonomic system.

# Complexities for control-affine systems

- We will use the sub-Riemannian metric to define the complexities.
- Since the system is not linear, we cannot reparametrize the trajectories, and hence

Tubular approximation complexity  $\longleftrightarrow$  Neighborhood approximation complexity

For any  $q \in M$ ,  $s \in \mathbb{N}$ , let

$$\Delta^s(q) = \text{span}\{[f_{i_1}, [f_{i_2}, [\dots, f_{i_k}] \dots]](q) \mid 1 \leq k \leq s, 1 \leq i_j \leq m\}.$$

$$\Delta^1(q) \subset \Delta^2(q) \subset \dots \subset \Delta^r(q) = T_q M$$

## Hypothesis

*Equiregularity:* for any  $s \in \mathbb{N}$ ,  $\dim \Delta^s$  does not depend on the point  $q \in M$ .

## Theorem

Let  $f_0 \in \Delta^s \setminus \Delta^{s-1}$ .

- Let  $\Gamma \subset M$  be a smooth curve. Let  $k$  such that  $T\Gamma \subset \Delta^k$  and  $T\Gamma \not\subset \Delta^{k-1}$ . Then, if  $\mathcal{T}$  is sufficiently small, we have

$$\Sigma_a(\Gamma, \varepsilon) \asymp \frac{1}{\varepsilon^k}$$

- Let  $\gamma : [0, T] \rightarrow M$  be a path and  $k$  such that  $\dot{\gamma} \in \Delta^k$  and  $\dot{\gamma} \notin \Delta^{k-1}$ . If, moreover,  $s = k$ , we assume that  $\dot{\gamma} \neq f_0(\gamma) \bmod \Delta^{s-1}(\gamma)$ . Then

$$\sigma_n(\gamma, \varepsilon) \asymp \frac{1}{\varepsilon^{\max\{s, k\}}}$$

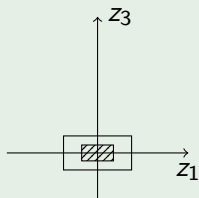
- The complexity of **curves** is not sensible to the drift.
- The complexity of **paths** depends on the drift. In particular, when  $f_0 \in \Delta^r \setminus \Delta^{r-1}$  where  $r$  is such that  $\Delta^r = T_q M$ , the complexity is always maximal, i.e.,  $\sigma_n(\gamma, \varepsilon) \asymp \varepsilon^{-r}$ .

# Techniques and Remarks

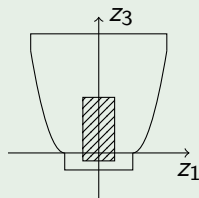
- Weak estimates of the value function near a point (generalization of the sub-Riemannian Ball-Box theorem).

## Example

- $f_1$  and  $f_2$  control vector fields on  $\mathbb{R}^3$  satisfying the Hörmander condition,
- Drift s.t.  $f_0 \notin \Delta^1 = \text{span}\{f_1, f_2\}$ .



Nonholonomic system.



Control-affine system.

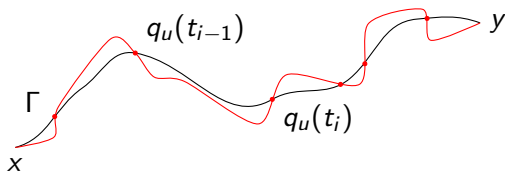
## Techniques and Remarks (continued)

- Estimates obtained by reducing the control system with drift to a **driftless** but **time-dependent** system.

$$\dot{q} = f_0(q) + \sum_{i=1}^m u_i f_i(q) \quad \longrightarrow \quad \dot{q} = \sum_{i=1}^m u_i (e^{-tf_0})_* f_i(q).$$

- For this system we can define a generalization of the nilpotent approximation, that yields the estimates.

- We studied also two other notions of complexity, where we track the curve/path by interpolation, and no metric is assumed.



- We studied also another cost

$$\mathcal{I}(u, T) = \int_0^T \sqrt{1 + \sum_{i=1}^m u_i(t)^2} dt.$$

Thank you for your attention.



Let  $\{\partial_{z_i}\}_{i=1}^n$  be the canonical basis of  $\mathbb{R}^n$  and  $\mathcal{R}_{f_0}(q, \varepsilon)$  the reachable set from  $q$  with cost  $\leq \varepsilon$ . We define

$$\Xi(\eta) = \bigcup_{0 \leq \xi \leq \mathcal{T}} (\xi \partial_{z_\ell} + \text{Box}(\eta))$$

$$\Pi(\eta) = \bigcup_{0 \leq \xi \leq \mathcal{T}} \{z \in \mathbb{R}^n : |z_\ell - \xi| \leq \eta^s, |z_i| \leq \eta^{w_i} + \eta \xi^{\frac{w_i}{s}} \text{ pour } w_i \leq s, i \neq \ell, \\ \text{et } |z_i| \leq \eta(\eta + \xi^{\frac{1}{s}})^{w_i-1} \text{ pour } w_i > s\},$$

## Theorem

Let  $z = (z_1, \dots, z_n)$  a privileged coordinate system at  $q$  for  $\{f_1, \dots, f_m\}$ , rectifying  $f_0$  as the  $k$ -th coordinate vector field  $\partial_{z_\ell}$ , for some  $1 \leq \ell \leq n$ . Then, there exist  $C, \varepsilon_0, T_0$  s.t., if  $\mathcal{T} < T_0$ , it holds

$$\Xi\left(\frac{1}{C}\varepsilon\right) \subset \mathcal{R}_{f_0}(q, \varepsilon) \subset \Pi(C\varepsilon), \quad \text{for } \varepsilon < \varepsilon_0.$$