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Complexity in control-affine systems

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We will consider affine-control systems, i.e., systems in the form

$$\dot{q}(t) = f_0(q(t)) + \sum_{i=1}^m u_i(t) f_i(q(t))$$

Here,

- the point q belongs to a smooth manifold M
- the f_i 's are smooth vector fields on M
- $u \in L^1([0, T], \mathbb{R}^m)$

This type of system appears in many applications

- Mechanical systems
- Quantum control
- Microswimmers (Tucsna, Alouges)
- Neuro-geometry of vision (Mumfor, Petitot)

- 1 Motion planning problem
- 2 Definitions of complexity
- 3 Asymptotic estimates in affine-control systems

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Problem

Given $x, y \in M$, find an admissible trajectory steering the system from x to y , possibly under some constraints.

Possible constraints:

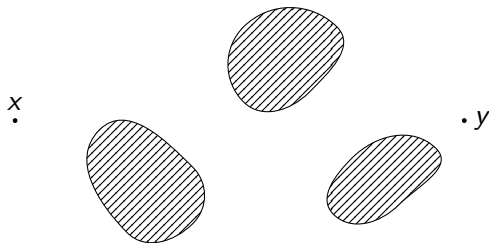
- 1 Avoiding some obstacles
- 2 Rendez-vous problem, i.e., being near certain places at certain times

Assumption

A metric with balls $B(q, \varepsilon)$ is fixed on M .

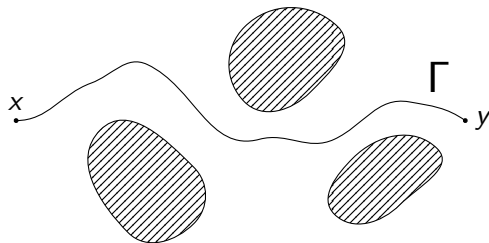
Method

Different approaches are possible. We consider the following method:



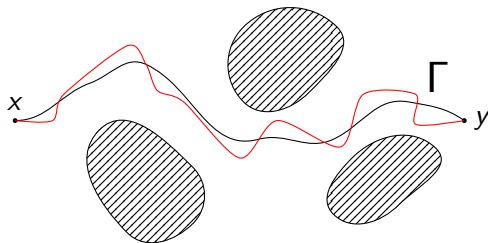
Different approaches are possible. We consider the following method:

- 1 Find an (non-admissible) curve $\Gamma \subset M$ or a path $\gamma : [0, T] \rightarrow M$ solving the problem.



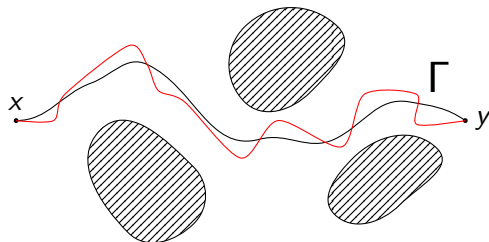
Different approaches are possible. We consider the following method:

- 1 Find an (non-admissible) curve $\Gamma \subset M$ or a path $\gamma : [0, T] \rightarrow M$ solving the problem.
- 2 Track Γ or γ with an admissible trajectory.



Different approaches are possible. We consider the following method:

- 1 Find an (non-admissible) curve $\Gamma \subset M$ or a path $\gamma : [0, T] \rightarrow M$ solving the problem. \rightarrow **global topology**
- 2 Track Γ or γ with an admissible trajectory. \rightarrow **local behavior of the control system**



We focus on quantifying the difficulty of the second step.

- 1 Motion planning problem
- 2 Definitions of complexity
- 3 Asymptotic estimates in affine-control systems

Let $J : \mathcal{U} \rightarrow [0, +\infty)$ be a cost function.

Definition (Complexity)

A measure of the cost of approximation of a given curve/path with a certain precision

In general:

- 1 we fix a set $\text{Adm}(\Gamma, \varepsilon)$ of admissible controls for precision ε
- 2 we define complexity as

$$\sigma(\gamma, \varepsilon) = \inf_{u \in \text{Adm}(\Gamma, \varepsilon)} \frac{\text{cost of } u}{\text{cost of an } \varepsilon \text{ piece of } u} = \frac{1}{\varepsilon} \inf_{u \in \text{Adm}(\Gamma, \varepsilon)} J(u, T).$$

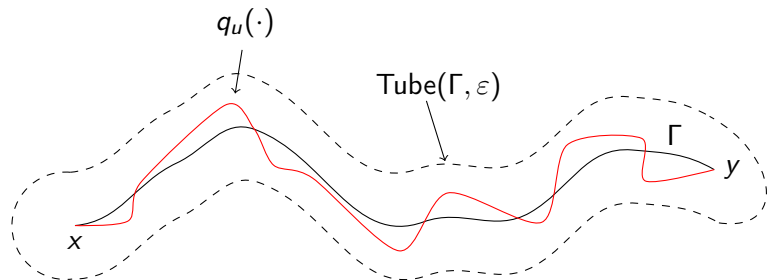
Obstacle-avoidance problem

Let $\Gamma \subset M$ be a curve, $\text{Tube}(\Gamma, \varepsilon) = \bigcup_{q \in \Gamma} B(q, \varepsilon)$, and

$$\mathcal{A}(\Gamma, \varepsilon) = \left\{ u \in L^1([0, T], \mathbb{R}^m) \mid \begin{array}{l} T > 0, q_u(T) = y, \\ q_u(\cdot) \subset \text{Tube}(\Gamma, \varepsilon) \end{array} \right\}.$$

With this set we define the *tubular approximation complexity*

$$\Sigma_a(\Gamma, \varepsilon) = \frac{1}{\varepsilon} \inf_{u \in \mathcal{A}(\Gamma, \varepsilon)} J(u, T).$$



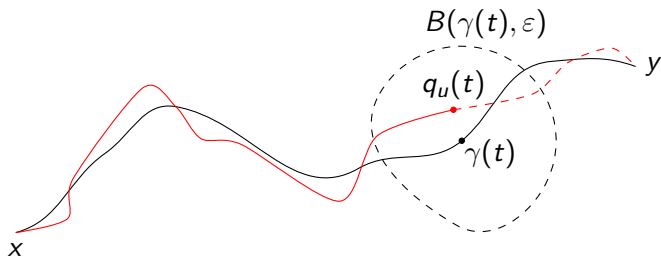
Rendez-vous problem

Let $\gamma : [0, T] \rightarrow M$ be a path and

$$\mathcal{N}(\gamma, \varepsilon) = \left\{ u \in L^1([0, T], \mathbb{R}^m) \mid \begin{array}{l} q_u(T) = y \text{ and } q_u(t) \in B(\gamma(t), \varepsilon) \\ \text{for any } t \in [0, T] \end{array} \right\}.$$

This set defines the *neighborhood approximation complexity*

$$\sigma_n(\gamma, \varepsilon) = \frac{1}{\varepsilon} \inf_{u \in \mathcal{N}(\gamma, \varepsilon)} J(u, T).$$



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Particular case: nonholonomic control systems

Nonholonomic control system = control-affine system without drift

$$\dot{q}(t) = \sum_{i=1}^m u_i(t) f_i(q(t)),$$

that satisfies the Hörmander condition, i.e., such that

$$\text{Lie}_q\{f_1, \dots, f_m\} = T_q M, \quad \text{for any } q \in M.$$

- 1 The value function associated to this system w.r.t. the L^1 cost is a distance, called sub-Riemannian distance.
- 2 Due to the linearity of the system, we can always reparametrize trajectories without changing their L^1 cost. Hence,

Tubular approximation
complexity



Neighborhood approximation
complexity

- Introduced by Gromov (1996) in a different context.

- *Weak equivalence:*

$$\sigma(\Gamma, \varepsilon) \asymp g(\varepsilon) \iff C_1 \leq \frac{\sigma(\Gamma, \varepsilon)}{g(\varepsilon)} \leq C_2 \quad \text{for } \varepsilon \downarrow 0.$$

Complete results (Jean 2003).

- *Strong equivalence:*

$$\sigma(\Gamma, \varepsilon) \simeq g(\varepsilon) \iff \lim_{\varepsilon \downarrow 0} \frac{\sigma(\Gamma, \varepsilon)}{g(\varepsilon)} = 1.$$

Results in particular cases (Gauthier, Zakalyukin, et al., 2004-2013)

Recall the general form of a control-affine system

$$\dot{q}(t) = f_0(q(t)) + \sum_{i=1}^m u_i(t) f_i(q(t)).$$

We will consider:

- *strong Hörmander condition*: $\text{Lie}_q\{f_1, \dots, f_m\} = T_q M$ for any $q \in M$.
- The set of controls is

$$\mathcal{U} = \bigcup_{T \in (0, T]} L^1([0, T], \mathbb{R}^m).$$

- The cost J is the L^1 -norm of u .

Consequences:

- 1 Small time local controllability.
- 2 The associated driftless system ($f_0 = 0$) is a nonholonomic system.

Complexities for control-affine systems

- We will use the sub-Riemannian metric to define the complexities.
- Since the system is not linear, we cannot reparametrize the trajectories, and hence

Tubular approximation complexity \longleftrightarrow Neighborhood approximation complexity

For any $q \in M$, $s \in \mathbb{N}$, let

$$\Delta^s(q) = \text{span}\{[f_{i_1}, [f_{i_2}, [\dots, f_{i_k}] \dots]](q) \mid 1 \leq k \leq s, 1 \leq i_j \leq m\}.$$

$$\Delta^1(q) \subset \Delta^2(q) \subset \dots \subset \Delta^r(q) = T_q M$$

Hypothesis

Equiregularity: for any $s \in \mathbb{N}$, $\dim \Delta^s$ does not depend on the point $q \in M$.

Theorem

Let $f_0 \in \Delta^s \setminus \Delta^{s-1}$.

- Let $\Gamma \subset M$ be a smooth curve. Let k such that $T\Gamma \subset \Delta^k$ and $T\Gamma \not\subset \Delta^{k-1}$. Then, if \mathcal{T} is sufficiently small, we have

$$\Sigma_a(\Gamma, \varepsilon) \asymp \frac{1}{\varepsilon^k}$$

- Let $\gamma : [0, T] \rightarrow M$ be a path and k such that $\dot{\gamma} \in \Delta^k$ and $\dot{\gamma} \notin \Delta^{k-1}$. If, moreover, $s = k$, we assume that $\dot{\gamma} \neq f_0(\gamma) \bmod \Delta^{s-1}(\gamma)$. Then

$$\sigma_n(\gamma, \varepsilon) \asymp \frac{1}{\varepsilon^{\max\{s, k\}}}$$

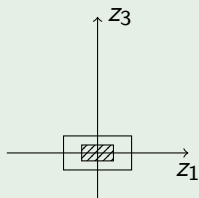
- The complexity of **curves** is not sensible to the drift.
- The complexity of **paths** depends on the drift. In particular, when $f_0 \in \Delta^r \setminus \Delta^{r-1}$ where r is such that $\Delta^r = T_q M$, the complexity is always maximal, i.e., $\sigma_n(\gamma, \varepsilon) \asymp \varepsilon^{-r}$.

Techniques and Remarks

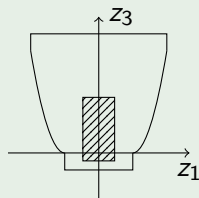
- Weak estimates of the value function near a point (generalization of the sub-Riemannian Ball-Box theorem).

Example

- f_1 and f_2 control vector fields on \mathbb{R}^3 satisfying the Hörmander condition,
- Drift s.t. $f_0 \notin \Delta^1 = \text{span}\{f_1, f_2\}$.



Nonholonomic system.



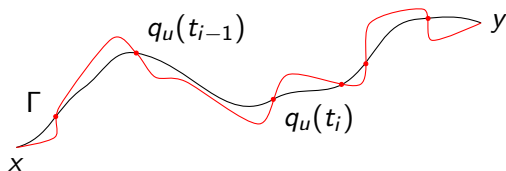
Control-affine system.

- Estimates obtained by reducing the control system with drift to a **driftless** but **time-dependent** system.

$$\dot{q} = f_0(q) + \sum_{i=1}^m u_i f_i(q) \quad \longrightarrow \quad \dot{q} = \sum_{i=1}^m u_i (e^{-tf_0})_* f_i(q).$$

- For this system we can define a generalization of the nilpotent approximation, that yields the estimates.

- We studied also two other notions of complexity, where we track the curve/path by interpolation, and no metric is assumed.



- We studied also another cost

$$\mathcal{I}(u, T) = \int_0^T \sqrt{1 + \sum_{i=1}^m u_i(t)^2} dt.$$

Thank you for your attention.

Let $\{\partial_{z_i}\}_{i=1}^n$ be the canonical basis of \mathbb{R}^n and $\mathcal{R}_{f_0}(q, \varepsilon)$ the reachable set from q with cost $\leq \varepsilon$. We define

$$\Xi(\eta) = \bigcup_{0 \leq \xi \leq \mathcal{T}} (\xi \partial_{z_\ell} + \text{Box}(\eta))$$

$$\Pi(\eta) = \bigcup_{0 \leq \xi \leq \mathcal{T}} \{z \in \mathbb{R}^n : |z_\ell - \xi| \leq \eta^s, |z_i| \leq \eta^{w_i} + \eta \xi^{\frac{w_i}{s}} \text{ pour } w_i \leq s, i \neq \ell, \\ \text{et } |z_i| \leq \eta(\eta + \xi^{\frac{1}{s}})^{w_i-1} \text{ pour } w_i > s\},$$

Theorem

Let $z = (z_1, \dots, z_n)$ a privileged coordinate system at q for $\{f_1, \dots, f_m\}$, rectifying f_0 as the k -th coordinate vector field ∂_{z_ℓ} , for some $1 \leq \ell \leq n$. Then, there exist C, ε_0, T_0 s.t., if $\mathcal{T} < T_0$, it holds

$$\Xi\left(\frac{1}{C}\varepsilon\right) \subset \mathcal{R}_{f_0}(q, \varepsilon) \subset \Pi(C\varepsilon), \quad \text{for } \varepsilon < \varepsilon_0.$$