



Optimal control problems on stratifiable state constraints sets.

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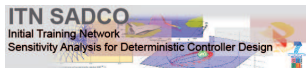
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Optimal control problems on stratifiable state constraints sets.

Cristopher HERMOSILLA
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UMA, ENSTA ParisTech

NetCo Conference 2014



Introduction

We consider an **infinite horizon** problem with **state constraints** \mathcal{K} :

$$(P) \quad \inf \left\{ \int_0^\infty e^{-\lambda t} \ell(y_{x,u}(t), u(t)) dt \mid \begin{array}{l} u : [0, +\infty) \rightarrow \mathcal{A} \text{ measurable} \\ y_{x,u}(t) \in \mathcal{K} \quad \forall t \geq 0 \end{array} \right\}.$$

where $\lambda > 0$ is fixed and $y_{x,u}(\cdot)$ is a trajectory of the control system

$$\begin{cases} \dot{y} = f(y, u) & \text{a.e. } t \geq 0 \\ y(0) = x \in \mathcal{K} \end{cases}$$

We are mainly concerned with a **characterization** of the value function of (P) as the **bilateral solution** to a Hamilton-Jacobi-Bellman equation.

Standing Hypothesis (SH)

- \mathcal{K} is **closed** and $\mathcal{A} \subseteq \mathbb{R}^m$ is nonempty and **compact**.
- $\ell : \mathbb{R}^N \times \mathcal{A} \rightarrow [0, +\infty)$ and $f : \mathbb{R}^N \times \mathcal{A} \rightarrow \mathbb{R}^N$ are **continuous and Lipschitz** w.r.t. the state :

$$\exists L > 0 \text{ such that } \left. \begin{array}{l} |f(x, u) - f(y, u)| \\ |\ell(x, u) - \ell(y, u)| \end{array} \right\} \leq L|x - y| \quad \forall u \in \mathcal{A}.$$

- We assume **convexity** of the augmented dynamics :

$$\left\{ \left(\begin{array}{c} f(x, u) \\ e^{-\lambda t} \ell(x, u) + r \end{array} \right) \mid \begin{array}{l} u \in \mathcal{A} \\ r \geq 0 \end{array} \right\}, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

The value function

Basic properties

$$v(x) := \inf_{u \in \mathbb{A}(x)} \int_0^\infty e^{-\lambda t} \ell(y_{x,u}(t), u(t)) dt, \quad \forall x \in \mathcal{K},$$

where

$$\mathbb{A}(x) = \{u : [0, +\infty) \rightarrow \mathcal{A} \text{ measurable} \mid y_{x,u}(t) \in \mathcal{K} \quad \forall t \geq 0\}.$$

Proposition

Suppose that (SH) holds. Then, $\exists \lambda_0 = \lambda_0(f, \ell) > 0$ so that if $\lambda > \lambda_0$

- if $v(x) \in \mathbb{R}$ then there exists $u \in \mathbb{A}(x)$ an *optimal control*.
- $v : \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$ is *lower semicontinuous*.
- v has *linear growth* on its domain :

$$\exists c_v > 0 \quad |v(x)| \leq c_v(1 + |x|) \quad \forall x \in \text{dom } \mathbb{A}.$$

An example

$$(\mathcal{P}) \quad \min \int_0^\infty e^{-\lambda t} u(t)^2 dt,$$

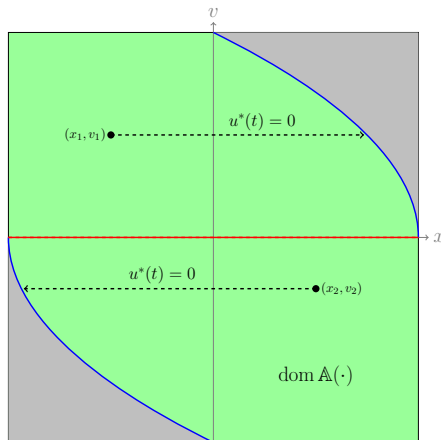
$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ u \end{pmatrix}$$

$$y_1(0) = x,$$

$$y_2(0) = v,$$

$$u \in \mathcal{A} := [-1, 1]$$

$$y_1(t), y_2(t) \in [-r, r]$$



Dynamic Programming Principle

Proposition

The value function *satisfies the DPP* : For any $T > 0$

$$v(x) = \inf_{u \in \mathbb{A}(x)} \left\{ \int_0^T e^{-\lambda t} \ell(y_{x,u}(t), u(t)) dt + e^{-\lambda T} v(y_{x,u}(T)) \right\}.$$

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Definition

Let $\varphi : \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$ be l.s.c.

i) φ is *weakly decreasing* provided $\forall x \in \text{dom } \varphi, \exists u \in \mathbb{A}(x)$ such that

$$e^{-\lambda t} \varphi(y_{x,u}(t)) + \int_0^t e^{-\lambda s} \ell(y_{x,u}(s), u(s)) ds \leq \varphi(x) \quad \forall t \geq 0.$$

ii) φ is *strongly increasing* provided $\text{dom } \mathbb{A} \subseteq \text{dom } \varphi$ and $\forall x \in \mathcal{K}, \forall u \in \mathbb{A}(x)$

$$e^{-\lambda t} \varphi(y_{x,u}(t)) + \int_0^t e^{-\lambda s} \ell(y_{x,u}(s), u(s)) ds \geq \varphi(x) \quad \forall t \geq 0.$$

The value function

Comparison principle

Proposition

Let $\varphi : \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$ be l.s.c. with *linear growth*.

- If φ is weakly decreasing, then $v(x) \leq \varphi(x)$ for all $x \in \mathcal{K}$.
- If φ is strongly increasing, then $v(x) \geq \varphi(x)$ for all $x \in \mathcal{K}$.

The value function

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- If φ is strongly increasing, then $v(x) \geq \varphi(x)$ for all $x \in \mathcal{K}$.

Corollary

Suppose that (SH) holds with $\lambda > \lambda_0$. Then $v(\cdot)$ is the *unique l.s.c. function with linear growth* defined on \mathcal{K} which is *weakly decreasing* and *strongly increasing* at the same time.

Weakly Decreasing Principle

Characterization of supersolutions

Definition (Subdifferentials)

Let $\varphi : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function. A vector $\zeta \in \mathbb{R}^N$ is called a **viscosity subgradient** of φ at $x \in \text{dom } \varphi$ if and only if :

$\exists g \in \mathcal{C}^1(\mathbb{R}^N)$ s.t. $\nabla g(x) = \zeta$ and $\varphi - g$ attains a local minimum at x .

Furthermore, ζ is called a **proximal subgradient** of φ at x if for some $\sigma > 0$,

$$g(y) := \langle \zeta, y - x \rangle - \sigma |y - x|^2.$$

The set of all proximal subgradients at x is denoted by $\partial_P \varphi(x)$.

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Proposition

Suppose that (SH) holds and let $\varphi : \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a **l.s.c. function**. Then φ is **weakly decreasing** if and only if

$$\lambda \varphi(x) + H(x, \zeta) \geq 0 \quad \forall x \in \mathcal{K}, \quad \forall \zeta \in \partial_P \varphi(x).$$

Strongly Increasing Principle

Characterization of subsolutions

Proposition

Suppose that (SH) holds and let $\varphi : \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a *l.s.c. function*. If φ is *strongly increasing* then

$$(1) \quad \lambda\varphi(x) + H(x, \zeta) \leq 0 \quad \forall x \in \text{int}(\mathcal{K}), \forall \zeta \in \partial_P \varphi(x).$$

Strongly Increasing Principle

Characterization of subsolutions

Proposition

Suppose that (SH) holds and let $\varphi : \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a *l.s.c. function*. If φ is *strongly increasing* then

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Remark

- The converse *does not hold* without additional hypothesis.
- (1) provides information only for trajectories that *touch* the boundary a *finite number* of times.
- There is *no information* of what happens on the boundary.

Feasible Neighboring Trajectories Approach

Soner, Frankowska-Vinter, Clarke-Stern, among many others

When does (1) become sufficient ??

- $v(\cdot)$ is **continuous** up to the boundary.
- **Interior approximation** of trajectories.
- Some **monotonicity properties** of the solutions to (1).

Feasible Neighboring Trajectories Approach

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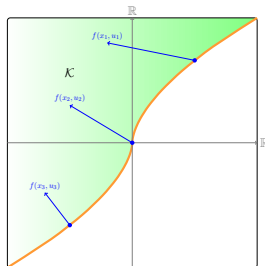
When does (1) become sufficient??

- $v(\cdot)$ is **continuous** up to the boundary.
- **Interior approximation** of trajectories.
- Some **monotonicity properties** of the solutions to (1).

What do we need to achieve one of these??

↪ $\mathcal{K} = \overline{\text{int}(\mathcal{K})}$ and tameness properties.

↪ Pointing Conditions
(Inward or Outward).



Our example

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ u \end{pmatrix}$$

$$y_1(0) = x,$$

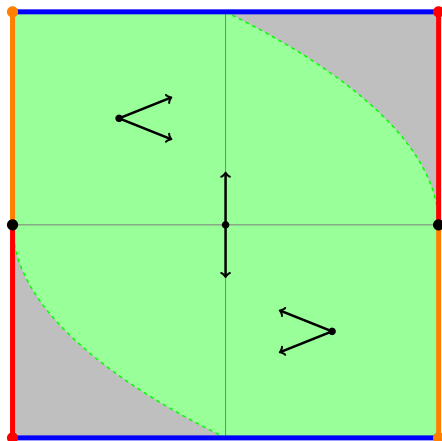
$$y_2(0) = v,$$

$$u \in \mathcal{A} := [-1, 1]$$

$$y_1(t), y_2(t) \in [-r, r]$$

Note that :

$$\left\langle \begin{pmatrix} 0 \\ u \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = 0$$

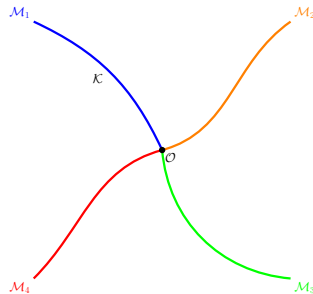
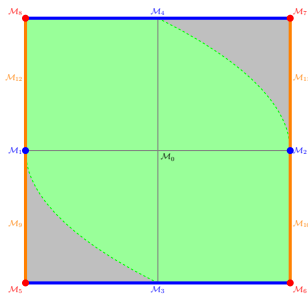


Stratifiable sets

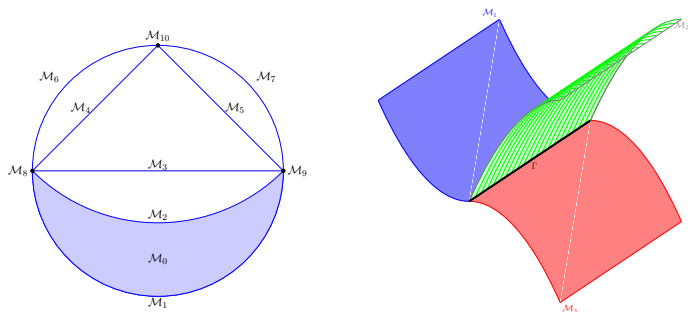
Definition

A closed set $\mathcal{K} \subseteq \mathbb{R}^N$ is said to be **stratifiable** if there exists a locally finite collection $\{\mathcal{M}_i : i \in \mathcal{I}\}$ of embedded manifolds of \mathbb{R}^N such that :

- $\mathcal{K} = \bigcup_{i \in \mathcal{I}} \mathcal{M}_i$ and $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ whenever $i \neq j$.
- $\mathcal{M}_i \cap \overline{\mathcal{M}_j} \neq \emptyset$, then $\mathcal{M}_i \subseteq \overline{\mathcal{M}_j}$ and $\dim(\mathcal{M}_i) < \dim(\mathcal{M}_j)$.



Stratifiable sets



The class of stratifiable sets on \mathbb{R}^N is wide, it includes :

- Semilinear sets \rightarrow finite union of open polyhedra.
- Semialgebraic sets \rightarrow finite union of polynomial manifolds.
- Subanalytic sets \rightarrow locally finite union of analytic manifolds.

Characterization of subsolutions

Some basic definitions and notation

Assume that \mathcal{K} is **stratifiable** and let $\{\mathcal{M}_i\}$ be its strata. Then, for each stratum \mathcal{M}_i we define

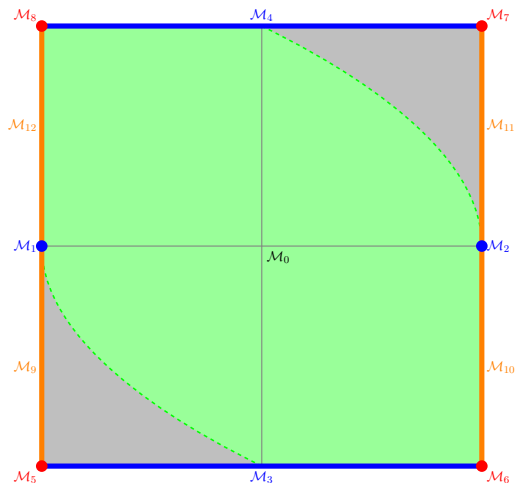
- the **set of tangent controls** as the map $\mathcal{A}_i : \mathcal{M}_i \rightrightarrows \mathcal{A}$ given by

$$\mathcal{A}_i(x) := \{u \in \mathcal{A} \mid f(x, u) \in \mathcal{T}_{\mathcal{M}_i}(x)\}.$$

- the **tangential Hamiltonian** as the map $H_i : \mathcal{M}_i \times \mathbb{R}^N \rightarrow \mathbb{R}$ given by

$$H_i(x, \zeta) = \max_{u \in \mathcal{A}_i(x)} \{-\langle \zeta, f(x, u) \rangle - \ell(x, u)\}.$$

Tangent controls



$$f(x_1, x_2, u) = (x_2, u)$$

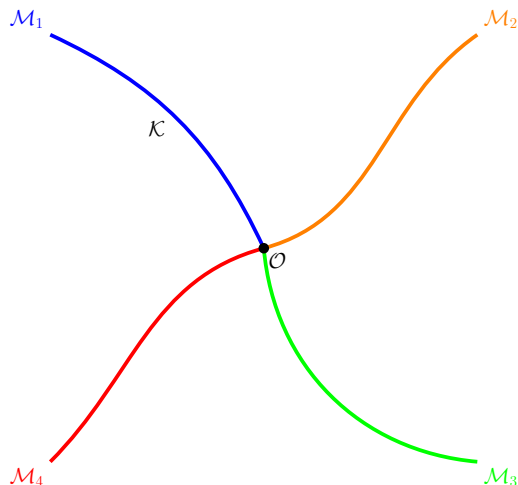
$$u \in [-1, 1]$$

$$\mathcal{A}_0(x) = \mathcal{A}$$

$$\mathcal{A}_i(x) = \{0\} \text{ for } i = 1, \dots, 4$$

$$\mathcal{A}_j(x) = \emptyset \text{ for } j = 5, \dots, 12$$

Tangent controls



$\exists \mathcal{A}_i \subseteq \mathcal{A}$ such that

$$f(x, \mathcal{A}) \cap \mathcal{T}_{\mathcal{M}_i}(x) = f(x, \mathcal{A}_i).$$

$$\mathcal{A}_i(x) = \mathcal{A}_i, \quad \forall i = 1, \dots, 4$$

Let $M_0 = \{\mathcal{O}\}$:

$$\mathcal{A}_0(\mathcal{O}) = \{u \in \mathcal{A} \mid f(\mathcal{O}, u) = 0\}.$$

Characterization of subsolutions

Proposition (CH - Zidani)

Suppose that (SH) holds with \mathcal{K} is *stratifiable* and let $\varphi : \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a *l.s.c. function*. Assume that

(H₀) \mathcal{A}_i is a *Lipschitz* set-valued map or has *empty images* on \mathcal{M}_i .

If φ is *strongly increasing* then for each $i \in \mathcal{I}$

$$(\star) \quad \lambda\varphi(x) + H_i(x, \zeta) \leq 0 \quad \forall x \in \mathcal{M}_i, \forall \zeta \in \partial_P \varphi_i(x),$$

where $\varphi_i(x) = \varphi(x)$ if $x \in \overline{\mathcal{M}_i}$ and $+\infty$ otherwise.

Remark

(\star) is equivalent to say : $\forall i \in \mathcal{I}, \forall x \in \mathcal{M}_i$ and $\forall g \in \mathcal{C}^1(\mathbb{R}^N)$ such that $\varphi - g$ attains a local minimum at x *relative to \mathcal{M}_i*

$$\lambda\varphi(x) + H_i(x, \nabla g(x)) \leq 0.$$

The converse ?

Strong Invariance Principle on $\text{int}(\mathcal{K})$

$$\lambda\varphi(x) + \max_{u \in \mathcal{A}} \{-\langle \zeta, f(x, u) \rangle - \ell(x, u)\} \leq 0 \quad \forall x \in \text{int}(\mathcal{K}), \quad \forall \zeta \in \partial_P \varphi(x).$$

\Downarrow

$y_{x,u}(s) \in \text{int}(\mathcal{K})$ for every $s \in (a, b)$, where $0 \leq a < b < +\infty$ then

$$\varphi(y_{x,u}(a)) \leq e^{-\lambda(b-a)}\varphi(y_{x,u}(b)) + e^{\lambda a} \int_a^b e^{-\lambda s} \ell(y_{x,u}, u) ds.$$

\Downarrow

Any admissible trajectory $y_{x,u}$ defined on $[0, T]$ with $y_{x,u}(s) \in \text{int}(\mathcal{K})$ for every $s \in (0, T)$ satisfies the **Strong Increasing Inequality**.

The converse ?

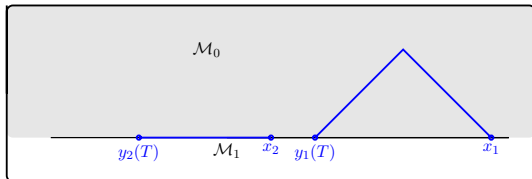
Strong Invariance Principle on each stratum

$$\lambda\varphi(x) + \max_{u \in \mathcal{A}_i(x)} \{-\langle \zeta, f(x, u) \rangle - \ell(x, u)\} \leq 0 \quad \forall x \in \mathcal{M}_i, \quad \forall \zeta \in \partial_P \varphi_i(x).$$

\Downarrow

$y_{x,u}(s) \in \mathcal{M}_i$ for every $s \in (a, b)$, where $0 \leq a < b < +\infty$ then

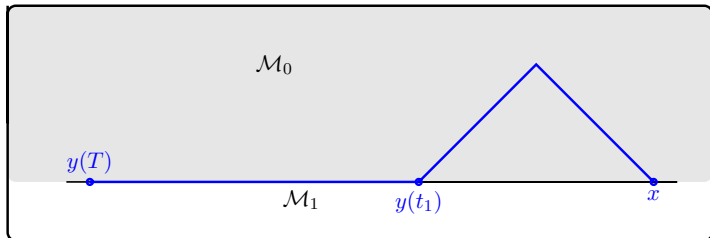
$$\varphi(y_{x,u}(a)) \leq e^{-\lambda(b-a)} \varphi(y_{x,u}(b)) + e^{\lambda a} \int_a^b e^{-\lambda s} \ell(y_{x,u}, u) ds.$$



Suppose that $y_{x,u}(s) \in \mathcal{K}$, $\forall s \in [0, T]$ and **there exists a partition** of $[0, T]$

$$\pi = \{0 = t_0 < t_1 < \dots < t_n < t_{n+1} = T\}$$

so that $\forall l \in \{0, \dots, n\}, \exists \mathcal{M}_l$ with $y_{x,u}(s) \in \mathcal{M}_l, \forall s \in (t_l, t_{l+1})$.



Whence, $\forall l \in \{0, \dots, n\}$ we have

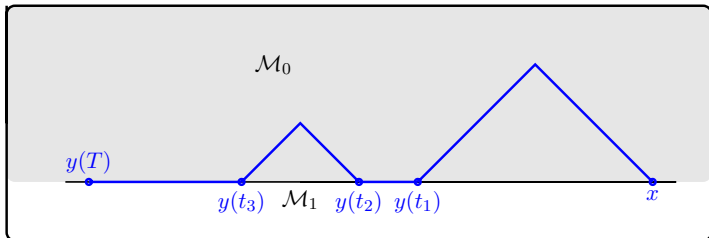
$$\varphi(y_{x,u}(t_l)) \leq e^{-\lambda(t_{l+1}-t_l)} \varphi(y_{x,u}(t_{l+1})) + e^{\lambda t_l} \int_{t_l}^{t_{l+1}} e^{-\lambda s} \ell(y_{x,u}, u) ds.$$

Therefore $y_{x,u}$ satisfies the **Strong Increasing Inequality !!**

Suppose that $y_{x,u}(s) \in \mathcal{K}$, $\forall s \in [0, T]$ and **there exists a partition** of $[0, T]$

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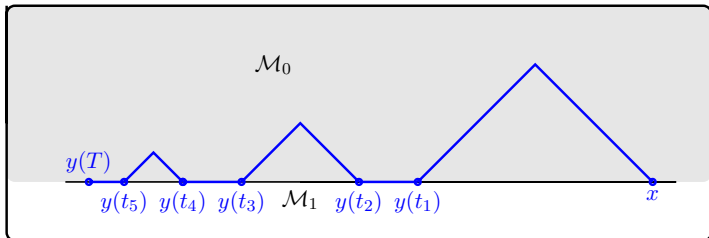
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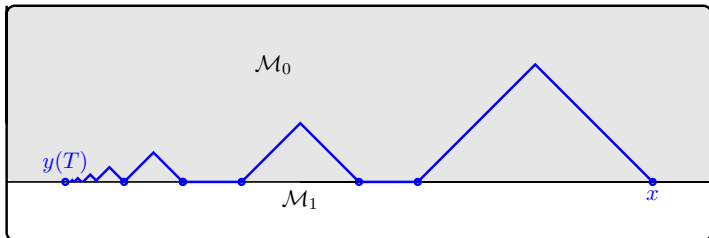
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Therefore $y_{x,u}$ satisfies the **Strong Increasing Inequality !!**

Suppose that $y_{x,u}(s) \in \mathcal{K}$, $\forall s \in [0, T]$ and **there is NO partition** of $[0, T]$

$$\pi = \{0 = t_0 < t_1 < \dots < t_n < t_{n+1} = T\}$$

so that $\forall l \in \{0, \dots, n\}, \exists \mathcal{M}_l$ with $y_{x,u}(s) \in \mathcal{M}_l, \forall s \in (t_l, t_{l+1})$.



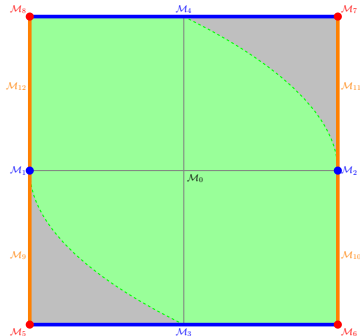
What if the trajectory "chatters" between two or more strata ???

Controllability assumption

$$(H_1) \quad \begin{cases} \forall i \in \mathcal{I} \text{ with } \text{dom } \mathcal{A}_i \neq \emptyset, \exists \delta_i, \Delta_i > 0 \text{ such that} \\ \mathcal{R}^{(t)}(x) \cap \overline{\mathcal{M}}_i \subseteq \bigcup_{s \in [0, \Delta_i t]} \mathcal{R}_i^{(s)}(x), \quad \forall x \in \mathcal{M}_i, \forall t \in [0, \delta_i]. \end{cases}$$

$\mathcal{R}^{(t)}(\cdot)$: reachable set at time t of $x \mapsto f(x, \mathcal{A})$.

$\mathcal{R}_i^{(t)}(\cdot)$: reachable set at time t of $x \mapsto f(x, \mathcal{A}_i(x))$.



$$f(x_1, x_2, u) = (x_2, u) \\ u \in [-1, 1]$$

$$\mathcal{A}_0(x) = \mathcal{A} \\ \mathcal{A}_i(x) = \{0\} \text{ for } i = 1, \dots, 4 \\ \mathcal{A}_j(x) = \emptyset \text{ for } j = 5, \dots, 12$$

Controllability assumption

$$(H_1) \quad \left\{ \begin{array}{l} \forall i \in \mathcal{I} \text{ with } \text{dom } \mathcal{A}_i \neq \emptyset, \exists \delta_i, \Delta_i > 0 \text{ such that} \\ \mathcal{R}^{(t)}(x) \cap \overline{\mathcal{M}}_i \subseteq \bigcup_{s \in [0, \Delta_i t]} \mathcal{R}_i^{(s)}(x), \quad \forall x \in \mathcal{M}_i, \forall t \in [0, \delta_i]. \end{array} \right.$$

$\mathcal{R}^{(t)}(\cdot)$: reachable set at time t of $x \mapsto f(x, \mathcal{A})$.

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Lemma

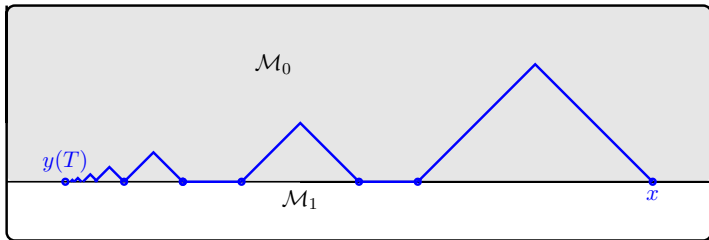
Suppose (SH) with \mathcal{K} stratifiable. Assume that (H_0) and (H_1) hold.
For any $x \in \mathcal{K}$ and $T > 0$ there exists $L > 0$ such that : $\forall u \in \mathbb{A}(x), \forall \varepsilon > 0$
if $y_{x,u}(b), y_{x,u}(a) \in \mathcal{M}_i$ with $0 \leq a < b \leq T$ for some $i \in \mathcal{I}$ then,

$$\varphi(y_{x,u}(a)) \leq e^{\lambda \varepsilon} \left(e^{-\lambda(b-a)} \varphi(y_{x,u}(b)) + e^{\lambda a} \int_a^b e^{-\lambda s} \ell(y_{x,u}, u) ds \right) + L \varepsilon.$$

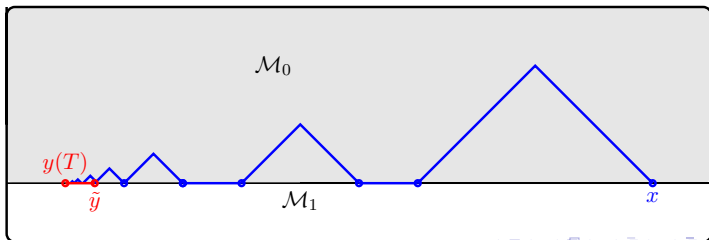
provided $\text{meas}(\{t \in [a, b] \mid y_{x,u}(t) \notin \mathcal{M}_i\}) < \varepsilon$.

Controllability assumption :

Chattering trajectory :



Approximated trajectory :



Characterization of the strong increasing principle

Proposition (CH - Zidani)

Suppose that (SH) with \mathcal{K} stratifiable. Assume that (H_0) and (H_1) hold. Let $\varphi : \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function with $\text{dom } \mathbb{A} \subseteq \text{dom } \varphi$ that satisfies

$$(\star) \quad \lambda\varphi(x) + H_i(x, \zeta) \leq 0 \quad \forall x \in \mathcal{M}_i, \forall \zeta \in \partial_P \varphi_i(x), \forall i \in \mathcal{I}.$$

Then φ is *strongly increasing*.

Recall

(\star) is equivalent to say : $\forall i \in \mathcal{I}, \forall x \in \mathcal{M}_i$ and $\forall g \in \mathcal{C}^1(\mathbb{R}^N)$ such that $\varphi - g$ attains a local minimum at x relative to \mathcal{M}_i

$$\lambda\varphi(x) + H_i(x, \nabla g(x)) \leq 0.$$

Characterization of the value function

Theorem (CH - Zidani)

Suppose (SH) with \mathcal{K} **stratifiable** and $\lambda > \lambda_0$. Assume (H_0) and (H_1) . Then the value function

$$v(x) := \inf \left\{ \int_0^\infty e^{-\lambda t} \ell(y_{x,u}(t), u(t)) dt \mid u \in \mathbb{A}(x) \right\}.$$

is the **unique l.s.c.** function with **linear growth** which is $+\infty$ on $\mathbb{R}^N \setminus \mathcal{K}$ and that satisfies :

$$\begin{aligned} \lambda v(x) + H(x, \zeta) &\geq 0 & \forall x \in \mathcal{K}, \forall \zeta \in \partial_P v(x), \\ \lambda v(x) + H_i(x, \zeta) &\leq 0 & \forall x \in \mathcal{M}_i, \forall \zeta \in \partial_P v_i(x), \forall i \in \mathcal{I}, \end{aligned}$$

$$\text{where } v_i(x) = \begin{cases} v(x) & \text{if } x \in \overline{\mathcal{M}}_i \\ +\infty & \text{otherwise.} \end{cases}$$

Characterization of the value function : $\mathcal{M}_0 = \text{int}(\mathcal{K})$.

Theorem (CH - Zidani)

Suppose (SH) with \mathcal{K} **stratifiable** with $\text{int}(\mathcal{K}) \neq \emptyset$ and $\lambda > \lambda_0$. Assume (H_0) and (H_1) . Then the value function

$$v(x) := \inf \left\{ \int_0^\infty e^{-\lambda t} \ell(y_{x,u}(t), u(t)) dt \mid u \in \mathbb{A}(x) \right\}.$$

is the **unique l.s.c.** function with **linear growth** which is $+\infty$ on $\mathbb{R}^N \setminus \mathcal{K}$ and that satisfies :

$$\begin{aligned} \lambda v(x) + H(x, \zeta) &\geq 0 & \forall x \in \mathcal{K}, \forall \zeta \in \partial_P v(x), \\ \lambda v(x) + H(x, \zeta) &\leq 0 & \forall x \in \text{int}(\mathcal{K}), \forall \zeta \in \partial_P v(x), \\ \lambda v(x) + H_i(x, \zeta) &\leq 0 & \forall x \in \mathcal{M}_i, \forall \zeta \in \partial_P v_i(x), \forall i \in \mathcal{I} \setminus \{0\}, \end{aligned}$$

$$\text{where } v_i(x) = \begin{cases} v(x) & \text{if } x \in \overline{\mathcal{M}}_i \\ +\infty & \text{otherwise.} \end{cases}$$

Applications to Networks

Suppose \mathcal{K} is a network with one junction \mathcal{O} and let $\mathcal{M}_1, \dots, \mathcal{M}_p$ be its branches.

Assume that for each $i \in \{1, \dots, p\}$, $\exists \mathcal{A}_i \subseteq \mathcal{A}$ s.t.

$$(H_3) \quad f(x, \mathcal{A}) \cap \mathcal{T}_{\mathcal{M}_i}(x) = f(x, \mathcal{A}_i), \quad \forall x \in \mathcal{M}_i.$$

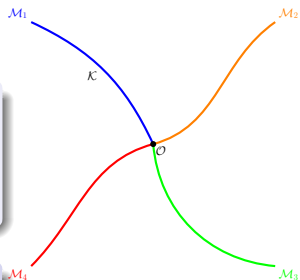
For instance, for some $v_i \in \mathbb{R}^N \setminus \{0\}$

$$\mathcal{M}_i = (0, +\infty)v_i \text{ and } f(x, u) = f_i(x, u)v_i, \quad \forall x \in \mathcal{M}_i$$

with f_i real-valued.

Claim

The hypothesis (H_0) and (H_1) are satisfied.



Theorem (CH - Zidani)

Suppose that (SH) with \mathcal{K} a *network* as before and $\lambda > \lambda_0$. Assume that (H_3) holds and let

$$\mathcal{A}_0 = \{u \in \mathcal{A} \mid f(\mathcal{O}, u) = 0\}.$$

Then the value function is the *unique l.s.c.* function with *linear growth* which is $+\infty$ on $\mathbb{R}^N \setminus \mathcal{K}$ and that satisfies :

$$\lambda v(x) + \max_{u \in \mathcal{A}_i} \{-\langle \zeta, f(x, u) \rangle - \ell(x, u)\} = 0 \quad \forall x \in \mathcal{M}_i, \quad \forall \zeta \in \partial_P v(x),$$

$$\lambda v(\mathcal{O}) + H(\mathcal{O}, \zeta) \geq 0 \quad \forall \zeta \in \partial_P v(\mathcal{O}),$$

$$\lambda v(\mathcal{O}) - \min_{u \in \mathcal{A}_0} \ell(\mathcal{O}, u) \leq 0.$$

Final remarks

- The interior of \mathcal{K} can always be taken as a stratum and so, the constrained Hamilton-Jacobi equation proposed by Soner is included in the set of equations proposed in the main theorem.
- The characterization of the value function neither requires its continuity nor that the state constraint set has empty interior.
- Under the continuity of the value function (on its domain) the controllability assumption can be dropped.
- The characterization for networks can be extended to a suitable notion of generalized network where the junction is replaced by a manifold.

Thanks for your attention !