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Asymptotic models for Hamilton–Jacobi–Bellman equations

A. Siconolfi

Università di Roma *La Sapienza*

partially in collaboration with Nguyen Thuong

Tours, June 2014

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[Our aim](#) is to recover their results adapting Alvarez–Bardi techniques to the nonperiodic case.

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- $f, g : \mathbb{R}^N \times \mathbb{R}^N \times A$ satisfy the usual conditions in order to have, for any initial datum (x_0, y_0) and control α unique solution $(x(t; x_0, y_0, \alpha), y(t; x_0, y_0, \alpha))$ to (CD_ε)

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$$\begin{aligned} x(\cdot) &= x(\cdot, x_0, y_0, \alpha) \\ y(\cdot) &= y(\cdot, x_0, y_0, \alpha) \end{aligned}$$

solutions to (CD_{ϵ}) with initial data (x_0, y_0) .

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The form of the [limit problem](#) is well understood thanks to a series of papers of [Arstein](#) and [Gaitsgory](#) (1998–2005) where they adapt, to describe it, the [Krylov–Bogolyubov](#) theory of invariant measures for flows.

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We define a probability measure on $\mathbb{R}^N \times A$ via

$$\frac{1}{t} \int_0^t \mathbb{I}_E(y(s), \alpha(s)) ds$$

where E is any Borel subset of $\mathbb{R}^N \times A$ and \mathbb{I}_E the corresponding indicator function.

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The **controls of the limit system** (depending on the state) are given by the closure $M(x_0)$ with respect to the **weak convergence** of the previous measures.

These are called **limit occupational** or **holonomic** measures.

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and the limit cost by

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Convergence of the value functions

We write the Hamiltonian

$$H(x, y, p, q) = \max_{a \in A} \{-p \cdot f(x, y, a) - q \cdot g(x, y, a) - \ell(x, y, a)\}$$

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$$\begin{cases} v_t^\varepsilon + H\left(x, y, D_x v^\varepsilon, \frac{D_y v^\varepsilon}{\varepsilon}\right) & = 0 \\ v^\varepsilon(x, y, 0) & = u_0(x) \end{cases} \quad (\text{Hj}_\varepsilon)$$

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Under our assumptions on ℓ , u_0 , the value functions v^ε of (Min_ε) uniquely solve (Hj_ε) .

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We aim at studying the asymptotic behavior v^ε in relation to the limit problem $(\bar{H}j)$.

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- Given x_0 , any pair y_1, y_2 of values the fast variable can be linked (up to a vector of \mathbb{Z}^N) by a trajectory of a fast dynamics, with x_0 fixed, in a time $t \leq T(x_0)$.

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Theorem

If the system satisfies the periodicity and controllability conditions, and, in addition, the limit equation satisfies a comparison principle, then the value functions $v^\varepsilon(x, y, t)$ [locally uniformly converge](#) to the value function $v(x, t)$ of the limit problem $(\overline{H_j})$.

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If the comparison principle does not hold for the limit equation then a weaker convergence result is obtained through [semilimits](#).

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The controllability condition can be viewed as a weak form of coercivity, which is needed for homogenization. There are examples of periodic **noncoercive** Hamiltonians for which **homogenization does not take place**.

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Proposition

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Periodic sub/super solutions of the cell problem play the role of **correctors** allowing to adapt Evans perturbed test function method to this setting.

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We also assume a local version of controllability

- Given x_0 , the previous controllability condition holds in any compact subset, say K of the fast variable and the time to link two points of K , say y_1 and y_2 , depends on x_0 and K .

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We obtain the following result:

Theorem

If the system satisfies the coercivity and local controllability conditions, and, in addition, the limit equation satisfies a comparison principle, then we get the same assertion of Alvarez–Bardi Theorem.

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- If yes, are the sub/supersolution so selected suitable for adapted perturbed test function method ?

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Answer to the other questions are positive, and this is actually the way of proving our convergence theorem.

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- Exploit local controllability to show that there is a **sequence of locally equibounded coercive solutions** u_k to $H_k = c_0^k$
- **pass to the semilimits**, as $k \rightarrow +\infty$ to get sub/supersolution to

$$H(x_0, y, p_0, Du) = c_0(x_0, p_0) =: \lim_k c_k$$

with the desired properties.