



**HAL**  
open science

## Mean field games on graphs

Olivier Guéant

► **To cite this version:**

| Olivier Guéant. Mean field games on graphs. NETCO 2014, 2014, Tours, France. hal-01024605

**HAL Id: hal-01024605**

**<https://inria.hal.science/hal-01024605>**

Submitted on 16 Jul 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Mean field games on graphs

Olivier Guéant  
Université Paris-Diderot.  
Laboratoire Jacques-Louis Lions.

June 2014

# Outline

- 1 Introduction
- 2 MFG on graphs: setup
- 3 Existence result
- 4 Uniqueness result
- 5 Potential games and Master equation

# Outline

- 1 Introduction
- 2 MFG on graphs: setup
- 3 Existence result
- 4 Uniqueness result
- 5 Potential games and Master equation

# Introduction

- Mean field games: introduced in 2005 by Pierre-Louis Lions and Jean-Michel Lasry
- Stochastic games when the number of (identical) players  $\rightarrow \infty$ .
- Mean field approach to model interactions between players/agents: each player considers the distribution of others instead of individual states.

## Usual framework

- Each individual solves a (stochastic) control problem. The objective function depends on the distribution of other players.  
→ HJB equation, backward in time.
- The distribution of players evolves according to players' optimal controls.  
→ transport equation, forward in time.

### MFG PDEs (continuous state space - usual framework)

$$(HJB) \quad \partial_t u + \frac{\sigma^2}{2} \Delta u + H(\nabla u, x, m) = -f(x, m)$$

$$(K) \quad \partial_t m - \frac{\sigma^2}{2} \Delta m + \nabla \cdot (m \partial_p H(\nabla u, x, m)) = 0$$

$$u(T, x) = g(x, m(T, \cdot)), \quad m(0, x) = m_0(x)$$

## Literature

Research can be divided into 3 strands:

# Literature

Research can be divided into 3 strands:

- Mean field game theory:
  - Existence and uniqueness results (Lasry-Lions, ...)
  - Special cases, mainly linear-quadratic MFGs
  - Long time behavior
  - Limit of  $N$ -player games
  - Probabilistic approach (Carmona and Delarue)



# Literature

Research can be divided into 3 strands:

- Mean field game theory:
  - Existence and uniqueness results (Lasry-Lions, ...)
  - Special cases, mainly linear-quadratic MFGs
  - Long time behavior
  - Limit of  $N$ -player games
  - Probabilistic approach (Carmona and Delarue)
- Numerical approximations of MFG PDEs:
  - General finite difference methods (see e.g. Achdou et al.)
  - Specific schemes with (unusual) monotonicity properties in the case of quadratic Hamiltonians
  - Gradient methods in the case of potential games
  - Talks today

# Literature

- Applications:
  - Many toy models (G., Lasry, Lions)
  - Growth models (e.g. Lucas and Moll)  
*Remark: Interesting paper “PDE Models in Macroeconomics”*
  - Competition between oil producers (Chan, Sircar)
  - Population dynamics (Achdou et al.)
  - Maybe applications to machine learning in the future...

Our goal here is to prove existence and (very general) uniqueness results for mean field games on a graph (discrete state space). In particular, we will consider congestion effects.

# Outline

- 1 Introduction
- 2 MFG on graphs: setup
- 3 Existence result
- 4 Uniqueness result
- 5 Potential games and Master equation

# Framework

- Mean Field Games: Continuum of anonymous and identical (indistinguishable) players
- Graph  $\mathcal{G}$ . Nodes indexed by integers from 1 to  $N$ .
- $\forall i \in \mathcal{N} = \{1, \dots, N\}$ :
  - $\mathcal{V}(i) \subset \mathcal{N} \setminus \{i\}$  the set of nodes  $j$  for which a directed edge exists from  $i$  to  $j$  (cardinal:  $d_i$ ).
  - $\mathcal{V}^{-1}(i) \subset \mathcal{N} \setminus \{i\}$  the set of nodes  $j$  for which a directed edge exists from  $j$  to  $i$ .

## Framework (continued)

- Each player's position: Markov chain  $(X_t)_t$  with values in  $\mathcal{G}$
- Distribution of players:  $t \mapsto m(t) = (m(1, t), \dots, m(N, t))$
- Instantaneous transition probabilities at time  $t$ :

$$\lambda_t(i, \cdot) : \mathcal{V}(i) \rightarrow \mathbb{R}_+ \quad (\forall i \in \mathcal{N})$$

- Running payoff:
  - Running costs:  $\mathcal{L}(i, (\lambda_{i,j})_{j \in \mathcal{V}(i)}, m(t))$  to set the value of  $\lambda_t(i, j)$  to  $\lambda_{i,j}$ .
  - Running gain:  $f(i, m(t))$ , where  $i$  is the current state.
- Terminal payoff:  $g(i, m(T))$ .

# Assumptions

- Continuity:  $\forall i \in \mathcal{N}$ , the functions  $\mathcal{L}(i, \cdot, \cdot)$ ,  $f(i, \cdot)$  and  $g(i, \cdot)$  are continuous.
- Convexity of the cost functions:  $\forall i \in \mathcal{N}, \forall \mu \in \mathbb{R}_+^N$ ,  $\mathcal{L}(i, \cdot, \mu)$  is a strictly convex function.
- Asymptotic super-linearity of the cost functions:

$$\forall i \in \mathcal{N}, \forall K > 0, \quad \lim_{\lambda \in \mathbb{R}_+^{d_i}, |\lambda| \rightarrow +\infty} \inf_{\mu \in [0, K]^N} \frac{\mathcal{L}(i, \lambda, \mu)}{|\lambda|} = +\infty.$$

We define the Hamiltonian functions  $\mathcal{H}(i, \cdot, \cdot)$  by:

$$\forall i \in \mathcal{N}, (p, \mu) \in \mathbb{R}^{d_i} \times \mathbb{R}_+^N \mapsto \mathcal{H}(i, p, \mu) = \sup_{\lambda \in \mathbb{R}_+^{d_i}} \lambda \cdot p - \mathcal{L}(i, \lambda, \mu).$$

# Nash-MFG equilibrium

- We denote by  $\mathcal{A}$  the set of admissible controls  $\lambda$  ( $L^\infty$  and deterministic).
- For a given differentiable function  $m : t \in [0, T] \mapsto (m(1, t), \dots, m(N, t)) \in \mathcal{P}_N$  (total mass=1) we define:

$$J_m(i, t, \lambda) = \mathbb{E} \left[ \int_t^T (-\mathcal{L}(X_s, \lambda_s(X_s, \cdot), m(s)) + f(X_s, m(s))) ds + g(X_T, m(T)) \right]$$

for  $(X_s)_{s \in [t, T]}$  a Markov chain on  $\mathcal{G}$ , starting from  $i$  at time  $t$ , with instantaneous transition probabilities given by  $(\lambda_s)_{s \in [t, T]}$ .

# Nash-MFG equilibrium

## Nash-MFG equilibrium

A differentiable function

$m : t \in [0, T] \mapsto (m(1, t), \dots, m(N, t)) \in \mathcal{P}_N$  is said to be a Nash-MFG equilibrium, if there exists an admissible control  $\lambda \in \mathcal{A}$  such that:

$$\forall \tilde{\lambda} \in \mathcal{A}, \forall i \in \mathcal{N}, J_m(i, 0, \lambda) \geq J_m(i, 0, \tilde{\lambda}),$$

and

$$\forall i \in \mathcal{N}, \frac{d}{dt} m(i, t) = \sum_{j \in \mathcal{V}^{-1}(i)} \lambda_t(j, i) m(j, t) - \sum_{j \in \mathcal{V}(i)} \lambda_t(i, j) m(i, t)$$



## $\mathcal{G}$ -MFG equations

The  $\mathcal{G}$ -MFG equations consist in a system of  $2N$  equations:

### $\mathcal{G}$ -MFG equations

$N$  Hamilton-Jacobi equations:

$$\frac{d}{dt}u(i, t) + \mathcal{H}\left(i, (u(j, t) - u(i, t))_{j \in \mathcal{V}(i)}, m(1, t), \dots, m(N, t)\right)$$

$$+ f(i, m(1, t), \dots, m(N, t)) = 0$$

with  $u(i, T) = g(i, m(1, T), \dots, m(N, T))$ .

and...

## $\mathcal{G}$ -MFG equations

### $\mathcal{G}$ -MFG equations (continued)

...  $N$  forward transport equations:

$$\begin{aligned} \frac{d}{dt} m(i, t) = & \sum_{j \in \mathcal{V}^{-1}(i)} \frac{\partial \mathcal{H}(j, \cdot, m(t))}{\partial p_i} \left( (u(k, t) - u(j, t))_{k \in \mathcal{V}(j)} \right) m(j, t) \\ & - \sum_{j \in \mathcal{V}(i)} \frac{\partial \mathcal{H}(i, \cdot, m(t))}{\partial p_j} \left( (u(k, t) - u(i, t))_{k \in \mathcal{V}(i)} \right) m(i, t) \end{aligned}$$

with  $(m(1, 0), \dots, m(N, 0)) = m^0$ .

*Remark: in the literature on MFG, people usually do not care about verification (why?). On graphs, there is no issue as we shall find  $C^1$  solutions.*

# Outline

- 1 Introduction
- 2 MFG on graphs: setup
- 3 Existence result**
- 4 Uniqueness result
- 5 Potential games and Master equation

# Existence

## Theorem (Existence)

*Under the assumptions made above, there exists a  $C^1$  solution  $(u, m)$  of the  $\mathcal{G}$ -MFG equations.*

To prove this result we need a Lemma. This lemma is a comparison principle.

## Comparison principle

### Lemma (Comparison principle)

Let  $m : [0, T] \rightarrow \mathcal{P}_N$  be a continuous function.

Let  $u : [0, T] \rightarrow \mathbb{R}^n$  be a  $C^1$  function that verifies:

$$\forall i, -\frac{d}{dt}u(i, t) - \mathcal{H}\left(i, (u(j, t) - u(i, t))_{j \in \mathcal{V}(i)}, m(t)\right) - f(i, m(t)) \leq 0$$

with  $u(i, T) \leq g(i, m(T))$ .

Let  $v : [0, T] \rightarrow \mathbb{R}^n$  be a  $C^1$  function that verifies:

$$\forall i, -\frac{d}{dt}v(i, t) - \mathcal{H}\left(i, (v(j, t) - v(i, t))_{j \in \mathcal{V}(i)}, m(t)\right) - f(i, m(t)) \geq 0$$

with  $v(i, T) \geq g(i, m(T))$ .

Then,  $\forall i \in \mathcal{N}, \forall t \in [0, T], v(i, t) \geq u(i, t)$ .

## Sketch of proof for the Theorem

- $N$  HJ equations defines a function  $B : m \mapsto u$
- Comparison principle leads a priori bounds on  $u$

$$\sup_{i \in \mathcal{N}} \|u(i, \cdot)\|_{\infty} \leq \sup_{i \in \mathcal{N}} \|g(i, \cdot)\|_{\infty} \\ + T \sup_{i \in \mathcal{N}, \mu \in \mathcal{P}_N} |\mathcal{H}(i, 0, \mu)| + T \sup_{i \in \mathcal{N}, \mu \in \mathcal{P}_N} |f(i, \mu)|.$$

- $N$  transport equations defines a function  $F : u \mapsto m$
- Bounds on  $u$  lead to bounds on  $\frac{dm}{dt}$
- Ascoli + Schauder permit to obtain a fixed point to  $m \mapsto F(B(m))$ .

# Outline

- 1 Introduction
- 2 MFG on graphs: setup
- 3 Existence result
- 4 Uniqueness result**
- 5 Potential games and Master equation

## Classical uniqueness result

It is possible to derive a uniqueness result equivalent to the one presented in Lasry-Lions' initial paper:

### Theorem (Classical Uniqueness)

Assume that  $g$  is such that:

$$\forall (\nu, \mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^N (g(i, \nu) - g(i, \mu))(\nu_i - \mu_i) \geq 0 \implies \nu = \mu.$$

Assume that  $f$  is such that:

$$\forall (\nu, \mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^N (f(i, \nu) - f(i, \mu))(\nu_i - \mu_i) \geq 0 \implies \nu = \mu$$



## Classical uniqueness result

### Theorem (Classical Uniqueness (continued))

Assume that  $\mathcal{H}$  **does not** depend on  $m$ .

Then, if  $(\hat{u}, \hat{m})$  and  $(\tilde{u}, \tilde{m})$  are two  $C^1$  solutions of the  $\mathcal{G}$ -MFG equations, we have  $\hat{m} = \tilde{m}$  and  $\hat{u} = \tilde{u}$ .

In words, this says that if people **do not** want to live together and if the cost to move **does not** depend on the number of people at specific nodes, then uniqueness is ensured.

How can this be extended? By adding  $m$  in the the Hamiltonian functions!

→ Congestion models for instance.

# General uniqueness result

## Theorem (Uniqueness)

Assume that  $g$  is such that:

$$\forall (\nu, \mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^N (g(i, \nu) - g(i, \mu))(\nu_i - \mu_i) \geq 0 \implies \nu = \mu.$$

Assume that  $f$  is such that:

$$\forall (\nu, \mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^N (f(i, \nu) - f(i, \mu))(\nu_i - \mu_i) \geq 0 \implies \nu = \mu$$

Assume that  $\mathcal{H}$  is such that  $\forall i \in \mathcal{N}, \forall j \in \mathcal{V}(i), \frac{\partial \mathcal{H}}{\partial p_j}(i, \cdot, \cdot)$  is a  $C^1$  function on  $\mathbb{R}^{d_i} \times \mathbb{R}_+^N$ .

# General uniqueness result

## Theorem (Uniqueness (continued))

Let us define

$A : (q_1, \dots, q_N, \mu) \in \prod_{i=1}^N \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto (\alpha_{ij}(q_i, \mu))_{i,j} \in \mathcal{M}_N$  by:

$$\alpha_{ij}(q_i, \mu) = -\frac{\partial \mathcal{H}}{\partial \mu_j}(i, q_i, \mu).$$

Let us also define

$\forall i \in \mathcal{N}, B^i : (q_i, \mu) \in \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto (\beta_{jk}^i(q_i, \mu))_{j,k} \in \mathcal{M}_{N,d_i}$  by:

$$\beta_{jk}^i(q_i, \mu) = \frac{\mu_i}{2} \frac{\partial^2 \mathcal{H}}{\partial \mu_j \partial q_{ik}}(i, q_i, \mu).$$

# General uniqueness result

## Theorem (Uniqueness (continued))

Let us also define,  $\forall i \in \mathcal{N}$ ,

$C^i : (q_i, \mu) \in \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto \left( \gamma_{jk}^i(q_i, \mu) \right)_{j,k} \in \mathcal{M}_{d_i, N}$  defined by:

$$\gamma_{jk}^i(q_i, \mu) = \frac{\mu_i}{2} \frac{\partial^2 \mathcal{H}}{\partial \mu_k \partial q_{ij}}(i, q_i, \mu).$$

Let us finally define,  $\forall i \in \mathcal{N}$ ,

$D^i : (q_i, \mu) \in \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto \left( \delta_{jk}^i(q_i, \mu) \right)_{j,k} \in \mathcal{M}_{d_i}$  defined by:

$$\delta_{jk}^i(q_i, \mu) = \mu_i \frac{\partial^2 \mathcal{H}}{\partial q_{ij} \partial q_{ik}}(i, q_i, \mu).$$

# General uniqueness result

## Theorem (Uniqueness (continued))

Assume that  $\forall (q_1, \dots, q_N, \mu) \in \prod_{i=1}^N \mathbb{R}^{d_i} \times \mathcal{P}_N$ :

$$\begin{pmatrix} A(q_1, \dots, q_N, \mu) & B^1(q_1, \mu) & \cdots & \cdots & \cdots & B^N(q_N, \mu) \\ C^1(q_1, \mu) & D^1(q_1, \mu) & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ C^N(q_N, \mu) & 0 & \cdots & \cdots & 0 & D^N(q_N, \mu) \end{pmatrix} \geq 0.$$

Then, if  $(\hat{u}, \hat{m})$  and  $(\tilde{u}, \tilde{m})$  are two  $C^1$  solutions of the  $\mathcal{G}$ -MFG equations, we have  $\hat{m} = \tilde{m}$  and  $\hat{u} = \tilde{u}$ .

# General uniqueness result – Proof

## Sketch of proof:

Classical idea of computing the value of

$$I = \sum_{i=1}^N \int_0^T \frac{d}{dt} ((\hat{u}(i, t) - \tilde{u}(i, t))(\hat{m}(i, t) - \tilde{m}(i, t))) dt$$

in two different ways.

We first know directly that

$$I = \sum_{i=1}^N (g(i, \hat{m}(T)) - g(i, \tilde{m}(T)))(\hat{m}(i, T) - \tilde{m}(i, T)).$$

# General uniqueness result – Proof

We introduce:

- $M(q_1, \dots, q_N, \mu)$  the matrix of the Theorem
- Convex combinations of solutions:  $u^\theta = \theta \hat{u} + (1 - \theta) \tilde{u}$  and  $m^\theta = \theta \hat{m} + (1 - \theta) \tilde{m}$
- A vector  $V$ :

$$V(t) = (\hat{m}(t) - \tilde{m}(t), ((\hat{u}(k, t) - \tilde{u}(k, t)) - (\hat{u}(1, t) - \tilde{u}(1, t))))_{k \in \mathcal{V}(1)}, \dots, ((\hat{u}(k, t) - \tilde{u}(k, t)) - (\hat{u}(N, t) - \tilde{u}(N, t))))_{k \in \mathcal{V}(N)}).$$

## General uniqueness result – Proof

If we differentiate the expression in  $I$  and rearrange the terms (not obvious), we have:

$$I = - \int_0^T \sum_{i=1}^N (f(i, \hat{m}(t)) - f(i, \tilde{m}(t))) (\hat{m}(i, t) - \tilde{m}(i, t)) dt + J,$$

where

$$J = \int_0^T \int_0^1 V(t) M((u^\theta(k, t) - u^\theta(1, t))_{k \in \mathcal{V}(1)}, \dots, (u^\theta(k, t) - u^\theta(N, t))_{k \in \mathcal{V}(N)}, m^\theta(t)) V(t)' d\theta dt.$$

Uniqueness then comes from  $M \geq 0$  and  $f$  and  $g$  decreasing.



# Outline

- 1 Introduction
- 2 MFG on graphs: setup
- 3 Existence result
- 4 Uniqueness result
- 5 Potential games and Master equation**

# Potential games

We consider the special case where:

- The Hamiltonian functions (or equivalently the cost functions) do not depend on  $m$ .
- There are two functions  $F$  and  $G$  such that:  
$$\frac{\partial F}{\partial m_i} = f(i, \cdot) \text{ and } \frac{\partial G}{\partial m_i} = g(i, \cdot)$$

## Potential games

We consider the problem of **one** planner:

$$\sup_{\lambda \in \mathcal{A}} \mathcal{J}(0, m^0, \lambda)$$

where  $\mathcal{J}(t, m^t, \lambda)$

$$= \int_t^T \left( F(m(s)) - \sum_{i=1}^N L(i, (\lambda_s(i, j))_{j \in \mathcal{V}(i)}) m(s, i) \right) ds + G(m(T))$$

where  $\forall i \in \mathcal{N}, m(t, i) = m_i^t$  and  $\forall i \in \mathcal{N}, \forall s \in [t, T]$ ,

$$\frac{d}{ds} m(s, i) = \sum_{j \in \mathcal{V}^{-1}(i)} \lambda_s(j, i) m(s, j) - \sum_{j \in \mathcal{V}(i)} \lambda_s(i, j) m(s, i).$$

## Potential games - HJ equation

The HJ equation associated to this problem is:

$$\frac{\partial \Phi}{\partial t}(t, m_1, \dots, m_N) + \mathcal{H}(m_1, \dots, m_N, \nabla \Phi) + F(m_1, \dots, m_N) = 0$$

Terminal conditions:  $\Phi(T, m_1, \dots, m_N) = G(m_1, \dots, m_N)$ .

The Hamiltonian function is  $\mathcal{H}(m_1, \dots, m_N, p) =$

$$\sup_{(\lambda_{i,j})_{i \in \mathcal{N}_j}} \sum_{i=1}^N \left[ \left( \sum_{j \in \mathcal{V}^{-1}(i)} \lambda_{j,i} m_j - \sum_{j \in \mathcal{V}(i)} \lambda_{i,j} m_i \right) p_i - L(i, (\lambda_{i,j})_{j \in \mathcal{V}(i)}) m_i \right]$$

$$= \sum_{i=1}^N m_i H \left( i, (p_j - p_i)_{j \in \mathcal{V}(i)} \right)$$

# The Master equation

Differentiating formally we obtain for  $U = \nabla\Phi$  the following equations:

$$0 = \frac{\partial U_i}{\partial t} + H\left(i, (U_j - U_i)_{j \in \mathcal{V}(i)}\right) \\ + \sum_{j=1}^N m_j \sum_{k \in \mathcal{V}(j)} \left( \frac{\partial U_i}{\partial m_k} - \frac{\partial U_i}{\partial m_j} \right) \frac{\partial H(j, \cdot)}{\partial p_k} \left( (U_l - U_j)_{l \in \mathcal{V}(j)} \right) + f(i, \cdot) \\ U_i(T, m) = g(i, m)$$

This would be the HJ equation of the initial MFG equation, had we considered  $m$  a state variable. This is the **Master equation!**

# Conclusion

With potential games:

- 1  $HJ$  equations
- $N$  equations (Master equation)
- $2N$  equations for the underlying MFG problem. A solution is:

$$m(0) = m^0, \quad \frac{d}{dt}m(t, i) =$$

$$\sum_{j \in \mathcal{V}^{-1}(i)} \frac{\partial H(j, \cdot)}{\partial p_i} \left( (U_k(t, m(t)) - U_j(t, m(t)))_{k \in \mathcal{V}(j)} \right) m(t, j)$$

$$- \sum_{j \in \mathcal{V}(i)} \frac{\partial H(i, \cdot)}{\partial p_j} \left( (U_k(t, m(t)) - U_i(t, m(t)))_{k \in \mathcal{V}(i)} \right) m(t, i)$$

$$\forall i \in \mathcal{N}, u(t, i) = U_i(t, m(t, 1), \dots, m(t, N))$$

Questions?