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► **To cite this version:**

| Olivier Guéant. Mean field games on graphs. NETCO 2014, 2014, Tours, France. hal-01024605

HAL Id: hal-01024605

<https://inria.hal.science/hal-01024605>

Submitted on 16 Jul 2014

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Mean field games on graphs

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June 2014

Outline

- 1 Introduction
- 2 MFG on graphs: setup
- 3 Existence result
- 4 Uniqueness result
- 5 Potential games and Master equation

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Introduction

- Mean field games: introduced in 2005 by Pierre-Louis Lions and Jean-Michel Lasry
- Stochastic games when the number of (identical) players $\rightarrow \infty$.
- Mean field approach to model interactions between players/agents: each player considers the distribution of others instead of individual states.

Usual framework

- Each individual solves a (stochastic) control problem. The objective function depends on the distribution of other players.
→ HJB equation, backward in time.
- The distribution of players evolves according to players' optimal controls.
→ transport equation, forward in time.

MFG PDEs (continuous state space - usual framework)

$$(HJB) \quad \partial_t u + \frac{\sigma^2}{2} \Delta u + H(\nabla u, x, m) = -f(x, m)$$

$$(K) \quad \partial_t m - \frac{\sigma^2}{2} \Delta m + \nabla \cdot (m \partial_p H(\nabla u, x, m)) = 0$$

$$u(T, x) = g(x, m(T, \cdot)), \quad m(0, x) = m_0(x)$$

Literature

Research can be divided into 3 strands:

Literature

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- Mean field game theory:
 - Existence and uniqueness results (Lasry-Lions, ...)
 - Special cases, mainly linear-quadratic MFGs
 - Long time behavior
 - Limit of N -player games
 - Probabilistic approach (Carmona and Delarue)

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 - Special cases, mainly linear-quadratic MFGs
 - Long time behavior
 - Limit of N -player games
 - Probabilistic approach (Carmona and Delarue)
- Numerical approximations of MFG PDEs:
 - General finite difference methods (see e.g. Achdou et al.)
 - Specific schemes with (unusual) monotonicity properties in the case of quadratic Hamiltonians
 - Gradient methods in the case of potential games
 - Talks today

Literature

- Applications:
 - Many toy models (G., Lasry, Lions)
 - Growth models (e.g. Lucas and Moll)
Remark: Interesting paper “PDE Models in Macroeconomics”
 - Competition between oil producers (Chan, Sircar)
 - Population dynamics (Achdou et al.)
 - Maybe applications to machine learning in the future...

Our goal here is to prove existence and (very general) uniqueness results for mean field games on a graph (discrete state space). In particular, we will consider congestion effects.

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Framework

- Mean Field Games: Continuum of anonymous and identical (indistinguishable) players
- Graph \mathcal{G} . Nodes indexed by integers from 1 to N .
- $\forall i \in \mathcal{N} = \{1, \dots, N\}$:
 - $\mathcal{V}(i) \subset \mathcal{N} \setminus \{i\}$ the set of nodes j for which a directed edge exists from i to j (cardinal: d_i).
 - $\mathcal{V}^{-1}(i) \subset \mathcal{N} \setminus \{i\}$ the set of nodes j for which a directed edge exists from j to i .

Framework (continued)

- Each player's position: Markov chain $(X_t)_t$ with values in \mathcal{G}
- Distribution of players: $t \mapsto m(t) = (m(1, t), \dots, m(N, t))$
- Instantaneous transition probabilities at time t :

$$\lambda_t(i, \cdot) : \mathcal{V}(i) \rightarrow \mathbb{R}_+ \quad (\forall i \in \mathcal{N})$$

- Running payoff:
 - Running costs: $\mathcal{L}(i, (\lambda_{i,j})_{j \in \mathcal{V}(i)}, m(t))$ to set the value of $\lambda_t(i, j)$ to $\lambda_{i,j}$.
 - Running gain: $f(i, m(t))$, where i is the current state.
- Terminal payoff: $g(i, m(T))$.

Assumptions

- Continuity: $\forall i \in \mathcal{N}$, the functions $\mathcal{L}(i, \cdot, \cdot)$, $f(i, \cdot)$ and $g(i, \cdot)$ are continuous.
- Convexity of the cost functions: $\forall i \in \mathcal{N}, \forall \mu \in \mathbb{R}_+^N$, $\mathcal{L}(i, \cdot, \mu)$ is a strictly convex function.
- Asymptotic super-linearity of the cost functions:

$$\forall i \in \mathcal{N}, \forall K > 0, \quad \lim_{\lambda \in \mathbb{R}_+^{d_i}, |\lambda| \rightarrow +\infty} \inf_{\mu \in [0, K]^N} \frac{\mathcal{L}(i, \lambda, \mu)}{|\lambda|} = +\infty.$$

We define the Hamiltonian functions $\mathcal{H}(i, \cdot, \cdot)$ by:

$$\forall i \in \mathcal{N}, (p, \mu) \in \mathbb{R}^{d_i} \times \mathbb{R}_+^N \mapsto \mathcal{H}(i, p, \mu) = \sup_{\lambda \in \mathbb{R}_+^{d_i}} \lambda \cdot p - \mathcal{L}(i, \lambda, \mu).$$

Nash-MFG equilibrium

- We denote by \mathcal{A} the set of admissible controls λ (L^∞ and deterministic).
- For a given differentiable function $m : t \in [0, T] \mapsto (m(1, t), \dots, m(N, t)) \in \mathcal{P}_N$ (total mass=1) we define:

$$J_m(i, t, \lambda) = \mathbb{E} \left[\int_t^T (-\mathcal{L}(X_s, \lambda_s(X_s, \cdot), m(s)) + f(X_s, m(s))) ds + g(X_T, m(T)) \right]$$

for $(X_s)_{s \in [t, T]}$ a Markov chain on \mathcal{G} , starting from i at time t , with instantaneous transition probabilities given by $(\lambda_s)_{s \in [t, T]}$.

Nash-MFG equilibrium

Nash-MFG equilibrium

A differentiable function

$m : t \in [0, T] \mapsto (m(1, t), \dots, m(N, t)) \in \mathcal{P}_N$ is said to be a Nash-MFG equilibrium, if there exists an admissible control $\lambda \in \mathcal{A}$ such that:

$$\forall \tilde{\lambda} \in \mathcal{A}, \forall i \in \mathcal{N}, J_m(i, 0, \lambda) \geq J_m(i, 0, \tilde{\lambda}),$$

and

$$\forall i \in \mathcal{N}, \frac{d}{dt} m(i, t) = \sum_{j \in \mathcal{V}^{-1}(i)} \lambda_t(j, i) m(j, t) - \sum_{j \in \mathcal{V}(i)} \lambda_t(i, j) m(i, t)$$

\mathcal{G} -MFG equations

The \mathcal{G} -MFG equations consist in a system of $2N$ equations:

\mathcal{G} -MFG equations

N Hamilton-Jacobi equations:

$$\frac{d}{dt}u(i, t) + \mathcal{H}\left(i, (u(j, t) - u(i, t))_{j \in \mathcal{V}(i)}, m(1, t), \dots, m(N, t)\right) \\ + f(i, m(1, t), \dots, m(N, t)) = 0$$

with $u(i, T) = g(i, m(1, T), \dots, m(N, T))$.

and...

\mathcal{G} -MFG equations

\mathcal{G} -MFG equations (continued)

... N forward transport equations:

$$\begin{aligned} \frac{d}{dt} m(i, t) = & \sum_{j \in \mathcal{V}^{-1}(i)} \frac{\partial \mathcal{H}(j, \cdot, m(t))}{\partial p_i} \left((u(k, t) - u(j, t))_{k \in \mathcal{V}(j)} \right) m(j, t) \\ & - \sum_{j \in \mathcal{V}(i)} \frac{\partial \mathcal{H}(i, \cdot, m(t))}{\partial p_j} \left((u(k, t) - u(i, t))_{k \in \mathcal{V}(i)} \right) m(i, t) \end{aligned}$$

with $(m(1, 0), \dots, m(N, 0)) = m^0$.

Remark: in the literature on MFG, people usually do not care about verification (why?). On graphs, there is no issue as we shall find C^1 solutions.

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Existence

Theorem (Existence)

Under the assumptions made above, there exists a C^1 solution (u, m) of the \mathcal{G} -MFG equations.

To prove this result we need a Lemma. This lemma is a comparison principle.

Comparison principle

Lemma (Comparison principle)

Let $m : [0, T] \rightarrow \mathcal{P}_N$ be a continuous function.

Let $u : [0, T] \rightarrow \mathbb{R}^n$ be a C^1 function that verifies:

$$\forall i, -\frac{d}{dt}u(i, t) - \mathcal{H}\left(i, (u(j, t) - u(i, t))_{j \in \mathcal{V}(i)}, m(t)\right) - f(i, m(t)) \leq 0$$

with $u(i, T) \leq g(i, m(T))$.

Let $v : [0, T] \rightarrow \mathbb{R}^n$ be a C^1 function that verifies:

$$\forall i, -\frac{d}{dt}v(i, t) - \mathcal{H}\left(i, (v(j, t) - v(i, t))_{j \in \mathcal{V}(i)}, m(t)\right) - f(i, m(t)) \geq 0$$

with $v(i, T) \geq g(i, m(T))$.

Then, $\forall i \in \mathcal{N}, \forall t \in [0, T], v(i, t) \geq u(i, t)$.

Sketch of proof for the Theorem

- N HJ equations defines a function $B : m \mapsto u$
- Comparison principle leads a priori bounds on u

$$\sup_{i \in \mathcal{N}} \|u(i, \cdot)\|_{\infty} \leq \sup_{i \in \mathcal{N}} \|g(i, \cdot)\|_{\infty} \\ + T \sup_{i \in \mathcal{N}, \mu \in \mathcal{P}_N} |\mathcal{H}(i, 0, \mu)| + T \sup_{i \in \mathcal{N}, \mu \in \mathcal{P}_N} |f(i, \mu)|.$$

- N transport equations defines a function $F : u \mapsto m$
- Bounds on u lead to bounds on $\frac{dm}{dt}$
- Ascoli + Schauder permit to obtain a fixed point to $m \mapsto F(B(m))$.

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Classical uniqueness result

It is possible to derive a uniqueness result equivalent to the one presented in Lasry-Lions' initial paper:

Theorem (Classical Uniqueness)

Assume that g is such that:

$$\forall (\nu, \mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^N (g(i, \nu) - g(i, \mu))(\nu_i - \mu_i) \geq 0 \implies \nu = \mu.$$

Assume that f is such that:

$$\forall (\nu, \mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^N (f(i, \nu) - f(i, \mu))(\nu_i - \mu_i) \geq 0 \implies \nu = \mu$$

Classical uniqueness result

Theorem (Classical Uniqueness (continued))

Assume that \mathcal{H} **does not** depend on m .

Then, if (\hat{u}, \hat{m}) and (\tilde{u}, \tilde{m}) are two C^1 solutions of the \mathcal{G} -MFG equations, we have $\hat{m} = \tilde{m}$ and $\hat{u} = \tilde{u}$.

In words, this says that if people **do not** want to live together and if the cost to move **does not** depend on the number of people at specific nodes, then uniqueness is ensured.

How can this be extended? By adding m in the the Hamiltonian functions!

→ Congestion models for instance.

General uniqueness result

Theorem (Uniqueness)

Assume that g is such that:

$$\forall (\nu, \mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^N (g(i, \nu) - g(i, \mu))(\nu_i - \mu_i) \geq 0 \implies \nu = \mu.$$

Assume that f is such that:

$$\forall (\nu, \mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^N (f(i, \nu) - f(i, \mu))(\nu_i - \mu_i) \geq 0 \implies \nu = \mu$$

Assume that \mathcal{H} is such that $\forall i \in \mathcal{N}, \forall j \in \mathcal{V}(i), \frac{\partial \mathcal{H}}{\partial p_j}(i, \cdot, \cdot)$ is a C^1 function on $\mathbb{R}^{d_i} \times \mathbb{R}_+^N$.

General uniqueness result

Theorem (Uniqueness (continued))

Let us define

$A : (q_1, \dots, q_N, \mu) \in \prod_{i=1}^N \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto (\alpha_{ij}(q_i, \mu))_{i,j} \in \mathcal{M}_N$ by:

$$\alpha_{ij}(q_i, \mu) = -\frac{\partial \mathcal{H}}{\partial \mu_j}(i, q_i, \mu).$$

Let us also define

$\forall i \in \mathcal{N}, B^i : (q_i, \mu) \in \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto (\beta_{jk}^i(q_i, \mu))_{j,k} \in \mathcal{M}_{N,d_i}$ by:

$$\beta_{jk}^i(q_i, \mu) = \frac{\mu_i}{2} \frac{\partial^2 \mathcal{H}}{\partial \mu_j \partial q_{ik}}(i, q_i, \mu).$$

General uniqueness result

Theorem (Uniqueness (continued))

Let us also define, $\forall i \in \mathcal{N}$,

$C^i : (q_i, \mu) \in \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto \left(\gamma_{jk}^i(q_i, \mu) \right)_{j,k} \in \mathcal{M}_{d_i, N}$ defined by:

$$\gamma_{jk}^i(q_i, \mu) = \frac{\mu_i}{2} \frac{\partial^2 \mathcal{H}}{\partial \mu_k \partial q_{ij}}(i, q_i, \mu).$$

Let us finally define, $\forall i \in \mathcal{N}$,

$D^i : (q_i, \mu) \in \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto \left(\delta_{jk}^i(q_i, \mu) \right)_{j,k} \in \mathcal{M}_{d_i}$ defined by:

$$\delta_{jk}^i(q_i, \mu) = \mu_i \frac{\partial^2 \mathcal{H}}{\partial q_{ij} \partial q_{ik}}(i, q_i, \mu).$$

General uniqueness result

Theorem (Uniqueness (continued))

Assume that $\forall (q_1, \dots, q_N, \mu) \in \prod_{i=1}^N \mathbb{R}^{d_i} \times \mathcal{P}_N$:

$$\begin{pmatrix} A(q_1, \dots, q_N, \mu) & B^1(q_1, \mu) & \cdots & \cdots & \cdots & B^N(q_N, \mu) \\ C^1(q_1, \mu) & D^1(q_1, \mu) & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ C^N(q_N, \mu) & 0 & \cdots & \cdots & 0 & D^N(q_N, \mu) \end{pmatrix} \geq 0.$$

Then, if (\hat{u}, \hat{m}) and (\tilde{u}, \tilde{m}) are two C^1 solutions of the \mathcal{G} -MFG equations, we have $\hat{m} = \tilde{m}$ and $\hat{u} = \tilde{u}$.

General uniqueness result – Proof

Sketch of proof:

Classical idea of computing the value of

$$I = \sum_{i=1}^N \int_0^T \frac{d}{dt} ((\hat{u}(i, t) - \tilde{u}(i, t))(\hat{m}(i, t) - \tilde{m}(i, t))) dt$$

in two different ways.

We first know directly that

$$I = \sum_{i=1}^N (g(i, \hat{m}(T)) - g(i, \tilde{m}(T)))(\hat{m}(i, T) - \tilde{m}(i, T)).$$

General uniqueness result – Proof

We introduce:

- $M(q_1, \dots, q_N, \mu)$ the matrix of the Theorem
- Convex combinations of solutions: $u^\theta = \theta \hat{u} + (1 - \theta) \tilde{u}$ and $m^\theta = \theta \hat{m} + (1 - \theta) \tilde{m}$
- A vector V :

$$V(t) = (\hat{m}(t) - \tilde{m}(t), ((\hat{u}(k, t) - \tilde{u}(k, t)) - (\hat{u}(1, t) - \tilde{u}(1, t))))_{k \in \mathcal{V}(1)}, \dots, ((\hat{u}(k, t) - \tilde{u}(k, t)) - (\hat{u}(N, t) - \tilde{u}(N, t))))_{k \in \mathcal{V}(N)}).$$

General uniqueness result – Proof

If we differentiate the expression in I and rearrange the terms (not obvious), we have:

$$I = - \int_0^T \sum_{i=1}^N (f(i, \hat{m}(t)) - f(i, \tilde{m}(t))) (\hat{m}(i, t) - \tilde{m}(i, t)) dt + J,$$

where

$$J = \int_0^T \int_0^1 V(t) M((u^\theta(k, t) - u^\theta(1, t))_{k \in \mathcal{V}(1)}, \dots, (u^\theta(k, t) - u^\theta(N, t))_{k \in \mathcal{V}(N)}, m^\theta(t)) V(t)' d\theta dt.$$

Uniqueness then comes from $M \geq 0$ and f and g decreasing.

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Potential games

We consider the special case where:

- The Hamiltonian functions (or equivalently the cost functions) do not depend on m .
- There are two functions F and G such that:
$$\frac{\partial F}{\partial m_i} = f(i, \cdot) \text{ and } \frac{\partial G}{\partial m_i} = g(i, \cdot)$$

Potential games

We consider the problem of **one** planner:

$$\sup_{\lambda \in \mathcal{A}} \mathcal{J}(0, m^0, \lambda)$$

where $\mathcal{J}(t, m^t, \lambda)$

$$= \int_t^T \left(F(m(s)) - \sum_{i=1}^N L(i, (\lambda_s(i, j))_{j \in \mathcal{V}(i)}) m(s, i) \right) ds + G(m(T))$$

where $\forall i \in \mathcal{N}, m(t, i) = m_i^t$ and $\forall i \in \mathcal{N}, \forall s \in [t, T]$,

$$\frac{d}{ds} m(s, i) = \sum_{j \in \mathcal{V}^{-1}(i)} \lambda_s(j, i) m(s, j) - \sum_{j \in \mathcal{V}(i)} \lambda_s(i, j) m(s, i).$$

Potential games - HJ equation

The HJ equation associated to this problem is:

$$\frac{\partial \Phi}{\partial t}(t, m_1, \dots, m_N) + \mathcal{H}(m_1, \dots, m_N, \nabla \Phi) + F(m_1, \dots, m_N) = 0$$

Terminal conditions: $\Phi(T, m_1, \dots, m_N) = G(m_1, \dots, m_N)$.

The Hamiltonian function is $\mathcal{H}(m_1, \dots, m_N, p) =$

$$\begin{aligned} \sup_{(\lambda_{i,j})_{i \in \mathcal{N}, j}} \sum_{i=1}^N \left[\left(\sum_{j \in \mathcal{V}^{-1}(i)} \lambda_{j,i} m_j - \sum_{j \in \mathcal{V}(i)} \lambda_{i,j} m_i \right) p_i - L(i, (\lambda_{i,j})_{j \in \mathcal{V}(i)}) m_i \right] \\ = \sum_{i=1}^N m_i H \left(i, (p_j - p_i)_{j \in \mathcal{V}(i)} \right) \end{aligned}$$

The Master equation

Differentiating formally we obtain for $U = \nabla\Phi$ the following equations:

$$0 = \frac{\partial U_i}{\partial t} + H\left(i, (U_j - U_i)_{j \in \mathcal{V}(i)}\right) \\ + \sum_{j=1}^N m_j \sum_{k \in \mathcal{V}(j)} \left(\frac{\partial U_i}{\partial m_k} - \frac{\partial U_i}{\partial m_j} \right) \frac{\partial H(j, \cdot)}{\partial p_k} \left((U_l - U_j)_{l \in \mathcal{V}(j)} \right) + f(i, \cdot) \\ U_i(T, m) = g(i, m)$$

This would be the HJ equation of the initial MFG equation, had we considered m a state variable. This is the **Master equation**!

Conclusion

With potential games:

- 1 HJ equations
- N equations (Master equation)
- $2N$ equations for the underlying MFG problem. A solution is:

$$m(0) = m^0, \quad \frac{d}{dt} m(t, i) =$$

$$\sum_{j \in \mathcal{V}^{-1}(i)} \frac{\partial H(j, \cdot)}{\partial p_i} \left((U_k(t, m(t)) - U_j(t, m(t)))_{k \in \mathcal{V}(j)} \right) m(t, j)$$

$$- \sum_{j \in \mathcal{V}(i)} \frac{\partial H(i, \cdot)}{\partial p_j} \left((U_k(t, m(t)) - U_i(t, m(t)))_{k \in \mathcal{V}(i)} \right) m(t, i)$$

$$\forall i \in \mathcal{N}, u(t, i) = U_i(t, m(t, 1), \dots, m(t, N))$$

Questions?