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► **To cite this version:**

Roberta Ghezzi, Marco Caponigro, Benedetto Piccoli, Emmanuel Trélat. Regularization of chattering phenomena via bounded variation controls. NETCO 2014, 2014, Tours, France. hal-01024604

HAL Id: hal-01024604

<https://inria.hal.science/hal-01024604>

Submitted on 16 Jul 2014

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Regularization of chattering phenomena via bounded variation controls

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joint work with
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NetCo 2014
Tours, June 23rd 2014

Outline

- 1 Motivation
 - what is chattering?
 - how often does it occur?
- 2 Convergence results
 - convergence for the perturbed problem
 - rate of convergence and switching times
- 3 Open problems
 - rate of convergence for the perturbed problem
 - generic rate of convergence

Fuller's Problem [Fuller 1961]

Control System

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u, \end{cases} \quad |u| \leq 1$$

Cost

$$\begin{aligned} &\text{Minimize} && \int_0^{t_f} x_1^2(s) ds \\ &x(0) = \bar{x} \\ &x(t_f) = 0 \end{aligned}$$

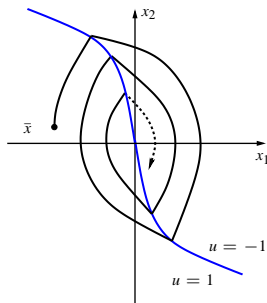
Fuller's Problem [Fuller 1961]

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- optimal controls are bang-bang with *infinitely many* switchings
- switchings take place on the curve $\{x_1 + C|x_2|x_2 = 0\}$, $C \sim 0.44$
- time intervals between consecutive switches decrease in a *geometric progression*

Examples of chattering

Control System

$$\dot{x} = f(x, u) \quad u \in \mathcal{U}$$

Cost

$$\begin{array}{l} \text{Minimize} \\ x(0) = \bar{x} \\ x(t_f) = 0 \end{array} \int_0^{t_f} L(s, x, u) ds$$

Examples of chattering: state constraints

Control System

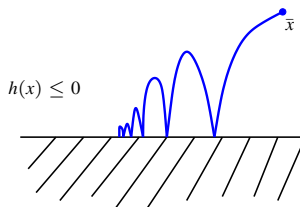
$$\dot{x} = f(x, u) \quad u \in \mathcal{U}$$

state constraints \implies

Cost

$$\begin{aligned} \text{Minimize} \quad & \int_0^{t_f} L(s, x, u) ds \\ & x(0) = \bar{x} \\ & x(t_f) = 0 \\ & h(x(s)) \leq 0 \end{aligned}$$

accumulation of contact points with the constraint's boundary



Examples of chattering

Control System

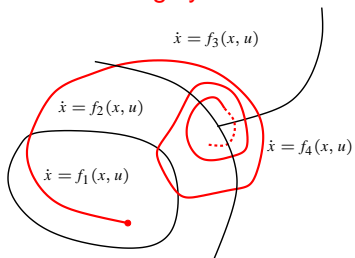
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Cost

$$\begin{array}{l} \text{Minimize} \\ x(0) = \bar{x} \\ x(t_f) = 0 \end{array} \int_0^{t_f} L(s, x, u) ds$$

Examples of chattering: switching systems

switching systems



Cost

$$\text{Minimize} \int_0^{t_f} L(s, x, u) ds$$

$$x(0) = \bar{x}$$

$$x(t_f) = 0$$

accumulation of switchings between different locations

Occurring of chattering

Optimal control problem

$$\dot{x} = f_0(x) + u f_1(x) \quad u \in \mathcal{U} \quad \text{Minimize} \quad \int_0^{t_f} L_0(x) + u L_1(x) ds$$

$$x(0) = \bar{x} \quad x(t_f) = 0$$

- $f_0, f_1 \in \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}^n)$
- $\mathcal{U} = \{v(\cdot) \text{ measurable } v(t) \in [-1, 1]\}$
- $L_0, L_1 \in \mathcal{C}^\infty(\mathbb{R}^n)$

Ubiquity of Fuller's phenomenon [Kupka 1990]

In sufficiently high state dimension, for single-input control affine problems with control affine cost solutions are *generically* chattering.

Regularization method

OCP

$$\left\{ \begin{array}{l} \text{Minimize } \int L(s, x, u) ds \\ x(0) = \bar{x} \\ x(t_f) = 0 \\ \dot{x} = f(x, u), \quad u \in \mathcal{U} \end{array} \right.$$

OCP_ε

$$\left\{ \begin{array}{l} \text{Minimize } \int L(s, x, u) ds + \varepsilon TV(u) \\ x(0) = \bar{x} \\ x(t_f) = 0 \\ \dot{x} = f(x, u), \quad u \in \mathcal{U} \end{array} \right.$$

$$- \varepsilon > 0$$

$$- TV(u) = \sup \sum_{i=1}^n |u(t_{i+1}) - u(t_i)|$$

- problem OCP_ε admits a solution u_ε which is “regular” ($TV(u_\varepsilon) < \infty$)
- convergence $\int L(s, x_\varepsilon, u_\varepsilon) \rightarrow \int L(s, x^*, u^*)$ as $\varepsilon \rightarrow 0$, where u^* is a chattering solution of OCP

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OCP $_{\varepsilon}$

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General setting and assumptions I

Optimal control problem

$$\dot{x} = f(x, u), \quad u \in \mathcal{U}$$

$$\text{Minimize}_{\substack{x(0) = \bar{x} \\ x(t_f) = 0}} \int_0^{t_f} L(s, x(s), u(s)) ds$$

where

- $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$, $f(0, 0) = 0$
- $\mathcal{U} = \{u(\cdot) \text{ measurable}, u(t) \in \mathbf{U}\}$, for a given subset $\mathbf{U} \subset \mathbb{R}^m$, $0 \in \overset{\circ}{\mathbf{U}}$
- $L \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m)$

General setting and assumptions II

Assumptions

Controllability

1. $\text{Lie}_0\{f(u, \cdot) \mid u \in \mathbf{U}\} = \mathbb{R}^n$
2. 0 is small time locally controllable

Weak convexity

3. for every $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ the set

$$\{(f(x, u), L(t, x, u) + \gamma) \mid u \in \mathbf{U}, \gamma \geq 0\} \text{ is convex}$$

Compactness

4. \mathbf{U} is compact
5. trajectories associated with admissible controls are uniformly bounded

Convergence

Theorem

Assume conditions 1-5 and assume that OCP has a solution (u^*, x^*) . Then, for every $\varepsilon > 0$, OCP_ε has a solution $(u_\varepsilon, x_\varepsilon)$ and

$$\lim_{\varepsilon \rightarrow 0} \int_0^{t_\varepsilon} L(s, x_\varepsilon(s), u_\varepsilon(s)) ds = \int_0^{t^*} L(s, x^*(s), u^*(s)) ds.$$

Moreover,

- $t_\varepsilon \rightarrow t^*$,
- $x_\varepsilon \rightarrow x^*$ uniformly.

Idea of the proof

Step 1.

controllability
 compactness of $\{u \in \mathcal{U} \mid TV(u) \leq 1\}$
 lower s.c. of $TV(\cdot)$

$$\left. \vphantom{\begin{array}{l} \text{controllability} \\ \text{compactness of } \{u \in \mathcal{U} \mid TV(u) \leq 1\} \\ \text{lower s.c. of } TV(\cdot) \end{array}} \right\} \Rightarrow \text{existence for OCP}_\varepsilon$$

Step 2.

convexity
 compactness

$$\left. \vphantom{\begin{array}{l} \text{convexity} \\ \text{compactness} \end{array}} \right\} \Rightarrow \text{existence of } w \in \mathcal{U} \text{ and } \gamma \in L^1, \gamma \geq 0 \text{ such that}$$

$$(f(x_\varepsilon, u_\varepsilon), L(\cdot, x_\varepsilon, u_\varepsilon)) \xrightarrow[w^* - L^\infty]{} (f(x_w, w), L(\cdot, x_w, w) + \gamma)$$

Step 3.

u_ε is optimal for OCP_ε
 density of bv controls

$$\left. \vphantom{\begin{array}{l} u_\varepsilon \text{ is optimal for } \text{OCP}_\varepsilon \\ \text{density of bv controls} \end{array}} \right\} \Rightarrow \begin{array}{l} w \text{ is optimal for OCP,} \\ \text{i.e., } \int L(s, x^*, u^*) = \int L(s, x_w, w) \end{array}$$

Step 4.

w is optimal for OCP
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$$\left. \vphantom{\begin{array}{l} w \text{ is optimal for OCP} \\ \text{density of bv controls} \end{array}} \right\} \Rightarrow \begin{array}{l} \text{convergence} \\ \int L(s, x_\varepsilon, u_\varepsilon) \text{ to } \int L(s, x^*, u^*) \end{array}$$

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Rate of convergence

Proposition

Assume that

- conditions 1, 2 (controllability), and 4,5 (compactness) are satisfied;
- OCP has a solution (u^*, x^*) and there exists $t_n \uparrow T^*$ such that $TV(u^*|_{[0, t_n]}) < \infty$;
- the time-optimal map is $C^{0, \alpha}$ for some $\alpha \in (0, 1]$ on a neighborhood of 0.

Then there exist a sequence $v_n : [0, T_n] \rightarrow \mathbf{U}$ of admissible controls such that

- $TV(v_n) < \infty$,
- $v_n \rightarrow u^*$ strongly in L^1 , and

$$\int_0^{T_n} L(t, x_n, v_n) dt - \int_0^{T^*} L(t, x^*, u^*) dt \leq C(T^* - t_n)^\alpha.$$

Remarks

Theorem	Proposition
<ul style="list-style-type: none">▶ general setting▶ computable controls u_ε▶ convergence only	<ul style="list-style-type: none">▶ general setting▶ controls v_n given by truncations of u^*▶ convergence + rate in terms of switching times

[Zelikin 1999] - [Manita 1996]

- specific OCP (small perturbation of Fuller's dynamics in higher dimension with Fuller's cost)
- controls v_n given by truncations of u^*
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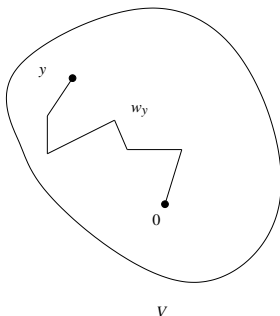
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Open problems I

1. Rate of convergence for the regularized problem OCP_ε

\Rightarrow uniform small time normal reachability property



Controllability assumptions \Rightarrow

- w_y is piecewise constant with **at most q switchings**,
- the time τ_y to steer y to 0 satisfies **$\tau_y \leq C|y|$** ,

where **q, C do not depend on y** ?

Open problems II

2. Generic rate of convergence

- Fuller's problem: switching times converge *geometrically*
- Kupka 1990 \Rightarrow exponential rate of convergence is "stable"

Can we expect lower rates of convergence? What rates are stable?