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# Homogenization Results for a Deterministic Multi-domains Periodic Control Problem

TOURS-HJnet-Netco

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June 23, 2014

# References-1

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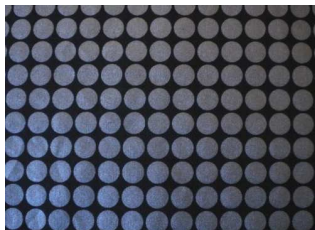
- 1 Hamilton-Jacobi equations and optimal control problem
  - Setting
  - Assumptions
  - The optimal control problems
  
- 2 The homogenization result for  $U_\varepsilon^-$ 
  - The cell problem and the definition of the effective Hamiltonian.
  - The homogenization result
  
- 3 The homogenization result for  $U_\varepsilon^+$ 
  - The cell problem and the definition of the effective Hamiltonian.
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- 4 The 1-D case: an example when  $\bar{H}^+(x, 0) \neq \bar{H}^-(x, 0)$

# Plan

- 1** Hamilton-Jacobi equations and optimal control problem
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# Geometry

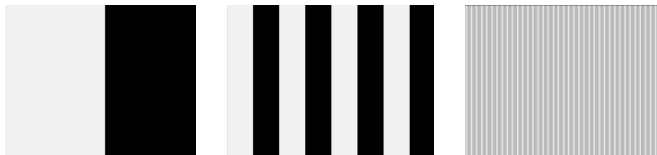
- $\mathbb{R}^N = \Omega_1 \cup \Omega_2 \cup \mathcal{H}$
- $\Omega_1, \Omega_2$  are open subsets of  $\mathbb{R}^N$
- $\Omega_1 \cap \Omega_2 = \emptyset$
- $\mathcal{H} = \partial\Omega_1 = \partial\Omega_2$  is regular
- $\Omega_i$  are  $\mathbb{Z}^N$ -periodic, i.e.  $x \in \Omega_i \Leftrightarrow x + z \in \Omega_i, \forall z \in \mathbb{Z}^N$ .

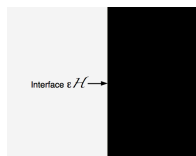


# Hamilton-Jacobi equations in $\varepsilon\Omega_i$

$$\begin{aligned} \lambda u_\varepsilon(x) + H_1\left(x, \frac{x}{\varepsilon}, Du_\varepsilon(x)\right) &= 0 & \text{in } \varepsilon\Omega_1, \\ \lambda u_\varepsilon(x) + H_2\left(x, \frac{x}{\varepsilon}, Du_\varepsilon(x)\right) &= 0 & \text{in } \varepsilon\Omega_2, \end{aligned} \tag{1.1}$$

- $\varepsilon \searrow 0$
- $\lambda > 0$
- $H_i(x, y, p) := \sup_{\alpha_i \in A_i} \{-b_i(x, y, \alpha_i) \cdot p - l_i(x, y, \alpha_i)\}$   
 $x \in \mathbb{R}^N, y \in \bar{\Omega}_i, p \in \mathbb{R}^N.$



Hamilton-Jacobi equations on  $\varepsilon\mathcal{H}$  : Ishii conditions

$$\min\{\lambda u_\varepsilon(x) + H_1(x, \frac{x}{\varepsilon}, Du_\varepsilon(x)), \lambda u_\varepsilon(x) + H_2(x, \frac{x}{\varepsilon}, Du_\varepsilon(x))\} \leq 0 \quad \text{on } \varepsilon\mathcal{H},$$

$$\max\{\lambda u_\varepsilon(x) + H_1(x, \frac{x}{\varepsilon}, Du_\varepsilon(x)), \lambda u_\varepsilon(x) + H_2(x, \frac{x}{\varepsilon}, Du_\varepsilon(x))\} \geq 0 \quad \text{on } \varepsilon\mathcal{H},$$
(1.2)

- (1.1)-(1.2) is **NOT** a well-posed problem
- $U_\varepsilon^+ :=$  maximal sub-solution
- $U_\varepsilon^- :=$  minimal super-solution



# Assumptions

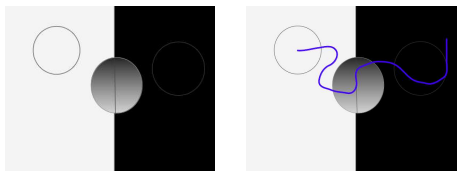
- Standard assumptions on costs and dynamics in  $\Omega_i$
- Convexity assumption:

$$\cup_{\alpha_i \in A_i} (b_i(x, y, \alpha_i), l_i(x, y, \alpha_i)) \text{ is convex}$$

- Strong controllability assumption:

$$\mathbf{B}_i(x, y) := \{b_i(x, y, \alpha_i) : \alpha_i \in A_i\} \supset \{|z| \leq \delta\}, \quad \delta > 0$$

# Optimal control characterization of extremal Ishii-solutions : Trajectories



$$\dot{X}_{x_0}^\varepsilon(t) \in \mathcal{B}\left(X_{x_0}^\varepsilon(t), \frac{X_{x_0}^\varepsilon(t)}{\varepsilon}\right) \quad \text{a.e. } t \in (0, +\infty); \quad X_{x_0}^\varepsilon(0) = x_0$$

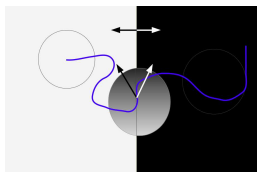
$$\mathcal{B}(x, y) := \begin{cases} \mathbf{B}_1(x, y) & \text{if } y \in \Omega_1 \\ \mathbf{B}_2(x, y) & \text{if } y \in \Omega_2 \\ \overline{\text{co}}(\mathbf{B}_1(x, y) \cup \mathbf{B}_2(x, y)) & \text{if } y \in \mathcal{H} \end{cases}$$

Dynamics on the interface:  $y \in \mathcal{H}$ ,  $a = (\alpha_1, \alpha_2, \mu)$   $\alpha_i \in A_i$ ,  $\mu \in [0, 1]$

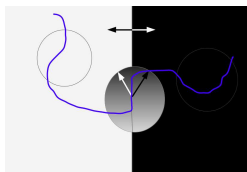
$$b_{\mathcal{H}}(x, y, a) := \mu b_1(x, y, \alpha_1) + (1 - \mu) b_2(x, y, \alpha_2)$$

# Optimal control characterization of extremal Ishii-solutions : regular and singular trajectories

$$b_{\mathcal{H}}(x, y, a) := \mu b_1(x, y, \alpha_1) + (1 - \mu) b_2(x, y, \alpha_2)$$



REGULAR



SINGULAR

- tangential :  $b_{\mathcal{H}}(x, y, a) \cdot \mathbf{n}_i(y) = 0$
  - **REGULAR**: tangential and  $b_i(x, y, \alpha_i) \cdot \mathbf{n}_i(y) \geq 0 \quad \forall i = 1, 2$
  - **SINGULAR**: tangential and  $b_i(x, y, \alpha_i) \cdot \mathbf{n}_i(y) \leq 0 \quad \forall i = 1, 2$
- $\mathbf{n}_i(y)$  unitary normal exterior vector to  $\Omega_i$  in  $y$

# Control characterization of extremal Ishii solutions: tangential Hamiltonians

Dynamics and cost at the interface  $y \in \mathcal{H}$ :

- $b_{\mathcal{H}}(x, y, a) := \mu b_1(x, y, \alpha_1) + (1 - \mu) b_2(x, y, \alpha_2)$
- $l_{\mathcal{H}}(x, y, a) := \mu l_1(x, y, \alpha_1) + (1 - \mu) l_2(x, y, \alpha_2)$

Tangential Hamiltonians:

- $H_T(x, y, p) := \sup_{a \in A_0(x, y)} \{- \langle b_{\mathcal{H}}(x, y, a), p \rangle - l_{\mathcal{H}}(x, y, a)\}$
  - $H_T^{\text{reg}}(x, y, p) := \sup_{a \in A_0^{\text{reg}}(x, y)} \{- \langle b_{\mathcal{H}}(x, y, a), p \rangle - l_{\mathcal{H}}(x, y, a)\}$
- $$H_T(x, y, p) \geq H_T^{\text{reg}}(x, y, p)$$

## Proposition

The tangential Hamiltonians are as regular as the data

# Infinite horizon optimal control problems

$$\text{Cost} = J(x_0; (X_{x_0}^\varepsilon, a)) := \int_0^{+\infty} l(X_{x_0}^\varepsilon(t), \frac{X_{x_0}^\varepsilon}{\varepsilon}(t), a(t)) e^{-\lambda t} dt$$

$$\begin{aligned} l(X_{x_0}^\varepsilon(t), \frac{X_{x_0}^\varepsilon}{\varepsilon}(t), a(t)) &:= l_1(X_{x_0}^\varepsilon(t), \frac{X_{x_0}^\varepsilon}{\varepsilon}(t), \alpha_1(t)) \mathbf{1}_{\mathcal{E}_1}(t) \\ &+ l_2(X_{x_0}^\varepsilon(t), \frac{X_{x_0}^\varepsilon}{\varepsilon}(t), \alpha_2(t)) \mathbf{1}_{\mathcal{E}_2}(t) \\ &+ l_{\mathcal{H}}(X_{x_0}^\varepsilon(t), \frac{X_{x_0}^\varepsilon}{\varepsilon}(t), a(t)) \mathbf{1}_{\mathcal{E}_{\mathcal{H}}}(t) \end{aligned}$$

Value functions:

$$V_\varepsilon^-(x_0) := \inf_{(X_{x_0}^\varepsilon, a) \in \mathcal{T}_{x_0}^\varepsilon} J(x_0; (X_{x_0}^\varepsilon, a))$$

$$V_\varepsilon^+(x_0) := \inf_{(X_{x_0}^\varepsilon, a) \in \mathcal{T}_{x_0}^{\text{reg}, \varepsilon}} J(x_0; (X_{x_0}^\varepsilon, a))$$

$$V_\varepsilon^-(x_0) \leq V_\varepsilon^+(x_0)$$

# Known facts: BBC 2013-2014

## Theorem

$V_\varepsilon^-$  and  $V_\varepsilon^+$  are viscosity solutions of the Hamilton-Jacobi-Bellman equations (1.1)-(1.2).

$V_\varepsilon^-$  is a subsolution of  $\lambda v(x) + H_T(x, \frac{x}{\varepsilon}, D_{\mathcal{H}}v(x)) \leq 0$  for  $x \in \varepsilon\mathcal{H}$ .

## Theorem

- $V_\varepsilon^- = U_\varepsilon^-$
- $V_\varepsilon^+ = U_\varepsilon^+$
- global (on  $\mathbb{R}^N$ ) (and local) strong comparison principle for
  - Lipschitz continuous subsolutions satisfying  $\lambda v(x) + H_T(x, \frac{x}{\varepsilon}, D_{\mathcal{H}}v(x)) \leq 0$
  - lsc supersolution

## Notation

$\lambda v(\cdot) + \mathbf{H}^-(\cdot, \cdot, Dv(\cdot)) = 0 \Leftrightarrow (1.1)-(1.2)$  *and*  $\lambda v(\cdot) + H_T(\cdot, \cdot, D_{\mathcal{H}}v(\cdot)) \leq 0$

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- └ The homogenization result for  $U_\varepsilon^-$
- └ The cell problem and the definition of the effective Hamiltonian.

### Theorem (The cell problem for $U_\varepsilon^-$ )

$\forall x \in \mathbb{R}^N, \forall p \in \mathbb{R}^N$  (frozen)  $\exists!$   $C := \bar{H}^-(x, p)$  such that

$$\mathbf{H}^-(x, y, Dv(y) + p) = C$$

has a Lipschitz continuous,  $\mathbb{Z}^N$ -periodic viscosity solution  $V^-$ .

### Proof.

Classical, uses a **stability result** for the existence and the **comparison result** for  $\mathbf{H}^-$  for the uniqueness ( $\rho$ -**problem**)... □

### Proposition

$\bar{H}^-(x, p)$  is regular (Lipschitz and coercive) and a comparison principle for the effective problem holds true.



- └ The homogenization result for  $U_\varepsilon^-$
- └ The homogenization result

### Theorem (Main theorem for $U_\varepsilon^-$ )

$(U_\varepsilon^-)_{\varepsilon>0}$  converges locally uniformly to  $U^-$  the unique solution of

$$\lambda U^-(x) + \bar{H}^-(x, DU^-) = 0$$

**Proof** The classical methods of Lions-Papanicolaou-Varadhan and Evans-perturbed test function apply (the **local comparison principle** for  $H^-$  is an essential tool).  $\square$

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- ↳ The homogenization result for  $U_\varepsilon^+$
- ↳ The cell problem and the definition of the effective Hamiltonian.

## Theorem (The cell problem for $U_\varepsilon^+$ )

$\forall x, p \in \mathbb{R}^N$  (frozen)  $\exists!$   $\bar{H}^+(x, p) \in \mathbb{R}$  such that  $\exists V^+$  Lipschitz continuous, periodic function  $\forall \tau \geq 0, y_0 \in \mathbb{R}^N$

$$(*) V^+(y_0) = \inf_{(Y_{y_0}, a) \in \mathcal{T}_{y_0}^{\text{reg}}} \left\{ \int_0^\tau (l(x, Y_{y_0}(t), a(t)) + b(x, Y_{y_0}(t), a(t)) \cdot p + \bar{H}^+(x, p)) dt + V^+(Y_{y_0}(\tau)) \right\}$$

## Proof.

**Dynamic Programming Principle** for the existence and uniqueness follows from the definition (\*) and regularity of  $V^+$  □

## Proposition

$\bar{H}^+(x, p)$  is regular (Lipschitz and coercive) and a comparison principle for the effective problem holds true.

Proof. Characterisation of the solution of the approximate cell problem as **maximal subsolution**. □

- └ The homogenization result for  $U_\varepsilon^+$

- └ The homogenization result

## Theorem (Main theorem for $U_\varepsilon^+$ )

$(U_\varepsilon^+)_{\varepsilon>0} \rightarrow U^+$  unique viscosity solution of

$$\lambda u(x) + \bar{H}^+(x, Du(x)) = 0 \text{ in } \mathbb{R}^N.$$

### Proof.

$U^+$  is a supersolution:

EDP proof as in the  $U^-$  case thanks to a local property of maximality.

$U^+$  is a subsolution:

a comparison principle concerning supersolutions and  $U_\varepsilon^+$  does not hold

- First case:  $b_i(x, y, a) = b_i(y, a)$ ,  $l_i(x, y, a) = l_i(y, a)$ .

Main tool: the perturbed test function *is almost super-optimal*

- The general case:

- Trajectories in the definition of correctors:  $\dot{Z}_\varepsilon(t) \in \mathcal{B}\left(\bar{x}, \frac{Z_\varepsilon(t)}{\varepsilon}\right)$

- Trajectories in the homogenization problem:  $\dot{X}_\varepsilon(t) \in \mathcal{B}\left(X_\varepsilon(t), \frac{X_\varepsilon(t)}{\varepsilon}\right)$

- Approximating problems with dynamics constant in the slow variable



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The 1-D case: an example when  $\bar{H}^+(x, 0) \neq \bar{H}^-(x, 0)$ 

$$\Omega_1 = \bigcup_{k \in \mathbb{Z}} ]2k, 2k + 1[ \quad \Omega_2 = \bigcup_{k \in \mathbb{Z}} ]2k + 1, 2k + 2[$$

$$b_1(x, y, \alpha_1) = \alpha_1, \quad b_2(x, y, \alpha_2) = \alpha_2, \quad A_1 = A_2 = [-1, +1]$$

$$l_1(x, y, \alpha_1) = |\alpha_1 - \cos(\pi y)| + 1 - |\cos(\pi y)|$$

$$l_2(x, y, \alpha_2) = |\alpha_2 + \cos(\pi y)| + 1 - |\cos(\pi y)|.$$

The data are independent of the  $x$  variable, then  $U^+$  and  $U^-$  are constants and only  $p = 0$  is relevant in  $\bar{H}^\pm$ .

Ergodic characterisation implies:

$$\bar{H}^\pm = \lim_{t \rightarrow +\infty} \left( - \inf_{(Y_0, a) \in \mathcal{T}_0^\circ} \left\{ \frac{1}{t} \int_0^t l(Y_0(s), a(s)) ds \right\} \right).$$

The best strategy is "singular"  $\alpha_1 = 1$   $\alpha_2 = -1$  and cannot be used in the  $\bar{H}^+$ -case. Finally  $\bar{H}^- = 0$ ,  $\bar{H}^+ < 0$

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