



HAL
open science

Degenerate second order mean field games systems

Daniela Tonon, Pierre Cardaliaguet, Philip J. Graber, Alessio Porretta

► **To cite this version:**

Daniela Tonon, Pierre Cardaliaguet, Philip J. Graber, Alessio Porretta. Degenerate second order mean field games systems. NETCO 2014 - New Trends on Optimal Control, Jun 2014, Tours, France. hal-01024600

HAL Id: hal-01024600

<https://inria.hal.science/hal-01024600>

Submitted on 18 Jul 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Degenerate second order mean field games systems

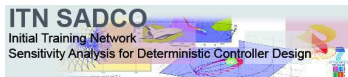
Daniela Tonon

joint work with P. Cardaliaguet, J. Graber and A. Porretta

CEREMADE, Université Paris-Dauphine, France

NETCO 2014

Tours, June 26th 2014



We consider the following second order mean field games (MFG) system with degenerate diffusion and a local coupling

$$\begin{cases} -\partial_t \phi - A_{ij} \partial_{ij} \phi + H(x, D\phi) = f(x, m(x, t)) \\ \partial_t m - \partial_{ij} (A_{ij} m) - \operatorname{div}(m D_p H(x, D\phi)) = 0 \\ m(0) = m_0, \phi(x, T) = \phi_T(x) \end{cases}$$

where

- $A : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ is symmetric and nonnegative
- the Hamiltonian $H : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is convex in the second variable
- the coupling $f : \mathbb{R}^d \times [0, +\infty) \rightarrow [0, +\infty)$ is increasing with respect to the second variable
- m_0 is a probability density
- $\phi_T : \mathbb{R}^d \rightarrow \mathbb{R}$ is a given function

Introduced by [Lasry and Lions](#) and by [Huang, Caines and Malhamé](#) to describe Nash equilibria in differential games with infinitely many players

The optimal control problem of a typical small player

$\phi(t, x)$ is the **value function**

Dynamics :

$$dX_s = \alpha_s ds + \Sigma(s, X_s) dB_s,$$

where (α_s) is the control, (B_s) is a d -dimensional Brownian motion and $\Sigma \Sigma^T = A$

Cost :

$$\mathbb{E} \left[\int_0^T H^*(s, X_s, -\alpha_s) + f(X_s, m(s, X_s)) ds + \phi_T(X_T) \right]$$

$\forall t \in [0, T]$ $m(t, x)$ denotes the **density** of population of small players at position x

the **optimal control** is formally given by the feedback
 $(t, x) \rightarrow -D_p H(x, D\phi(t, x))$

the second equation in (MFG) is the **Kolmogorov equation** of the process (X_s) when the small player plays in an optimal way

In the MFG systems with **uniformly parabolic diffusions** (typically $A_{ij}\partial_{ij}\phi = \Delta\phi$) the solutions are **smooth**, (at least if the coupling is nonlocal and regularizing or if it has a "small growth")

Analysis by **PDE methods** : see Cardaliaguet, Lasry, Lions and Porretta (2012), Lasry and Lions (2006, 2007), Gomes, Pimentel and Sánchez-Morgado (2013), Lions (2011), Porretta (2013)

Analysis by **stochastic techniques** : see Carmona and Delarue (2013), Huang, Malhamé and Caines (2006)

The case of **couplings with an arbitrary growth** has been discussed in Cardaliaguet, Lasry, Lions and Porretta (2012) (for quadratic hamiltonians and smooth solutions) and in Porretta (2013) (for more general hamiltonians but weak solutions)

We consider **degenerate parabolic equations** : lack of regularity \Rightarrow break down of the uniformly parabolic techniques

IDEA : use convex optimization methods from optimal transport problems (see Benamou and Brenier (2000), Cardaliaguet, Carlier and Nazaret (2012))

These techniques were already used to study first order MFG systems (i.e., $A \equiv 0$): see Cardaliaguet (2013), Cardaliaguet and Graber (2013), Graber (2013)

We show the existence and uniqueness of a weak solution for the degenerate MFG system as well as the stability of solutions with respect to perturbation of the data

Assumptions

- (H1) $f : \mathbb{T}^d \times [0, +\infty) \rightarrow \mathbb{R}$ is continuous in both variables, **increasing** wrt the second variable m , and $\exists q > 1, C_1$ s.t.

$$\frac{1}{C_1}|m|^{q-1} - C_1 \leq f(x, m) \leq C_1|m|^{q-1} + C_1 \quad \forall m \geq 0$$

Moreover $f(x, 0) = 0 \quad \forall x \in \mathbb{T}^d$

- (H2) $H : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is continuous in both variables, **convex** and differentiable in the second variable, with $D_p H$ continuous in both variable, and $\exists r > 1, C_2 > 0$ s.t.

$$\frac{1}{rC_2}|\xi|^r - C_2 \leq H(x, \xi) \leq \frac{C_2}{r}|\xi|^r + C_2 \quad \forall (x, \xi) \in \mathbb{T}^d \times \mathbb{R}^d$$

Assumptions

(H3) $A : \mathbb{T}^d \rightarrow \mathbb{R}^{d \times d}$ is Lipschitz continuous with **symmetric nonnegative** values: $\exists C_3 > 0$ s.t.

$$|A(x) - A(y)| \leq C_3|x - y| \quad \forall x, y \in \mathbb{T}^d, \xi \in \mathbb{R}^d$$

Moreover, either $r \geq p$ or $A \equiv 0$, where $\frac{1}{p} + \frac{1}{q} = 1$

(H4) $\phi_T : \mathbb{T}^d \rightarrow \mathbb{R}$ is of class C^2 , while $m_0 : \mathbb{T}^d \rightarrow \mathbb{R}$ is a C^1 positive density

The **Fenchel conjugate** H^* of H wrt the second variable is continuous and

$$\frac{1}{r' C_2} |\xi|^{r'} - C_2 \leq H^*(x, \xi) \leq \frac{C_2}{r'} |\xi|^{r'} + C_2 \quad \forall (x, \xi) \in \mathbb{T}^d \times \mathbb{R}^d,$$

where r' is s.t. : $\frac{1}{r} + \frac{1}{r'} = 1$

Assumptions

(H3) $A : \mathbb{T}^d \rightarrow \mathbb{R}^{d \times d}$ is Lipschitz continuous with **symmetric nonnegative** values: $\exists C_3 > 0$ s.t.

$$|A(x) - A(y)| \leq C_3|x - y| \quad \forall x, y \in \mathbb{T}^d, \xi \in \mathbb{R}^d$$

Moreover, either $r \geq p$ or $A \equiv 0$, where $\frac{1}{p} + \frac{1}{q} = 1$

(H4) $\phi_T : \mathbb{T}^d \rightarrow \mathbb{R}$ is of class C^2 , while $m_0 : \mathbb{T}^d \rightarrow \mathbb{R}$ is a C^1 positive density

The **Fenchel conjugate** H^* of H wrt the second variable is continuous and

$$\frac{1}{r'C_2}|\xi|^{r'} - C_2 \leq H^*(x, \xi) \leq \frac{C_2}{r'}|\xi|^{r'} + C_2 \quad \forall (x, \xi) \in \mathbb{T}^d \times \mathbb{R}^d,$$

where r' is s.t. : $\frac{1}{r} + \frac{1}{r'} = 1$

Let

$$F(x, m) = \begin{cases} \int_0^m f(x, \tau) d\tau & \text{if } m \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

Then F is continuous on $\mathbb{T}^d \times (0, +\infty)$, derivable and **strictly convex** in m and

$$\frac{1}{qC_1}|m|^q - C_1 \leq F(x, m) \leq \frac{C_1}{q}|m|^q + C_1 \quad \forall m \geq 0$$

Let F^* be the **Fenchel conjugate** of F wrt the second variable then $F^*(x, a) = 0$ for $a \leq 0$, moreover,

$$\frac{1}{pC_1}|a|^p - C_1 \leq F^*(x, a) \leq \frac{C_1}{p}|a|^p + C_1 \quad \forall a \geq 0,$$

where p is s.t. : $1/p + 1/q = 1$

Key point : integral estimates

Assume that (H2) and (H3) hold true and let ϕ satisfy

$$\begin{cases} -\partial_t \phi - A_{ij}(x) \partial_{ij} \phi + H(x, D\phi) \leq \alpha(t, x) \\ \phi(x, T) \leq \phi_T(x) \end{cases}$$

in the **sense of distribution**, with $\alpha \in L^p((0, T) \times \mathbb{T}^d)$, $\phi_T \in L^\infty(\mathbb{T}^d)$

i.e. for any nonnegative test function $\zeta \in C_c^\infty((0, T] \times \mathbb{T}^d)$,

$$\begin{aligned} - \int_{\mathbb{T}^d} \zeta(T) \phi_T + \int_0^T \int_{\mathbb{T}^d} \phi \partial_t \zeta + \langle D\zeta, AD\phi \rangle + \zeta(\partial_i A_{ij} \partial_j \phi + H(x, D\phi)) \leq \\ \int_0^T \int_{\mathbb{T}^d} \alpha \zeta \end{aligned}$$

Key point : integral estimates

Theorem

Then, if ϕ is bounded below, we have

$$\|\phi\|_{L^\infty((0,T),L^m(\mathbb{T}^d))} + \|\phi\|_{L^\gamma((0,T)\times\mathbb{T}^d)} \leq C \left(\|\alpha\|_{L^p((0,T)\times\mathbb{T}^d)} + \|\phi_T\|_{L^m(\mathbb{T}^d)} \right)$$

where $m = \frac{d(r(p-1)+1)}{d-r(p-1)}$ and $\gamma = \frac{rp(1+d)}{d-r(p-1)}$ if $p < 1 + \frac{d}{r}$
and $m = \gamma = +\infty$ if $p > 1 + \frac{d}{r}$,

with a constant C depending on T, p, d, r, C_2, C_3 and on $\|\alpha\|_{L^p((0,T)\times\mathbb{T}^d)}, \|\phi_T\|_{L^m(\mathbb{T}^d)}$ and $\|\phi_-\|_\infty$.

When $r > 2$ and $p > 1 + d/r$, Cardaliaguet and Silvestre 2012 proved that the solution ϕ is locally Hölder continuous, with Hölder modulus depending only on the growth condition of H , the L^∞ bound on u and the L^p norm of α

Key point : integral estimates

Theorem

Then, if ϕ is bounded below, we have

$$\|\phi\|_{L^\infty((0,T),L^m(\mathbb{T}^d))} + \|\phi\|_{L^\gamma((0,T)\times\mathbb{T}^d)} \leq C \left(\|\alpha\|_{L^p((0,T)\times\mathbb{T}^d)} + \|\phi_T\|_{L^m(\mathbb{T}^d)} \right)$$

where $m = \frac{d(r(p-1)+1)}{d-r(p-1)}$ and $\gamma = \frac{rp(1+d)}{d-r(p-1)}$ if $p < 1 + \frac{d}{r}$
and $m = \gamma = +\infty$ if $p > 1 + \frac{d}{r}$,

with a constant C depending on T, p, d, r, C_2, C_3 and on $\|\alpha\|_{L^p((0,T)\times\mathbb{T}^d)}, \|\phi_T\|_{L^m(\mathbb{T}^d)}$ and $\|\phi_-\|_\infty$.

When $r > 2$ and $p > 1 + d/r$, Cardaliaguet and Silvestre 2012 proved that the solution ϕ is locally Hölder continuous, with Hölder modulus depending only on the growth condition of H , the L^∞ bound on u and the L^p norm of α

Optimal control problems in duality

An optimal control problem for a **backward HJ equation** :

the state variable ϕ is controlled by a distributed control $\alpha : (0, T) \times \mathbb{T}^d \rightarrow \mathbb{R}$ in order to minimize the criterium

$$\int_0^T \int_{\mathbb{T}^d} F^*(x, \alpha(t, x)) \, dx dt - \int_{\mathbb{T}^d} \phi(0, x) dm_0(x)$$

when the state ϕ is driven by

$$\begin{cases} -\partial_t \phi - A_{ij}(x) \partial_{ij} \phi + H(x, D\phi) = \alpha(t, x) \\ \phi(x, T) = \phi_T(x) \end{cases}$$

Precisely :

$$\inf_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi)$$

where $\mathcal{K}_0 := \{\phi \in \mathcal{C}^2([0, T] \times \mathbb{T}^d) \mid \phi(T, x) = \phi_T(x)\}$ and

$$\begin{aligned} \mathcal{A}(\phi) = & \int_0^T \int_{\mathbb{T}^d} F^*(x, -\partial_t \phi(t, x) - A_{ij} \partial_{ij} \phi + H(x, D\phi(t, x))) \, dx dt \\ & - \int_{\mathbb{T}^d} \phi(0, x) dm_0(x) \end{aligned}$$

The second optimal control problem is an optimal control problem for a **Fokker-Plank equation** :

we control the state variable m through a vector field $v : (0, T) \times \mathbb{T}^d \rightarrow \mathbb{R}^d$ in order to minimize

$$\int_0^T \int_{\mathbb{T}^d} m(t, x) H^*(x, -v(t, x)) + F(x, m(t, x)) \, dx dt + \int_{\mathbb{T}^d} \phi_T(x) m(T, x) \, dx$$

when m solves the Fokker-Plank equation

$$\partial_t m - \partial_{ij}(A_{ij}(x)m) + \operatorname{div}(mv) = 0$$

Precisely :

$$\inf_{(m,w) \in \mathcal{K}_1} \mathcal{B}(m, w)$$

where

$$\mathcal{K}_1 := \left\{ (m, w) \in L^1((0, T) \times \mathbb{T}^d) \times L^1((0, T) \times \mathbb{T}^d, \mathbb{R}^d) \mid \right. \\ \left. m(t, x) \geq 0 \text{ a.e.}, \int_{\mathbb{T}^d} m(t, x) dx = 1 \text{ for a.e. } t \in (0, T), \text{ and} \right. \\ \left. \partial_t m - \partial_{ij}(A_{ij}(x)m) + \operatorname{div}(w) = 0 \text{ in } (0, T) \times \mathbb{T}^d, \quad m(0) = m_0 \right. \\ \left. \text{in the sense of distribution} \right\}$$

$$\mathcal{B}(m, w) = \int_0^T \int_{\mathbb{T}^d} m(t, x) H^* \left(x, -\frac{w(t, x)}{m(t, x)} \right) + F(x, m(t, x)) \, dx dt \\ + \int_{\mathbb{T}^d} \phi_T(x) m(T, x) dx$$

$$\text{for } m(t, x) = 0, \quad m(t, x) H^* \left(x, -\frac{w(t, x)}{m(t, x)} \right) = \begin{cases} +\infty & \text{if } w(t, x) \neq 0 \\ 0 & \text{if } w(t, x) = 0 \end{cases}$$

Lemma

We have

$$\inf_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi) = - \min_{(m, w) \in \mathcal{K}_1} \mathcal{B}(m, w)$$

Moreover, the minimum in the right-hand side is achieved by a **unique** pair $(\bar{m}, \bar{w}) \in \mathcal{K}_1$ satisfying

$$(\bar{m}, \bar{w}) \in L^q((0, T) \times \mathbb{T}^d) \times L^{\frac{r'q}{r'+q-1}}((0, T) \times \mathbb{T}^d)$$

IDEA : Use the Fenchel-Rockafellar duality Theorem

Relaxation

Let

$$\mathcal{K} := \left\{ (\phi, \alpha) \in L^\gamma((0, T) \times \mathbb{T}^d) \times L^p((0, T) \times \mathbb{T}^d) \mid D\phi \in L^r((0, T) \times \mathbb{T}^d) \right. \\ \left. \text{and which satisfies in the sense of distribution} \right. \\ \left. - \partial_t \phi - A_{ij}(x) \partial_{ij} \phi + H(x, D\phi) \leq \alpha, \quad \phi(T, \cdot) \leq \phi_T \right\}$$

Due to the presence of second order derivatives we do not expect the function ϕ to be BV (as in the first order case)

What about **trace properties**? ϕ has a trace in a weak sense

Lemma

Let $(\phi, \alpha) \in \mathcal{K}$. Then, for any Lipschitz continuous map $\zeta : \mathbb{T}^d \rightarrow \mathbb{R}$, the map $t \rightarrow \int_{\mathbb{T}^d} \zeta(x)\phi(t, x)dx$ has a **BV representative** on $[0, T]$.

Moreover, if we denote by $\int_{\mathbb{T}^d} \zeta(x)\phi(t^+, x)dx$ its right limit at $t \in [0, T)$, then the map $\zeta \rightarrow \int_{\mathbb{T}^d} \zeta(x)\phi(t^+, x)dx$ is continuous in $L^{m'}(\mathbb{T}^d)$.

\Rightarrow for any nonnegative C^1 map $\zeta : [0, T] \times \mathbb{T}^d \rightarrow \mathbb{R}$, one can write the integration by part formula: for any $0 \leq t_1 \leq t_2 \leq T$,

$$\begin{aligned}
 - \left[\int_{\mathbb{T}^d} \zeta \phi \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \phi \partial_t \zeta + \langle D\zeta, AD\phi \rangle + \zeta(\partial_i A_{ij} \partial_j \phi + H(x, D\phi)) \leq \\
 \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \alpha \zeta
 \end{aligned}$$

Lemma

Let $(\phi, \alpha) \in \mathcal{K}$. Then, for any Lipschitz continuous map $\zeta : \mathbb{T}^d \rightarrow \mathbb{R}$, the map $t \rightarrow \int_{\mathbb{T}^d} \zeta(x)\phi(t, x)dx$ has a **BV representative** on $[0, T]$.

Moreover, if we denote by $\int_{\mathbb{T}^d} \zeta(x)\phi(t^+, x)dx$ its right limit at $t \in [0, T)$, then the map $\zeta \rightarrow \int_{\mathbb{T}^d} \zeta(x)\phi(t^+, x)dx$ is continuous in $L^{m'}(\mathbb{T}^d)$.

\Rightarrow for any nonnegative C^1 map $\zeta : [0, T] \times \mathbb{T}^d \rightarrow \mathbb{R}$, one can write the integration by part formula: for any $0 \leq t_1 \leq t_2 \leq T$,

$$\begin{aligned}
 - \left[\int_{\mathbb{T}^d} \zeta \phi \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \phi \partial_t \zeta + \langle D\zeta, A D\phi \rangle + \zeta (\partial_i A_{ij} \partial_j \phi + H(x, D\phi)) \leq \\
 \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \alpha \zeta
 \end{aligned}$$

Proof :

For any Lipschitz continuous, nonnegative map ζ ,

$$-\frac{d}{dt} \int_{\mathbb{T}^d} \zeta \phi(t) + \int_{\mathbb{T}^d} \langle D\zeta, AD\phi(t) \rangle + \zeta(\partial_i A_{ij} \partial_j \phi + H(x, D\phi) - \alpha) \leq 0,$$

in the sense of distribution

As the second integral is in $L^1((0, T))$, the map $t \rightarrow \int_{\mathbb{T}^d} \zeta \phi(t)$ is BV

If now ζ is Lipschitz continuous and changes sign, one can write $\zeta = \zeta^+ - \zeta^-$ and the map $t \rightarrow \int_{\mathbb{T}^d} \zeta \phi(t) = \int_{\mathbb{T}^d} \zeta^+ \phi(t) - \int_{\mathbb{T}^d} \zeta^- \phi(t)$ is still BV

The continuity with respect to ζ comes from the $L^\infty((0, T), L^m(\mathbb{T}^d))$ estimate on ϕ

We extend the functional \mathcal{A} to \mathcal{K}

$$\mathcal{A}(\phi, \alpha) = \int_0^T \int_{\mathbb{T}^d} F^*(x, \alpha(x, t)) \, dx dt - \int_{\mathbb{T}^d} \phi(x, 0) m_0(x) \, dx \quad \forall (\phi, \alpha) \in \mathcal{K}$$

Proposition

We have

$$\inf_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi) = \inf_{(\phi, \alpha) \in \mathcal{K}} \mathcal{A}(\phi, \alpha)$$

Lemma

Let $(\phi, \alpha) \in \mathcal{K}$ and $(m, w) \in \mathcal{K}_1$.

Assume that $mH^*(\cdot, -w/m) \in L^1((0, T) \times \mathbb{T}^d)$ and $m \in L^q$.

Then, for a.e. $t \in [0, T]$, $m(t)\phi(t)$ is integrable and

$$\left[\int_{\mathbb{T}^d} m\phi \right]_t^T + \int_t^T \int_{\mathbb{T}^d} m \left(\alpha + H^*(x, -\frac{w}{m}) \right) \geq 0$$

and

$$\left[\int_{\mathbb{T}^d} m\phi \right]_0^t + \int_0^t \int_{\mathbb{T}^d} m \left(\alpha + H^*(x, -\frac{w}{m}) \right) \geq 0.$$

Moreover, if equality holds in the first inequality for $t = 0$, then $w = -mD_p H(x, D\phi)$ a.e.

Proof :

Inequality $\inf_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi) \geq \inf_{(\phi, \alpha) \in \mathcal{K}} \mathcal{A}(\phi, \alpha)$ holds obviously

Let $(\phi, \alpha) \in \mathcal{K}$, then $\forall (m, w) \in \mathcal{K}_1$ with $mH^*(\cdot, -w/m) \in L^1$ we have,

$$\begin{aligned} \mathcal{A}(\phi, \alpha) &= \int_0^T \int_{\mathbb{T}^d} F^*(\alpha) - \int_{\mathbb{T}^d} \phi(0) m_0 \\ &\geq \int_0^T \int_{\mathbb{T}^d} \alpha m - F(m) - \int_{\mathbb{T}^d} m_0 \phi(0) \\ &\geq \int_0^T \int_{\mathbb{T}^d} -mH^*(x, -\frac{w}{m}) - F(m) - \int_{\mathbb{T}^d} m(T) \phi_T = -\mathcal{B}(m, w) \end{aligned}$$

Taking the sup with respect to (m, w) in the right-hand side

$$\mathcal{A}(\phi, \alpha) \geq - \inf_{(m, w) \in \mathcal{K}_1} \mathcal{B}(m, w) = \inf_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi)$$

Proposition

The relaxed problem has *at least one solution* $(\bar{\phi}, \bar{\alpha}) \in \mathcal{K}$ which is bounded below by a constant depending on $\|\phi_T\|_{C^2}$, on $\|A_{ij}\|_{C^0}$ and on $\|H(\cdot, D\phi_T)\|_{\infty}$.

Therefore, thanks to the equivalence, we have a infimum for \mathcal{A} over \mathcal{K}_0 and a minimum for \mathcal{B} over \mathcal{K}_1

$$\min_{(\phi, \alpha) \in \mathcal{K}} \mathcal{A}(\phi, \alpha) = \inf_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi) = - \min_{(m, w) \in \mathcal{K}_1} \mathcal{B}(m, w)$$

Existence and uniqueness

We say that a pair $(\phi, m) \in L^\gamma((0, T) \times \mathbb{T}^d) \times L^q((0, T) \times \mathbb{T}^d)$ is a **weak solution to (MFG)** if

- (i) $D\phi \in L^r$, $mH^*(\cdot, D_p H(\cdot, D\phi)) \in L^1$ and $mD_p H(\cdot, D\phi) \in L^1$
- (ii) The following inequality holds in the sense of distribution

$$-\partial_t \phi - \partial_i (A_{ij}(x) \partial_j \phi) + (\partial_i A_{ij}) \partial_j \phi + H(x, D\phi) \leq f(x, m) \quad \text{in } (0, T) \times \mathbb{T}^d,$$

with $\phi(T, \cdot) \leq \phi_T$

- (iii) The following equation holds in the sense of distribution

$$\partial_t m - \partial_{ij} (A_{ij}(x) m) - \operatorname{div}(m D_p H(x, D\phi)) = 0 \quad \text{in } (0, T) \times \mathbb{T}^d, \quad m(0) = m_0$$

- (iv) The following equality holds :

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^d} m(t, x) (f(x, m(t, x)) + H^*(x, D_p H(x, D\phi)(t, x))) \, dx dt \\ & + \int_{\mathbb{T}^d} m(T, x) \phi_T(x) - m_0(x) \phi(0, x) \, dx = 0 \end{aligned}$$

Theorem

There **exists** a weak solution $(\bar{\phi}, \bar{m})$ to (MFG).

Moreover this solution is **unique** in the following sense: if $(\bar{\phi}, \bar{m})$ and $(\bar{\phi}', \bar{m}')$ are two solutions, then $\bar{m} = \bar{m}'$ a.e. and $\bar{\phi} = \bar{\phi}'$ in $\{\bar{m} > 0\}$.

Finally, there exists a solution which is **bounded below** by a constant depending on $\|\phi_T\|_{C^2}$, on $\|A_{ij}\|_{C^0}$ and on $\|H(\cdot, D\phi_T)\|_{\infty}$.

Proof :

Let $(\bar{m}, \bar{w}) \in \mathcal{K}_1$ be a minimizer of $\min_{(m,w) \in \mathcal{K}_1} \mathcal{B}(m, w)$
 and $(\bar{\phi}, \bar{\alpha}) \in \mathcal{K}$ be a minimizer of $\min_{(\phi, \alpha) \in \mathcal{K}} \mathcal{A}(\phi, \alpha)$

We have $-\mathcal{B}(\bar{m}, \bar{w}) = \mathcal{A}(\bar{\phi}, \bar{\alpha})$

$$\int_0^T \int_{\mathbb{T}^d} F^*(x, \bar{\alpha}) + F(x, \bar{m}) + \bar{m} H^* \left(x, -\frac{\bar{w}}{\bar{m}} \right) dx dt + \int_{\mathbb{T}^d} \phi_T \bar{m}(T) - \bar{\phi}(0) m_0 dx = 0$$

By convexity of F ,

$$F^*(x, \bar{\alpha}) + F(x, \bar{m}) - \bar{\alpha} \bar{m} \geq 0,$$

hence

$$\int_0^T \int_{\mathbb{T}^d} \bar{\alpha} \bar{m} + \bar{m} H^* \left(x, -\frac{\bar{w}}{\bar{m}} \right) dx dt + \int_{\mathbb{T}^d} \phi_T \bar{m}(T) - \bar{\phi}(0) m_0 dx \leq 0$$

Therefore $\bar{w} = -\bar{m} D_p H(\cdot, D\bar{\phi})$ a.e. and $\bar{\alpha}(t, x) = f(x, \bar{m}(t, x))$ a.e.

and $(\bar{\phi}, \bar{m})$ is a weak solution

Recall that :

Lemma

Let $(\phi, \alpha) \in \mathcal{K}$ and $(m, w) \in \mathcal{K}_1$. Assume that $mH^*(\cdot, -w/m) \in L^1((0, T) \times \mathbb{T}^d)$ and $m \in L^q$.

Then, for a.e. $t \in [0, T]$, $m(t)\phi(t)$ is integrable and

$$\left[\int_{\mathbb{T}^d} m\phi \right]_t^T + \int_t^T \int_{\mathbb{T}^d} m \left(\alpha + H^*(x, -\frac{w}{m}) \right) \geq 0$$

and

$$\left[\int_{\mathbb{T}^d} m\phi \right]_0^t + \int_0^t \int_{\mathbb{T}^d} m \left(\alpha + H^*(x, -\frac{w}{m}) \right) \geq 0.$$

Moreover, if equality holds in the first inequality for $t = 0$, then $w = -mD_p H(x, D\phi)$ a.e.

Conversely, using the convexity of F and the Lemma, we can show that any weak solution $(\bar{\phi}, \bar{m})$ of (MFG) is such that :

the pair $(\bar{m}, -\bar{m}D_p H(\cdot, D\bar{\phi}))$ is the minimizer of $\min_{(m,w) \in \mathcal{K}_1} \mathcal{B}(m, w)$

while $(\bar{\phi}, f(\cdot, \bar{m}))$ is a minimizer of $\min_{(\phi, \alpha) \in \mathcal{K}} \mathcal{A}(\phi, \alpha)$

Uniqueness : Since the solution of $\min_{(m,w) \in \mathcal{K}_1} \mathcal{B}(m, w)$ is unique \bar{m} is unique

The proof of the uniqueness of $\bar{\phi}$ is more technical

Stability

Assume that (A^n) , (H^n) , (f^n) , m_0^n and ϕ_T^n satisfy conditions (H1)...(H4) uniformly with respect to n and converge to A , H , f , m_0 and ϕ_T locally uniformly.

Theorem

Let (ϕ^n, m^n) be a weak solution of (MFG) associated with A^n , H^n , f^n and with the initial and terminal conditions m_0^n and ϕ_T^n . Assume also that the sequence ϕ^n is uniformly bounded below.

Then (m^n) converges strongly to m in L^q while ϕ^n converges weakly and up to a subsequence to a map ϕ in L^γ , where the pair (ϕ, m) is a weak solution to (MFG).

In particular the weak solution is stable with respect to **viscous approximation**

P. Cardaliaguet, J. Graber, A. Porretta and D. Tonon, Second order mean field games with degenerate diffusion and local coupling, in preparation