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# Necessary Conditions in Dynamic Optimization

Richard B. Vinter

Imperial College London

**NetCo 2014**

**New Trends in Optimal Control**

**Tours, France**

**23-27 June 2014**

# NetCo (HJNet + SADCO)



# Outline of the talk

- Early Necessary Conditions in Classical C. of V.
- Dynamic Constraints and The Maximum Principle
- **Differential Inclusion Problems**
- Clarke's Hamiltonian Inclusion for Convex Diff. Inclusions
- Euler Lagrange Inclusion for Non-convex Diff. Inclusions
- Return to the Hamiltonian Inclusion
- Uses of the Hamiltonian Inclusion and Open Questions

# Some History

## Calculus of Variations:

$$(Q) \begin{cases} \text{Minimize } \int_0^T L(t, x(t), \dot{x}(t)) dt \\ \text{over } x(\cdot) \in W^{1,1}[S, T]; \mathbb{R}^n \\ \text{satisfying} \\ x(0) = x_0 \text{ and } x(T) = x_1 . \end{cases}$$

( $L : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  and points  $x_0, x_1 \in \mathbb{R}^n$ .)

$W^{1,1} \equiv$  **abs. continuous functions**

Take a minimizer  $\bar{x}(\cdot)$ .

Assume

- $L(\cdot, \cdot, \cdot)$  is continuous and  $L(t, \cdot, \cdot)$  is  $C^1$
- There exist  $k(\cdot) \in L^1$  and  $\epsilon > 0$  s.t.

$$|\nabla L(x, v)| \leq k(t)$$

for all  $(x, v) \in (\bar{x}(t), \dot{\bar{x}}(t)) + \epsilon \mathbb{B}$      $\mathbb{B} \equiv$  'closed unit ball'.

# Necessary Conditions

Take a minimizer  $\bar{x}(\cdot)$  for

$$\text{Minimize } \left\{ \int_0^T L(t, x(t), \dot{x}(t)) dt \mid x(\cdot) \in W^{1,1}, x(0) = x_0, x(T) = x_1 \right\}$$

There exists an absolutely continuous  $p(\cdot) : [0, 1] \rightarrow \mathbb{R}^n$  such that

- $(\dot{p}(t), p(t)) = \nabla L(t, \bar{x}(t), \dot{\bar{x}}(t))$  a.e

(The Euler-Lagrange Condition)

- $p(t) \cdot \dot{\bar{x}}(t) - L(t, \bar{x}(t), \dot{\bar{x}}(t)) = \max_{v \in \mathbb{R}^n} \{p(t) \cdot v - L(t, \bar{x}(t), v)\}$  a.e.

(The Weierstrass condition)

- If  $L(t, x, v)$  is indep. of  $t$ ,

$$p(t) \cdot \dot{\bar{x}}(t) - L(t, \bar{x}(t), \dot{\bar{x}}(t)) = \text{const. a.e.}$$

(Constancy of the Hamiltonian)

# Hamilton's Equations

Assume

- $L(t, \cdot, \cdot)$  is  $C^2$
- $L_{vv}(t, x, v)$  is invertible.

Hamiltonian  $H(t, x, p) : [0, T] \times R^n \times R^n \rightarrow R$ :

$$H(t, x, p) = p \cdot v(t, x, p) - L(t, x, v(t, x, p)) \quad (\text{Legendre transformation})$$

$$(v(t, x, p) := \text{'solution function of } p \cdot v - L_v(t, x, v) = 0 \text{'})$$

Hamilton's equations:

$$(-\dot{p}(t), \dot{\bar{x}}(t)) = \nabla H(t, \bar{x}(t), p(t))$$

**Permits study of minimizers via systems of differential equations.**

# The Optimal Control Problem with Dynamic Constraints

Aerospace applications ('optimal flight trajectories') motivated:

## Pontryagin Formulation

$$\left\{ \begin{array}{l} \text{Minimize } g(x(0), x(T)) \\ \text{over } x(\cdot) \in W^{1,1}([0, T]; \mathbb{R}^n) \\ \quad \text{and measurable functions } u(\cdot) : [0, T] \rightarrow \mathbb{R}^m \text{ s.t.} \\ \dot{x}(t) \in f(t, x(t), u(t)) \quad \textit{a.e.} \\ u(t) \in U(t) \quad \textit{a.e.} \\ (x(0), x(T)) \in C \end{array} \right.$$

**Data:**  $f(\cdot, \cdot, \cdot) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ ,  $U(\cdot) : [0, T] \rightsquigarrow \mathbb{R}^m$ ,  
 $C \subset \mathbb{R}^n \times \mathbb{R}^n$ .





# Dynamic constraint, cont.

Take minimizer  $(\bar{x}(\cdot), \bar{u}(\cdot))$ . Assume

- $f(\cdot, \cdot, \cdot)$  is continuous,  $f(t, \cdot, u)$  is  $C^1$ ,  $C$  are closed
- $\text{Gr } U(\cdot)$  is a Borel set
- there exist  $c(\cdot) \in L^1$  and  $k_f(\cdot) \in L^1$  such that

$$|f(t, x, u) - f(t, x', u)| \leq k(t)|x - x'|,$$

$$|f(t, x, u)| \leq c(t)$$

for all  $x, x' \in \bar{x}(t) + \epsilon B$ ,  $u \in U$ , a.e.  $t \in [0, T]$ .

Write  $H(\cdot, \cdot, \cdot, \cdot) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$

$$H(t, x, p, u) = p \cdot f(t, x, u).$$

(Un-maximized Hamiltonian)

# Maximum Principle (L S Pontragin et al.) 1959

Take a minimizer  $(\bar{x}(\cdot), \bar{u}(\cdot))$ .

There exist  $\lambda \geq 0$  and  $p(\cdot) \in W^{1,1}$  such that

- $(\lambda, p(\cdot)) \neq (0, 0)$  (Nontriviality of multipliers)
- $-\dot{p}(t) = H_x(t, \bar{x}(t), \bar{u}(t))$  a.e. (Costate Equation)
- $H(t, \bar{x}(t), \bar{u}(t)) = \max\{H(t, \bar{x}(t), u) \mid u \in U\}$  a.e.

(Weierstrass Condition)

- $(p(0), -p(T)) = \lambda g_x(\bar{x}(0), \bar{x}(T)) + \xi$

where  $\xi$  is 'normal' to  $C$  at  $(\bar{x}(0), \bar{x}(T))$ .

(Transversality Condition)

- $H(t, \bar{x}(t), p(t), \bar{u}(t)) = \max\{H(t, \bar{x}(t), p(t), u) \mid u \in U(t)\}$  a.e.

(Constancy of Maximized Hamiltonian)

# Problem Reformulation

The dynamic constraint

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \\ u(t) \in U(t) \end{cases}$$

is a constraint on possible velocities for a given state, and can be captured as a differential inclusion

$$\dot{x}(t) \in F(t, x(t)) \quad \text{a.e.}$$

where

$$F(t, x) := \{f(t, x, u) \mid u \in U\}.$$

**The optimal control problem can be expressed directly in terms of  $F(\cdot)$ :**

$$\begin{cases} \text{Minimize } g(x(0), x(T)) \\ \text{over } x(\cdot) \in W^{1,1}([0, T] : \mathbb{R}^n) \text{ satisfying} \\ \dot{x}(t) \in F(t, x(t)) \\ (x(0), x(T)) \in C \end{cases}$$

# Reformulation cont.

Since 1970's: interest in deriving necessary conditions based on a 'differential inclusion' formulation of the dynamic constraint:

**Goal:** Express both hypotheses and necessary conditions directly in terms of  $F(\cdot)$  instead of ingredients of parameterization:

$$F(t, x) := f(t, x, U) .$$

## Reasons:

- New kinds of necessary conditions
- Improvements on the Maximum Principle (validity under reduced hypotheses)
- Better tools for studying properties of optimal trajectories, value functions, etc.

# First Ideas (R T Rockafellar)



# First Ideas (R T Rockafellar)

Re-write the 'differential inclusion' problem

$$\left\{ \begin{array}{l} \text{Minimize } \int_0^T L(t, x(t), \dot{x}(t)) dt + g(x(0), x(T)) \\ \text{over } x(\cdot) \in W^{1,1}([0, T], \mathbb{R}^n) \text{ satisfying} \\ \dot{x}(t) \in F(t, x(t)) \\ (x(0), x(T)) \in C \end{array} \right.$$

as a **Generalized Problem of Bolza**:

$$\left\{ \begin{array}{l} \text{Minimize } \tilde{g}(x(0), x(T)) \\ \text{over } x(\cdot) \in W^{1,1}([0, T], \mathbb{R}^n) \end{array} \right.$$

in which

('hard penalization')

$$L(t, x, v) := \begin{cases} 0 & \text{if } v \in F(t, x) \\ +\infty & \text{otherwise} \end{cases}, \quad \tilde{g}(x_0, x_1) = \begin{cases} g(x_0, x_1) & \text{if } (x_0, x_1) \in C \\ +\infty & \text{otherwise} \end{cases}$$

('Generalized' because it involves extended valued functions)

# Optimality Conditions for Fully Convex Problems

## Theorem (Rockafellar 1970, Fully Convex Case)

Let  $\bar{x}(\cdot)$  be a solution to the **Generalized Bolza Problem**. Assume

- $\tilde{L}(t, \cdot, \cdot)$  and  $\tilde{g}(\cdot, \cdot)$  are proper convex functions. ('full convexity')
- $\tilde{L}(t, x, \cdot)$  has superlinear growth,  $g(\cdot, \cdot)$  linearly bounded below.
- $\text{rel interior } \{\tilde{\mathcal{C}}\} \cap \text{rel interior } \{\mathcal{R}\} \neq \emptyset$  ('controllability')

$$(\tilde{\mathcal{C}} = \{(x_0, x_1) \mid \tilde{g}(x_0, x_1) < \infty\} \text{ and } \mathcal{R} = \text{'reachable set'}).$$

Then there exist  $p(\cdot) \in AC$  such that

- $(\dot{p}(t), p(t)) \in \partial L(t, \bar{x}(t), \dot{\bar{x}}(t))$  a.e. (Euler-Lagrange Condition)
- $(p(0), -p(T)) \in \partial \tilde{g}(\bar{x}(0), \bar{x}(T))$

$\partial f(\bar{x})$  is subdifferential of convex analysis:

$$\partial f(\bar{x}) := \{\xi \mid f(x) - f(\bar{x}) \geq \xi \cdot (x - \bar{x}) \text{ for all } x \in R^n\}$$



# The Fully Convex Case Cont.

## Approach:

For a convex optimization problem, **Fenchel Duality** provides dual problem, conditions for 'strong duality', 'multiplier' interpretations of solutions to dual problem, etc.

- Compute subdifferentials of integral functionals involving extended valued integrands
- Check no duality gap and Fenchel Dual has solutions, which can be identified with co-state arcs.

# Hamiltonian Conditions

For  $L(t, x, \cdot)$  convex, the Hamiltonian:

$$H(t, x, p) = p \cdot v(t, x, p) - \tilde{L}(t, x, v(t, x, p)) \quad (\text{Legendre transformation})$$

can be expressed

$$H(t, x, p) = \sup\{p \cdot v - L(t, x, v) \mid v \in \mathbb{R}^n\}$$

- $L(t, \cdot, \cdot)$  convex  $\implies H(t, x, \cdot)$  is concave, and  $H(t, \cdot, p)$  is convex

For  $L(t, \cdot, \cdot)$  convex, Rockafellar showed under earlier conditions:

$$-\dot{p}(t) \in \tilde{\partial}_x H(t, \bar{x}(t), p(t)), \quad \dot{\bar{x}}(t) \in \partial_p H(t, \bar{x}(t), p(t)).$$

‘Hamiltonian inclusion’

$(\tilde{\partial}_x H := -\partial_x(-H))$  is ‘concave’ subdifferential

# Generalization to Non-convex Problems?

$$(P) \begin{cases} \text{Minimize } g(x(0), x(T)) \\ \text{over } x(\cdot) \in W^{1,1}([0, T]; \mathbb{R}^n) \text{ satisfying} \\ \dot{x}(t) \in F(t, x(t)) \\ (x(0), x(T)) \in C \end{cases}$$

Major research theme (1973 - ): **Derive necessary conditions for this non-convex problem**, e.g.

$$(\dot{p}(t), p(t)) \in \partial_{x,v} \tilde{L}(t, \bar{x}(t), \dot{\bar{x}}(t)) \quad \text{Euler Lagrange inclusion}$$

or 
$$(\tilde{L}(x, v) = \mathbb{I}_{\text{Graph } F(\cdot)}(x, v))$$

$$(-\dot{p}(t), \dot{\bar{x}}(t)) \in \partial \tilde{H}(t, \bar{x}(t), p(t)) \quad \text{Hamiltonian inclusion}$$

- **How to interpret  $\partial \tilde{L}$  and  $\partial H$  for non-convex functions?**

# Concepts of Local Minimizer

' $x(\cdot)$  feasible'  $F$ -trajectory  $x(\cdot)$  means ' $(x(0), x(T)) \in C$ '.

**Definition.** A feasible  $F$ -trajectory  $\bar{x}(\cdot)$  is a:

- $L^\infty$ -local minimizer if there exists  $\epsilon > 0$  such that

$$g(\bar{x}(0), \bar{x}(T)) \leq g(x(0), x(T))$$

for all feasible  $x(\cdot)$ 's s.t.  $\|x(\cdot) - \bar{x}(\cdot)\|_{L^\infty} \leq \epsilon$

- $W^{1,1}$ -local minimizer if there exists  $\epsilon > 0$  such that

$$g(\bar{x}(0), \bar{x}(T)) \leq g(x(0), x(T))$$

for all feasible  $x(\cdot)$ 's s.t.  $\|x(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}} \leq \epsilon$ .

# Choice of Subdifferentials, etc.

Take a close set  $A \subset X$  ( $X$  top. vector space) and  $\bar{x} \in A$ .

How to define tangent and normal cones ' $T_A$ '( $\bar{x}$ ) and ' $N_A$ '( $\bar{x}$ ) ?

Then define the corresponding subdifferential of l.s.c.

$$f : X \rightarrow R \cup \{+\infty\} \text{ at } x \in \text{dom } f(.):$$

$$' \partial f(\bar{x}) := \{ \xi \mid (\xi, -1) \in 'N_{\text{epi } f(.)}'(\bar{x}, f(\bar{x})) \}.$$

## Requirements:

- ' $N_A$ '( $\bar{x}$ ) coincides with normal cone of convex analysis, etc, for convex sets/functions, etc.
- There is a suitably rich calculus  
( $' \partial(f_1 + f_2)(\bar{x}) \subset ' \partial f_1(\bar{x}) + f_2(\bar{x}), \dots$ )
- 'robustness'
- $f(\bar{x}) \leq \inf\{f(x) \mid x \in X\} \implies 0 \in ' \partial f(\bar{x})$ .

# Nonsmooth Analysis: Francis Clarke



# Clarke's Tangent Cone and Normal Cone

Key advance: **Clarke's tangent cone and normal cone**

$$T_A^C(\bar{x}) := \{ \xi \mid \forall y_i \xrightarrow{A} \bar{x}, \exists x_i \xrightarrow{A} \bar{x} \text{ and } \epsilon \downarrow 0 \text{ s.t. } \epsilon^{-1}(y_i - x_i) \rightarrow \xi \}$$

$$N_A^C(\bar{x}) = \{ \eta \mid \eta \cdot \xi \leq 0 \text{ for all } \xi \in T_A^C(\bar{x}) \}.$$

**'polar cone'** of Clarke tangent cone

$$\partial^C f(\bar{x}) := \{ \xi \mid (\xi, -1) \in N_{\text{epi}(\cdot)}^C(\bar{x}, f(\bar{x})) \}.$$

- **Satisfies main 'requirements'**
- **Suitable vehicle for derivation of necessary conditions in optimal control.**

# The Hamiltonian Inclusion

$$(P) \begin{cases} \text{Minimize } g(x(0), x(T)) \\ \text{over } x(\cdot) \in W([0, T] : \mathbb{R}^n) \text{ satisfying} \\ \dot{x}(t) \in F(t, x(t)) \\ (x(0), x(T)) \in C \end{cases}$$

Standard Hypotheses (for given  $F$ -trajectory  $\bar{x}(\cdot)$ ): for some  $c_F(\cdot), k_F(\cdot) \in L^1$  s.t.

**(H1)** :  $F(t, x)$  is closed for each  $(t, x)$ ,  $F(\cdot, x)$  is meas.,  
 $C$  is closed,  $g(\cdot, \cdot)$  is locally Lipschitz

**(H2)**:  $F(t, x) \subset c_F(t)B$

for all  $x \in \bar{x}(t) + \epsilon B$ , a.e.  $t \in [0, T]$ .

**(H3)**:  $F(t, x) \subset F(t, x') + k_F(t)|x - x'|B$

for all  $x, x' \in \bar{x}(t) + \epsilon B$ , a.e.  $t \in [0, T]$ .



# The Hamiltonian Inclusion, cont.

**Theorem** (Clarke 1973)

Take an  $L^\infty$ -local minimizer  $\bar{x}(\cdot)$ . Assume standard hypotheses and

**(C):**  $F(t, x)$  is convex for all  $x$ .

Then there exists  $p(\cdot) \in W^{1,1}$  and  $\lambda \geq 0$  such that

**(i):**  $(p(\cdot), \lambda) \neq (0, 0)$

**(ii):**  $(-\dot{p}(t), \dot{\bar{x}}(t)) \in \partial_{x,p}^{\mathcal{C}} H(t, \bar{x}(t), p(t))$  a.e.

**(iii)**  $(p(0), -p(T)) \in \lambda \partial^{\mathcal{C}} g(\bar{x}(0), \bar{x}(T)) + N_{\mathcal{C}}^{\mathcal{C}}(\bar{x}(0), \bar{x}(T))$

# Nonconvex Normal Cones: A.Ya Kruger and B. Mordukhovich



Alexander Ya. Kruger



Boris Mordukhovich

# Non-Convex Normal Cones and Subdifferentials

A non-convex normal cone (Kruger/Mordukhovich):

$$N_A(\bar{x}) := \limsup_{x' \rightarrow \bar{x}} \{\xi \mid \xi \cdot (x - x') \leq o(|x - x'|)\}$$

Hence, subdifferential  $\partial f(\bar{x})$ :

$$\partial f(\bar{x}) = \{\xi \mid (\xi, -1) \in N_{\text{epi } F(\cdot)}(\bar{x}, f(\bar{x}))\}.$$

Relation with Clarke normal cone and subdifferential:

$$N_A^C(\bar{x}) = \text{co } N_A(\bar{x}) \text{ and } \partial^C f(\bar{x}) = \text{co } \partial f(\bar{x})$$

(if  $f(\cdot)$  Lipschitz near  $\bar{x}$ ).

**Normal Cones and Subdifferentials (in this sense) have a rich calculus, and can be used to formulate refined necessary conditions.**

# Hamiltonian Inclusion, First Refinement

**Theorem** (First Refinement: **improved transversality condition**)

The assertions of the Hamiltonian Inclusion can be strengthened as follows:

There exists  $p(\cdot) \in W^{1,1}$  and  $\lambda \geq 0$  such that

- (i):  $(p(\cdot), \lambda) \neq (0, 0)$
- (ii):  $(-\dot{p}(t), \dot{\bar{x}}(t)) \in \text{co } \partial H(t, \bar{x}(t), p(t))$  a.e.
- (iii)  $(p(0), -p(T)) \in \lambda \partial g(\bar{x}(0), \bar{x}(T)) + N_C(\bar{x}(0), \bar{x}(T))$

Example: for  $f(x) = -|x|$  and  $\bar{x} = 0$ ,

$$\partial f(\bar{x}) = \{-1\} \cup \{1\} \stackrel{\text{strict}}{\subset} [-1, +1] = \partial^C f(\bar{x})$$

# Two Further Refinements

## 1. Improvement of of Convexity Condition:

The Hamiltonian inclusion

$$(-\dot{p}(t), \dot{\bar{x}}(t)) \in \text{co } \partial_{x,p} H(t, \bar{x}(t), p(t)) \text{ a.e.}$$

Note: **convexification w.r.t. two variables** can be replaced by

$$(-\dot{p}(\cdot) \in \text{co}\{(\xi \mid (\xi, \dot{\bar{x}}(t)) \in \partial_{x,p} H(t, \bar{x}(t), p(t)) \text{ a.e.}$$

which involves **convexification w.r.t. only one variable**

## 2. Strengthen Topology Defining ‘Local Minimizer’

The Hamiltonian inclusion is valid for  $W^{1,1}$ -local minimizers (not just  $L^\infty$ -local minimizers)

# Refined Hamiltonian Inclusion

**Theorem** (Loewen Rockafellar, 1994)

Take an  $W^{1,1}$ -local minimizer  $\bar{x}(\cdot)$ . Assume standard hypotheses and

**(C):**  $F(x)$  is convex for all  $x$ .

Then there exists  $p(\cdot) \in AC$  and  $\lambda \geq 0$  such that

**(i):**  $(p(\cdot), \lambda) \neq (0, 0)$

**(ii):**  $(-\dot{p}(t) \in \text{co}\{(\xi \mid (\xi, \dot{\bar{x}}(t)) \in \partial_{x,p}H(t, \bar{x}(t), p(t)) \text{ a.e.}$

**(iii)**  $(p(0), -p(T)) \in \lambda \partial g(\bar{x}(0), \bar{x}(T)) + N_C(\bar{x}(0), \bar{x}(T))$

**Note:** **(iii)**  $\implies p(t) \cdot \dot{\bar{x}}(t) = \max\{p(t) \cdot v \mid v \in F(\bar{x}(t))\}$

**(Weierstrass Condition)**

# The Generalized Euler-Lagrange Inclusion

Research direction since early 1990's: Find necessary conditions when  $F(x)$  is non-convex.

Generalizations of the classical Euler-Lagrange condition have been the most fruitful.

What should such conditions look like?

Here, generalizations of the classical Euler-Lagrange condition have been the most fruitful.

**(Mordukhovich, Ioffe, Rockafellar, Vinter)**

# E-L inclusion: B. Mordukhovich and A.D. Ioffe





# Generalized E-L Condition, Cont.

Differential inclusion problem formulated as extended Bolza problem:

Minimize  $\{\int_0^T L(t, x(t), \dot{x}(t)) + g(x(0), x(1))\}$  over  $x(\cdot) \in AC$

where

$$L(x, v) = \mathbb{I}_{\text{Gr } F(t, \cdot)}(x, v)$$

But

$$\partial \mathbb{I}_{\text{Gr } F(t, \cdot)}(x, v) = N_{\text{Gr } F(t, \cdot)}(x, v)$$

So natural generalization of Euler Lagrange condition is

$$(\dot{p}(t), p(t)) \in \text{co } N_{\text{Gr } F(\cdot)}(x, v) \quad ?$$

# Generalized E-L Condition, Cont.

Fully convexified E-L Inclusion is inadequate condition!

**Example:**  $F(x) = \{|x|\}$ ,  $\bar{x} = 0$  and  $\bar{v} = 0$  (Lipschitz data)

$$\text{Gr } F(\cdot) = \{(x, |x|) \mid x \in R\}$$

$$N_{\text{Gr } F(\cdot)}(\bar{x}, \bar{v}) = \{\alpha(1, 1) : \alpha \in R\} \cup \{\alpha(-1, 1) : \alpha \in R\}$$

$$\text{co } N_{\text{Gr } F(\cdot)}(\bar{x}, \bar{v}) = R \times R$$

$$\text{So } (\dot{p}, p) \in \text{co } N_{\text{Gr } F(\cdot)}(\bar{x}, \bar{v}) \implies (\dot{p}, p) \in R \times R$$

**But** **no useful information**

$$\dot{p} \in \text{co}\{\xi \mid (\xi, p) \in N_{\text{Gr } F(\cdot)}(0, 0)\} \implies |\dot{p}| \leq |p|$$

**(partially convexification gives much more information)**

**'right condition'**

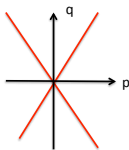
# Normal cones to $\text{Gr } F(\cdot)$

Take  $F(x) = \{|x|\}$

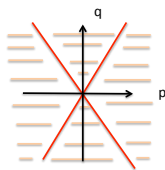
(a):  $(q, p) \in N_{\text{Gr } F(\cdot)}(0, 0)$

(b):  $(q, p) \in \text{co } N_{\text{Gr } F(\cdot)}(0, 0)$

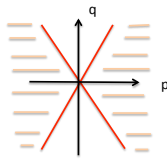
(c):  $(q, p)$  s.t.  $q \in \text{co}\{\xi \mid (\xi, p) \in N_{\text{Gr } F(\cdot)}(0, 0)\}$



(a)



(b)



(c)

# Nec. Condition for Non-Convex $F(x)$

## Theorem (loffe 1997)

Take an  $W^{1,1}$ -local minimizer  $\bar{x}(\cdot)$ . Assume standard hypotheses

Then there exists  $p(\cdot) \in AC$  and  $\lambda \geq 0$  such that

- (i):  $(p(\cdot), \lambda) \neq (0, 0)$
- (ii):  $(-\dot{p}(t) \in \text{co}\{(\xi \mid (\xi, p(t)) \in N_{\text{Gr } F(t, \cdot)}(\bar{x}(t), \bar{x}(t)) \text{ a.e.}$
- (iii)  $(p(0), -p(T)) \in \lambda \partial g(\bar{x}(0), \bar{x}(T)) + N_C(\bar{x}(0), \bar{x}(T))$
- (iv):  $p(t) \cdot \dot{\bar{x}}(t) = \max\{p(t) \cdot v \mid v \in F(t, \bar{x}(t))\} \text{ a.e.}$

(Earlier version without Weierstrass condition due to Morduckhovich 1995. Extensions allow unbounded  $F(\cdot)$ , state constraints etc. loffe, Vinter, Clarke . . .)

**Notes: No convexity of  $F(\cdot)$ . Must add Weierstrass Cond.**

Different necessary conditions vary according to costate equation or inclusion.

Euler Lagrange inclusion subsumes earlier 'Hamiltonian inclusion' conditions

**Theorem** (Rockafellar 1996) Assume standard conditions and

**(C):**  $F(x)$  is convex for each  $x$  (convexity)

Then

$$\text{co} \{q \mid (q, p(t)) \in N_{\text{Gr}\{F(t, \cdot)\}}(\bar{x}(t), \dot{\bar{x}}(t))\} \subset \\ \text{co} \{-q \mid (q, \bar{x}(t)) \in \partial_{x,p} H(t, \bar{x}(t), p(t))\} .$$

# Necessary Conditions: State of the Art 2005

Assume:

- Standard hypotheses
- Assume  $\bar{x}(\cdot)$  is a  $W^{1,1}$  local minimizer

Then there exists (non-trivial)  $(p(\cdot)) \in W^{1,1}$ ,  $\lambda \geq 0$ , satisfying transversality and Weierstrass conditions, s.t.

$$(-\dot{p}(t) \in \text{co}\{(\xi \mid (\xi, p(t)) \in N_{\text{Gr } F(t, \cdot)}(\bar{x}(t), \dot{\bar{x}}(t)) \text{ a.e. (1)}$$

If also (Euler-Lagrange Inclusion)

- (C):  $F(t, x)$  is convex for all  $x$

Then (1) implies

$$(-\dot{p}(t) \in \text{co}\{(\xi \mid (\xi, \dot{\bar{x}}(t)) \in \partial H(\bar{x}(t)'p(t)) \text{ a.e.}$$

(Partially Convexified Hamiltonian Inclusion)

Rockafellar Duality is only valid for convex  $F(x)$ 's.

**Open the question:** Are the conditions

$$(-\dot{p}(\cdot), \dot{\bar{x}}(t) \in \text{co } \partial_{x,p}H(t, \bar{x}(t), p(t)) \text{ a.e.}$$

or **(Fully Convexified Hamiltonian Inclusion)**

$$(-\dot{p}(t) \in \text{co}\{(q \mid (q, \dot{\bar{x}}(t)) \in \partial_{x,p}H(t, \bar{x}(t), p(t)) \text{ a.e.}$$

**(Partially Convexified Hamiltonian Inclusion)**

valid, when  $F(x)$  is not convex?

This was partly resolved by Clarke in 2005.

# Hamiltonian Inclusion for Non-Convex $F(x)$

## Theorem (Clarke 2005)

Assume Standard Hypotheses and

- $\bar{x}(\cdot)$  is  $L^\infty$ -local minimizer.

Then there exists non-trivial  $p(\cdot) \in W^{1,1}$  and  $\lambda \geq 0$  such that

(i):  $(-\dot{p}(t), \dot{\bar{x}}(t)) \in \text{co } \partial_{x,p} H(t, \bar{x}(t), p(t)) \quad \text{a.e.}$

(Fully Convexified Hamiltonian inclusion)

(ii)  $(p(0), -p(T)) \in \lambda \partial g(\bar{x}(0), \bar{x}(T)) + N_C(\bar{x}(0), \bar{x}(T))$

**Notes:  $L^\infty$ -local minimizer and fully convexified Hamiltonian Inclusion**



# Remaining Open Questions

This work validates (for non-Convex  $F(x)$ 's)

The Fully Convexified Hamiltonian Inclusion for  $L^\infty$ -minimizers.

BUT, It appeared to be an intermediate result to proving

The Partially Convexified Hamiltonian Inclusion for  $W^{1,1}$ -minimizers.

The status of these conditions is clarified by a recent counter-example.

## Proposition. (Vinter 2014)

For some choice of data for  $(P)$  satisfying the Standard Hypotheses such that

$F(x)$  is not convex,

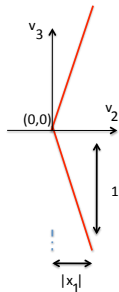
and some  $F$ -trajectory  $\bar{x}(\cdot)$ , we have

- $(\bar{x}(\cdot))$  is a  $W^{1,1}$  local minimizer,
- $(\bar{x}(\cdot))$  is not a  $L^\infty$  local minimizer,
- There exist non-trivial multipliers satisfying (the transversality condition and) the fully convexified Hamiltonian inclusion.
- There exist do not exist non-trivial multipliers satisfying t the fully convexified Hamiltonian inclusion.

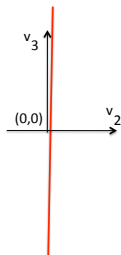
# Example

$$(E) \begin{cases} \text{Minimize } \frac{1}{2}x_1(1) - x_2(1) \\ \text{over } x(\cdot) = (x_1(\cdot), x_2(\cdot), x_3(\cdot)) \text{ satisfying} \\ \dot{x}_1(t) = 0, (\dot{x}_2, \dot{x}_3) \in \tilde{F}(x_1) \\ x_1(0) \geq 0 \end{cases}$$

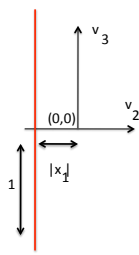
(i):  $x_1 > 0$



(ii):  $x_1 = 0$

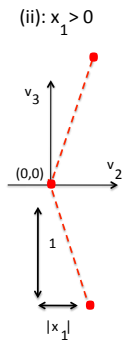


(iii):  $x_1 < 0$



$(v_2, v_3)$  coordinates of points  $(v_1, v_2, v_3) \in F(x)$

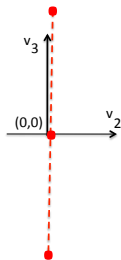
# Differential Inclusion $\tilde{F}(x_1), x_1 > 0$



$(v_2, v_3)$  coordinates of points  $(v_1, v_2, v_3) \in F(x)$

# Differential Inclusion $\tilde{F}(x_1), x_1 = 0$

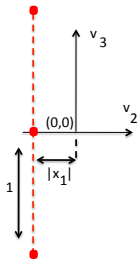
(ii):  $x_1 = 0$



$(v_2, v_3)$  coordinates of points  $(v_1, v_2, v_3) \in F(x)$

# Differential Inclusion $\tilde{F}(x_1)$ , $x_1 < 0$

(iii):  $x_1 < 0$



$(v_2, v_3)$  coordinates of points  $(v_1, v_2, v_3) \in F(x)$

# Example , continued

The example is very benign-seeming:

- Free right end-point problem.
- Can be re-formulated as a 'finite Lagrangian problem'
- $F(x)$  is not convex for all  $x$ , **BUT  $F(\bar{x}(t))$  is convex**

Note that

- **'Non-smooth' of  $x \rightarrow F(x)$  is essential feature of example**

## Fully convexified Hamiltonian Inclusion is 2nd Class Citizen!

- The Hamilton Inclusion is valid, in general, only if  $\bar{x}(\cdot)$  is a  $L^\infty$  minimizer
- Gives new information in some cases:  
'Hamiltonian inclusion is not satisfied'  
 $\implies$  ' $\bar{x}(\cdot)$  is not a minimizer.'
- **Degeneracy:** Conditions can be given for the Hamiltonian inclusion to be non-degenerate when all other known conditions are degenerate.
- **Relaxation:** 'normality of Hamiltonian inclusion'  $\implies$  'relaxation does not reduce minimum cost'

(not known for other necessary conditions)



# Some Concluding Remarks

- We have focused on necessary conditions for the most basic of differential inclusion problems. These are the building blocks for studying other problems - state constrained, impulse, hybrid., retarded . .
- The significance of 'differential inclusion' necessary conditions is to study structural properties of optimal controls.
- Pontryagin Maximum Principle is usually better for solving specific problems/ inspiring numerical methods
- The search for differential inclusions necessary conditions has had an enormous influence on development of techniques of nonlinear analysis.
- **Open problem: validity of the partially convexified HI for non-convex  $F(t, x)$ ,**

Thank you

and

Happy holidays!