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# SOME OLD AND NEW PERSPECTIVES FOR DIFFERENTIAL GAMES

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# 1. ZERO SUM DIFFERENTIAL GAMES

## DYNAMICS:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \alpha, \beta) & (0 \leq t \leq \tau) \\ \mathbf{x}(0) = \mathbf{x}, \end{cases}$$

## CONTROLS:

$$\begin{cases} \alpha : [0, \tau] \rightarrow A = \text{control for the maximizing player} \\ \beta : [0, \tau] \rightarrow B = \text{control for the minimizing player} \end{cases}$$

## PAYOFF:

$$J^{\alpha, \beta}(\mathbf{x}) = \int_0^{\tau} r(\mathbf{x}, \alpha, \beta) dt + g(\mathbf{x}(\tau)).$$

$r$  is the **running payoff** and  $g$  is the **terminal payoff**.

## MINIMAX CONDITION:

$$\min_{b \in B} \max_{a \in A} \{f(x, a, b) \cdot p + r(x, a, b)\} = \max_{a \in A} \min_{b \in B} \{f(x, a, b) \cdot p + r(x, a, b)\}$$

for all  $x, p$ .

## VALUE FUNCTION:

$u(x)$  = payoff to max player, if each plays optimally

## THEOREM

The value function  $u$ , if continuous, solves the **Hamilton–Jacobi–Isaacs (HJI)** equation

$$H(Du, x) = 0$$

in the viscosity sense, for the **game theory Hamiltonian**

$$H(p, x) = - \min_{b \in B} \max_{a \in A} \{f(x, a, b) \cdot p + r(x, a, b)\}.$$

# References

- M. Bardi and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuser, 1997.
- A. Bressan, *Noncooperative Differential Games. A tutorial*, [www.math.psu.edu/bressan/PSPDF/game-lnew.pdf](http://www.math.psu.edu/bressan/PSPDF/game-lnew.pdf)
- P. Cardaliaguet, *Introduction to Differential Games*, [www.ceremade.dauphine.fr/~cardalia/CoursJeuxDiff.pdf](http://www.ceremade.dauphine.fr/~cardalia/CoursJeuxDiff.pdf)
- R. Isaacs, *Differential Games. A mathematical theory with applications to warfare and pursuit, control and optimization*, Wiley, 1965.
- J. Lewin, *Differential Games. Theory and methods for solving game problems with singular surfaces*, Springer, 1994.
- A. Melikyan, *Generalized Characteristics of First Order PDEs. Applications in optimal control and differential games*, Birkhäuser, 1998.

# Problems for theory applied to “real” examples

- Hamiltonian  $H = H(p, x)$  is nonconvex in  $p$ .
- Hamiltonian  $H$  is not  $C^1$  (and therefore problems with characteristics = optimal trajectories).
- The value function  $u$  need not be continuous.
- Boundary conditions are unclear and/or hard to interpret in viscosity sense.

Failure of minimax condition is usually not a problem, at least for examples in Isaacs' and Lewin's books.

# AN (UNNAMED) EXAMPLE OF ISAACS

## DYNAMICS:

$$\begin{cases} \dot{x} = c(y) + \cos \alpha \\ \dot{y} = 2\beta + \sin \alpha, \end{cases}$$

where  $0 \leq \alpha \leq 2\pi$  and  $-1 \leq \beta \leq 1$ .

$y \mapsto c(y)$  is positive, increasing.

The game is played in the upper half plane  $U := \mathbb{R}^2 \cap \{y \geq 0\}$ .

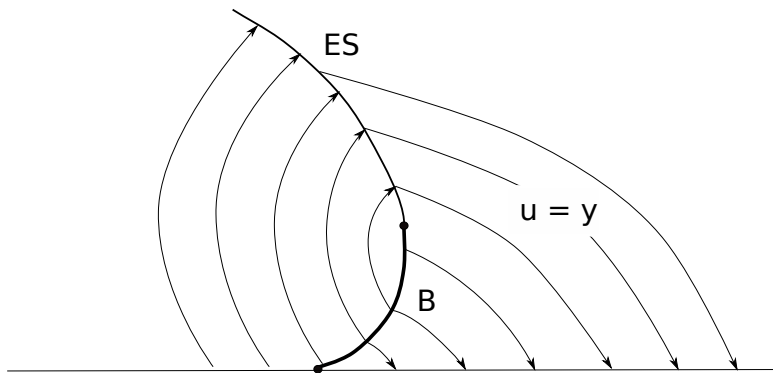
## PAYOFF:

$J^{\alpha, \beta} = \tau =$  exit time from  $U$  along the positive  $x$ -axis.

**HJI EQUATION:** The value function  $u$  (= time to exit) solves

$$H(Du, y) = -c(y)u_x - |Du| + 2|u_y| - 1 = 0 \quad \text{in } U.$$

Boundary condition is  $u = 0$  on  $\{x \geq 0, y = 0\}$ .



**B** = barrier; **ES** = "equivocal surface"

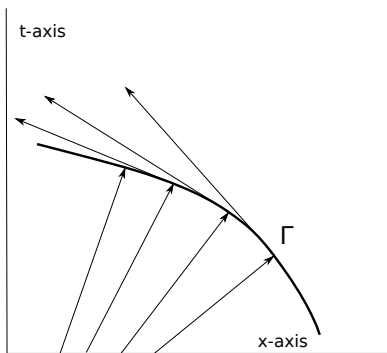


## 2. NONCONVEX HAMILTON-JACOBI PDE

Study the initial-value problem

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

where  $H$  is smooth, but not convex.



The central difficulty for nonconvex Hamilton–Jacobi PDE is that backwards characteristics can hit “shocks”, along which the gradient is discontinuous.

New work of F. Rezakhanlou: Backwards characteristics as martingales.

## FORMULAS FOR SOLUTIONS:

- If  $H$  is convex,

$$u(x, t) := \inf_{y \in \mathbb{R}^n} \left\{ g(y) + tL\left(\frac{x-y}{t}\right) \right\} \quad (\text{Hopf–Lax})$$

where  $L = H^*$

- If  $g$  is convex and has linear growth,

$$u(x, t) := \sup_{z \in \mathbb{R}^n} \{ x \cdot z - g^*(z) - tH(z) \} \quad (\text{Hopf})$$

## THEOREM (ENVELOPE REPRESENTATION FORMULA)

For  $\mathcal{L}^{n+1}$ -a.e. point  $(x, t) \in \mathbb{R}^n \times (0, \infty)$  there exists a Radon probability measure  $\gamma_{x,t}$  on  $\mathbb{R}^n$  such that

$$u(x, t) = \int_{\mathbb{R}^n} g(y) + (x - y) \cdot Dg(y) - tH(Dg(y)) d\gamma_{x,t}.$$

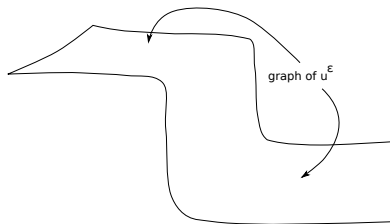
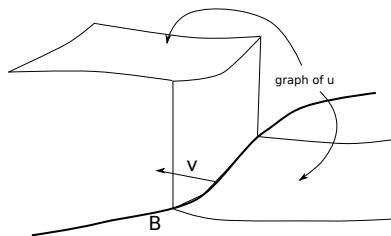
Furthermore,

$$u_t(x, t) = - \int_{\mathbb{R}^n} H(Dg(y)) d\gamma_{x,t}, \quad Du(x, t) = \int_{\mathbb{R}^n} Dg(y) d\gamma_{x,t},$$

and

$$H\left(\int_{\mathbb{R}^n} Dg(y) d\gamma_{x,t}\right) = \int_{\mathbb{R}^n} H(Dg(y)) d\gamma_{x,t}.$$

# BARRIERS (= surfaces of discontinuity)



$$H(Du, x) = 0 \quad \text{in } U.$$

## APPROXIMATION:

$$-\epsilon \Delta u^\epsilon + H(Du^\epsilon, x) = 0 \quad \text{in } U.$$

## Assume:

$$u^\epsilon \leq u^+ + o(1), \quad u^\epsilon \geq u^- - o(1).$$

The **recession function** associated with  $H = H(p, x)$  is

$$K(p, x) := \lim_{\lambda \rightarrow \infty} \frac{H(q + \lambda p, x)}{\lambda}.$$

## THEOREM

(i)

$$\begin{cases} H(Du^+ + \lambda \nu, x) \leq 0 \\ H(Du^- + \lambda \nu, x) \geq 0 \end{cases} \quad \text{for all } \lambda \geq 0.$$

(ii) *In particular,*

$$K(\nu, x) = 0.$$

**This is a PDE for the barrier (Isaacs).** We can in principle solve for the location of the discontinuities, before we solve the full HJI equation!

### 3. NONZERO SUM GAMES: PRINCIPAL-AGENT PROBLEMS

#### SANNIKOV'S MODEL (simplified)

#### DYNAMICS:

$$dX = A dt + dB,$$

where  $B$  is a one-dimensional Brownian motion.

#### CONTROLS:

$$\begin{cases} C = \text{principal's control (= payments to agent)} \\ A = \text{agent's control of dynamics.} \end{cases}$$

#### PAYOFFS:

$$\begin{cases} J^P = \text{principal's payoff} = E \left( \int_0^\infty e^{-t} (A - C) dt \right) \\ J^A = \text{agent's payoff} = E \left( \int_0^\infty e^{-t} (g(C) - h(A)) dt \right) \end{cases}$$

# How can we make the agent do what we want?

In particular, how can the principal encourage the agent to perform the action  $A^*$ ?

**KEY POINT #1:** The agent's actions and the Brownian motion are unobservable by the principal; she can only observe  $X$ .

**DYNAMICS REINTERPRETED:**

$$\begin{aligned}dX &= A^* dt + dB \\ &= A dt + dB^A\end{aligned}$$

**KEY POINT #2:**  $B$  is a Brownian motion under the original probability measure  $P$  and  $B^A$  is a Brownian motion under a new probability measure  $P^A$ .

**PAYOFFS REINTERPRETED:**

$$\begin{cases} J^P = E^A \left( \int_0^\infty e^{-t} (A - C) dt \right) \\ J^A = E^A \left( \int_0^\infty e^{-t} (g(C) - h(A)) dt \right) \end{cases}$$

**NEW DYNAMICS:** Given  $A^* = a(W)$  and  $C^* = c(W)$ , solve

$$\begin{aligned}dW &= (W - g(C^*) + h(A^*))dt + h'(A^*)dB \\ &= (W - g(C^*) + h(A^*) - A^* h'(A^*))dt + h'(A^*)dX \\ W(0) &= w.\end{aligned}$$

The principal will therefore pay the agent  $C^* = c(W)$ , depending upon the values of  $W$ , which she can compute in terms of  $X$ .

**THEOREM** (Sannikov optimality condition for agent)

*Assume*

*$h$  is convex.*

*Then it is optimal for the agent to use the control  $A^*$ .*



## New proof of optimality condition for agent

$$\begin{aligned}d(e^{-t}W) &= e^{-t} [(-g(C^*) + h(A^*))dt + h'(A^*)dB] \\ &= e^{-t} [(-g(C^*) + h(A^*) + h'(A^*)(A - A^*))dt + h'(A^*)dB^A]\end{aligned}$$

Integrate from 0 to  $\infty$ :

$$\begin{aligned}-w &= \int_0^\infty e^{-t}(-g(C^*) + h(A^*) + h'(A^*)(A - A^*)) dt \\ &\quad + \int_0^\infty e^{-t}h'(A^*) dB^A\end{aligned}$$

Take expected value with respect to  $P^A$  and rewrite:

$$\begin{aligned}w &= E^A \left( \int_0^\infty e^{-t}(g(C^*) - h(A)) dt \right) \\ &\quad + E^A \left( \int_0^\infty e^{-t}(h(A) - h(A^*) - h'(A^*)(A - A^*)) dt \right)\end{aligned}$$

By convexity, the last term is nonnegative, and is zero for  $A = A^*$ .



**EXERCISE:** Use dynamic programming to rederive this optimality condition. What is the agent's value function?

**THEOREM** (Optimality condition for principal)

The principal's value function  $u = u(w)$  solves the HJB equation

$$\max_{a,c} \left\{ \frac{(h'(a))^2}{2} u'' + (w - g(c) + h(a))u' - u + a - c \right\} = 0$$

Select

$$a(w), c(w)$$

giving the max in the HJB equation.

- I.Ekeland, *How to build stable relationships with people who lie and cheat*, [www.ceremade.dauphine.fr/~ekeland/Articles/13-03-08-ArticleMilano.pdf](http://www.ceremade.dauphine.fr/~ekeland/Articles/13-03-08-ArticleMilano.pdf)

Many fascinating PDE/optimality issues remain open.