Regular and singular points for linear minimum time problems

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Joint works with

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Consider the controlled dynamics

$$\dot{x} = f(x) + g(x)u, \qquad x(0) = \xi \tag{1}$$

$\xi \in \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$, U compact and convex.

We are given a closed target S.

Let $\mathcal{T}(\xi)$ be the minimum time to reach the target S from ξ, i.e.,

$$\mathcal{T}(\xi) = \text{inf}\{t : y(t) \in S,\ y \text{ is a solution of (1)}\} \Big(\in [0, +\infty] \Big).$$

$\mathcal{T}$ is practically never everywhere differentiable, however in all examples it is differentiable at “most” points. We have been trying to understand precisely where and why $\mathcal{T}$ is not differentiable and study the set of singularities (i.e., the nondifferentiability set), as well as identifying suitable regularity properties satisfied by $\mathcal{T}$.

The regularity of the minimum time function is in fact a widely studied topic under several viewpoints.

- Analytic stratification (Brunovský, Sussmann, '70).
- Semiconcavity/semiconvexity (or similar properties which do not imply local Lipschitz continuity) of $\mathcal{T}$ (Cannarsa and Sinestrari (Calc. Var. (1995) and book by Birkhäuser (2004), C.-Marigonda-Wolenski (2006), C.-Nguyen T. Khai (2010), Khai-Cannarsa and others), which gives information on the structure of the singular set as well as on higher order a.e. differentiability and on representation of the generalized gradient, together with a reasonable feedback concept. (Semi-concavity/-convexity $\sim$ quadratic perturbation of concavity/convexity $\Rightarrow$ structured singularities.)
- Variational analysis: computing generalized gradients of $\mathcal{T}$ (C.-Wolenski, Mordukhovich, C.-Khai, . . .).
- Complete description of $\mathcal{T}$ as well as of optimal synthesis, under generic conditions, in 2 D with the origin as target (Boscain-Piccoli); also results by Schaettler in 3 D.

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- nonsmoothness of the target
- nonuniqueness of optimal trajectories
- discontinuities of optimal controls (=switchings)

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In particular, the two last facts give raise to different types of singularities (downward or upward kinks/cusps).

Switchings are connected with higher order controllability assumptions and singularities of $\mathcal{T}$ may be of two types: non-Lipschitz points (i.e., the graph has at least one horizontal normal ray) and kinks (i.e., the epi (hypo)graph has multiple normals).

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A general message: that common believe is not to be taken as always true.

In this talk I will focus on linear and (for simplicity) single input dynamics, under normality assumptions. Namely we consider the problem of reaching the origin in minimum time from ξ subject to the dynamics

$$\dot{x} = Ax + bu, \qquad |u| \leq 1, \quad x \in \mathbb{R}^N,$$

such that the Kalman rank condition holds

$$\text{rk}\left[b, Ab, \ldots, A^{N-1}b\right] = N$$

(A is a $N \times N$ matrix, $b \in \mathbb{R}^N$).

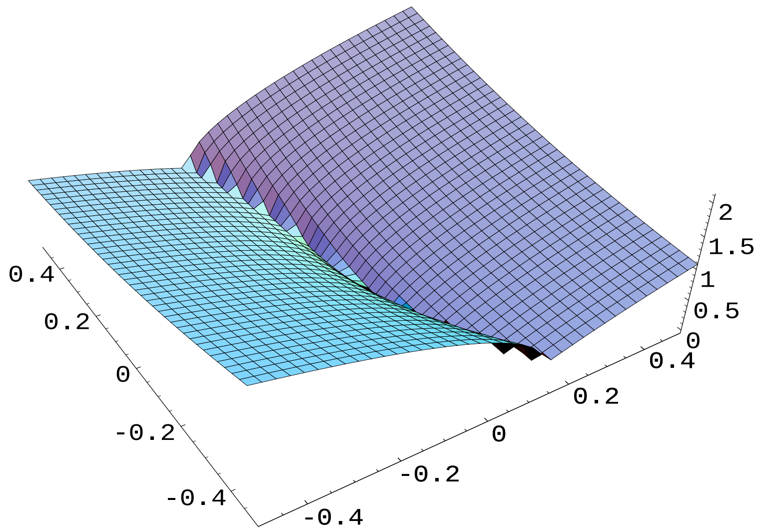
This is a very classical topic, which was studied a lot in the ’70. We try to bring some new idea and new result, considering it as a model problem for possible future developments.

Example 1 (rocket car). Consider the problem of reaching in minimum time the origin subject to the dynamics

$$\ddot{x} = u \in [-1, 1].$$

The minimum time function is:

$$\mathcal{T}(x,y) = \begin{cases} y + 2\sqrt{y^2/2 + x} & \text{for} \quad x \geq -y|y|/2, \\ -y + 2\sqrt{y^2/2 - x} & \text{for} \quad x < -y|y|/2 \end{cases}$$

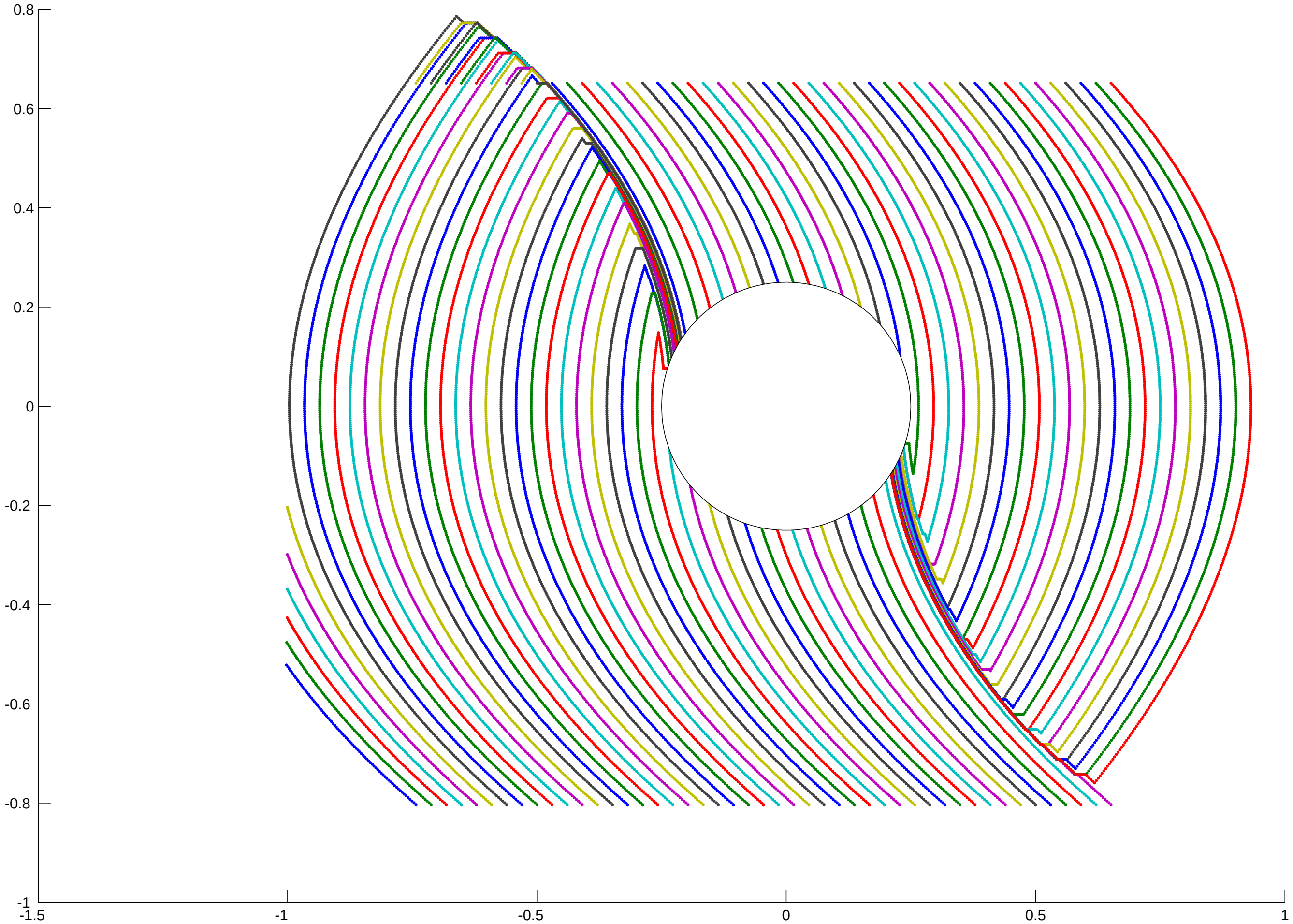


Example 2 (rocket car). Consider the problem of reaching in minimum time a small ball centered at the origin subject to the dynamics

$$\ddot{x} = u \in [-1, 1].$$

In the next slide we will see that computational problems seem to occur close to the non-Lipschitz/non-differentiability locus (algorithm and computation by Le T.T. Thuy: this is an advertisement of her poster! [Thursday]).

I feel it makes sense trying to identify where one can find such points, in order to refine the computation only where it is needed.



0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.5
-1
-0.5
0
0.5
1

What is known “since ever” on this problem:

– small time controllability holds (i.e., $\mathcal{T}$ is finite and ($\frac{1}{N}$-Hölder)-continuous in a ("big") neighborhood of the origin); the lack of Lipschitz continuity of $\mathcal{T}$ is due to higher order controllability conditions;

– sublevels of $\mathcal{T}$ ($\mathcal{R}_T = \{x : \mathcal{T}(x) \leq T\}$) are strictly convex;

– the optimal control steering ξ to the origin is unique and bang-bang;

– (Brunovský) there exists a regular time optimal synthesis : general proof, constructive in $\mathbb{R}^2$;

– (Hájek) there exists a time $\epsilon > 0$ such that $\mathcal{R}_\epsilon$ contains an open dense set where $\mathcal{T}$ is analytic and where a time optimal feedback $u(x)$ is well defined, with kind of explicit form (and the corresponding [nonlinear, discontinuos] ODE $\dot{x} = Ax + bu(x)$ has a solution).

C.+Nguyen T. Khai have identified a modulus of strict convexity for $\mathcal{R}_T$ (MCRF 2013).

From the above results one can understand in which sense our linear minimum time problem is a model one: it is focused on singularities due to switchings, no other singularities can occur.

Let $x \in \partial\mathcal{R}_T$ (i. e., $\mathcal{T}(x) = T$) and let ζ be normal to $\mathcal{R}_T$ at x. Pontryagin's Maximum Principle implies that the optimal control steering x to the origin is

$$u_x(t) = -\mathrm{sign}\Big(\langle \zeta, \mathrm{e}^{\mathrm{A}t}\mathrm{b}\rangle\Big).$$

In other words, the driving force of such a minimum time problem is the switching function

$$g_\zeta(t) = \langle \zeta, e^{At} b\rangle.$$

In our case, g is analytic and is not $\equiv 0$ (rank condition), whence every time optimal control is bang-bang and has finitely many switchings. Moreover, zeros of the switching function impose conditions on normals to reachable sets (linear conditions on normal vectors): actually the *normal cone* to reachable sets is characterized by such zeros. In fact, Hájek's results are based on the fact that for small time all zeros of the switching function can be taken of first order and imposing linearly independent conditions on the normal vector ζ (i.e., $e^{At_i}b$ are linearly independent, t_i are the zeros).

The (minimized) Hamiltonian:

$$\begin{aligned} h(x,p) &= \min_{u\in\mathcal{U}}\Big(\langle f(x),p\rangle + \langle g(x)u,p\rangle\Big) \\ &= \langle Ax,p\rangle + \min_{|u|\le 1} u\langle b,p\rangle \\ &= \langle Ax,p\rangle - |\langle b,p\rangle|. \end{aligned}$$

Classical result (Bardi): the minimum time function is the unique bounded below (viscosity) solution of the boundary value problem

$$h(x,\nabla\mathcal{T}(x)) = -1,\ \mathcal{T}(0)=0,\ \lim_{x\to\partial\mathcal{R}}\mathcal{T}(x)=+\infty$$

on the reachable set $\mathcal{R}$, which is open.

RESULTS

The Hamiltonian detects points around which $\mathcal{T}$ is not Lipschitz:

Theorem 1. (Colombo, Nguyen T. Khai, Nguyen V. Luong (Calc. Var. PDE's, in print)).
$\mathcal{T}$ is not Lipschitz around x if and only if there exists a nonzero normal ζ to $\mathcal{R}_{\mathcal{T}(x)}$ at x such that $h(x,\zeta) = 0$.
More precisely, ζ is normal to $\mathcal{R}_T$ at x if and only if $(\zeta, h(x,\zeta))$ is normal to the epigraph of $\mathcal{T}$. In particular: corner singularities of $\mathcal{T}$ (kinks) are due only to kinks of reachable sets.

Recall that such normal cone is everywhere nontrivial (i.e., contains non-zero vectors) and $\mathcal{T}$ is a.e. (twice) differentiable.
(This last statement was proved, as a consequence of other regularities properties, in a paper by C., Marigonda & Wolenski (SICON 2006)).

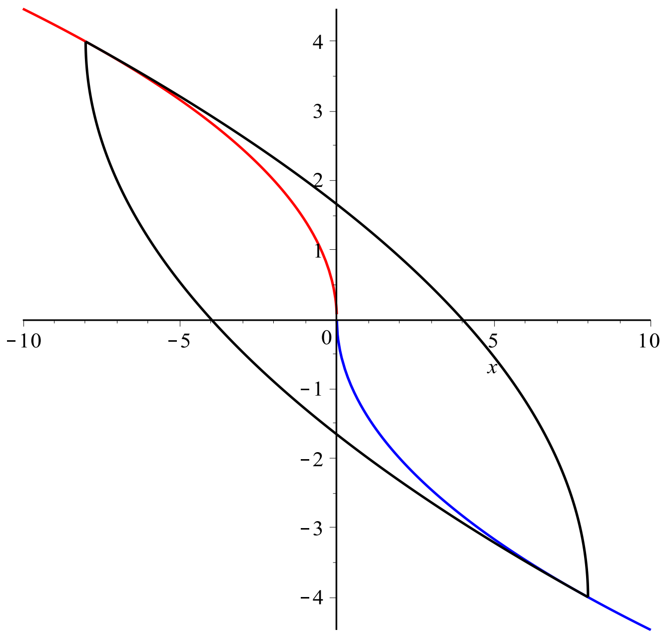
Since the Hamiltonian is constant along optimal trajectories, one can explicitly compute the non-Lipschitz set (that we call $\mathcal{S}$), through a connection with the location of zeros of the switching function: x is a non-Lipschitz point if and only if there exists a normal to $\mathcal{R}_T$ at x such that the corresponding switching function vanishes at $t = 0$. Moreover, $\mathcal{S}$ is invariant for optimal trajectories having vanishing Hamiltonian:

$$\mathcal{S} = \Big\{ x \in \mathbb{R}^N : \exists\, r > 0,\ \zeta \in \mathbb{S}^{N-1} \text{ such that } x = \int_0^r e^{A(t-r)} b \,\mathrm{sign}\left(\langle \zeta, e^{At} b \rangle\right) dt \text{ and } \langle \zeta, b \rangle = 0 \Big\}.$$

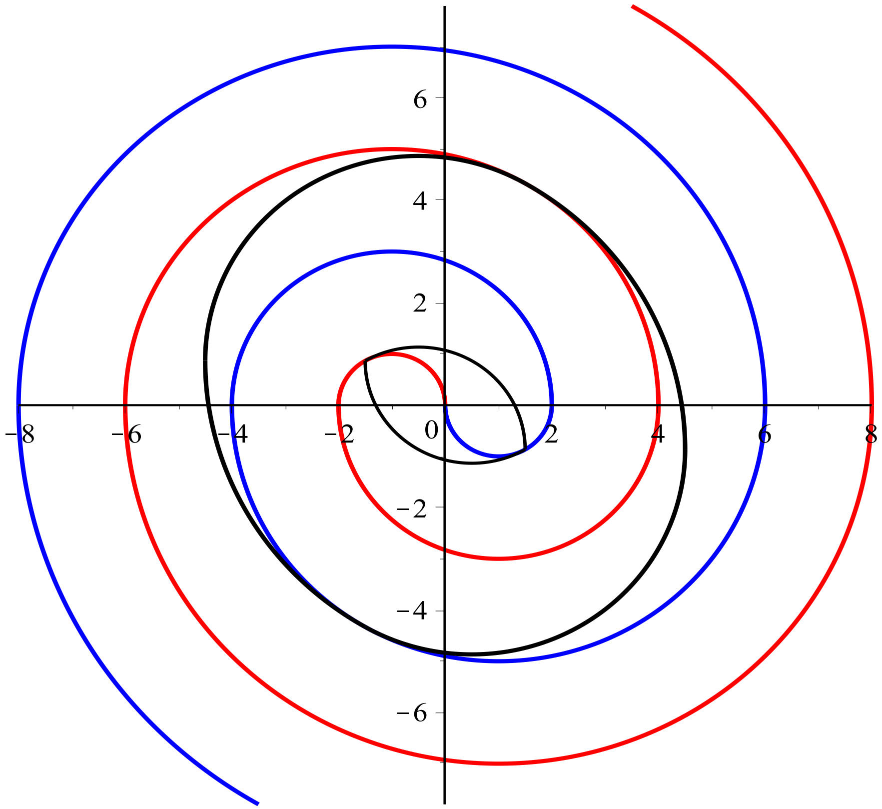
This representation permits to prove that $\mathcal{S}$ is countably $N-1$-rectifiable.

Two examples. The non-Lipschitz set $\mathcal{S}$ is the blue and red set, corresponding to the two optimal trajectories having vanishing Hamiltonian.

The rocket car: $\ddot{x} = u, \ |u| \leq 1$.



The controlled harmonic oscillator: $\ddot{x} + x = u, \ |u| \le 1$.



This is an interesting example: for $\mathcal{T} \ge \pi$ the graph of $\mathcal{T}$ is smooth, but $\mathcal{T}$ is nonsmooth (it has infinite partial derivatives at all points of $\mathcal{S}$).

In two dimensions there are essentially no other possibilities. In higher dimensions the situation is much more intricate, due to zeros of the switching function which may be of higher order (order k, $1 < k \leq N - 1$) and linearly dependent (i.e., $e^{At_1}, \ldots, e^{At_j}$ are linearly dependent). Apparently it is very difficult/hopeless describing precisely what happens in connection with zeros of the switching function.

Sussmann proved (1977, Helsinki) that $\mathcal{T}$ is piecewise analytic, using the theory of subanalytic sets and stratifications. He proved that the reachable set can be decomposed into a locally finite partition of analytic manifolds on each of which $\mathcal{T}$ is analytic. In particular, singularities occur only at smaller dimensional sets.

The proof relies on famous results on the stratification of subanalytic sets, which are not of constructive type.

Theorem 2. (C., Nguyen V. Luong) Consider a normal linear time optimal problem in $\mathbb{R}^N$. Then there exists an open set Ω such that

- $\mathcal{R} \setminus \Omega$ is countably $\mathcal{H}^{N-1}$-rectifiable;
- $\mathcal{R} \setminus \Omega$ can be described rather precisely using the exponential matrix e^{At} and the switching function (technical);
- $\mathcal{T}$ is analytic in Ω;
- in Ω, $\nabla\mathcal{T}$ is a classical solution of the system of PDE's

$$A\,\nabla T(x) + \Big(Ax - \mathrm{sign}(\langle \nabla T(x), b\rangle)\, b\Big)^{\top} \nabla^2 T(x) = 0.$$

Boundary conditions can also be specified.

Sussmann's result is on the one hand more precise ($\mathcal{R}$ can be decomposed into a locally finite union of analytic submanifolds, with $\mathcal{T}$ analytic in each of them), on the other we give information on the singular set and establish a relation between singularities and the set of points which can be reached by controls corresponding to a switching function with higher order zeros.

The difficulty in making an explicit construction of a stratification of the reachable set such that $\mathcal{T}$ is analytic on each stratum is handling higher order and/or linearly dependent zeros of the switching function.

Essentially we can prove that points whose corresponding switching function has higher order zeros (or more in general has some “bad behavior”) lie on a small (countably $N-1$-rectifiable) closed set and singular points lie there.

Technical remark: I’m not saying that the set of singularities is closed. I’m saying that it is contained in a (small) closed set.

One technical slide on proving $\mathcal{H}^{N-1}$-rectifiability for various exceptional sets.

A set E is $\mathcal{H}^k$-countably rectifiable if it is contained in the union of countably many images of Lipschitz functions of k real variables.

$$\mathcal{S} = \Big\{ x \in \mathbb{R}^N \ : \ \text{there exist } r > 0 \text{ and } \zeta \in \mathbb{S}^{N-1} \text{ such that}$$
$$x = \int_0^r e^{A(t-r)} b \,\text{sign}\left(\langle \zeta, e^{At} b\rangle\right) dt$$
$$\text{and } \langle \zeta, b\rangle = 0 \Big\}.$$

The set $\mathcal{S}$ is described using $N-1$ free parameters: $r \in \mathbb{R}$ and $\zeta \in \mathbb{S}^{N-1}$ subject to one linear condition, so $1 + N - 2$ parameters. The point is showing that one can split $\mathcal{S}$ into countably many Lipschitz graphs. Note that if $\bar{t}$ is a higher order zero of $g_{\bar{\zeta}}(\cdot)(= \langle \bar{\zeta}, e^{A\cdot} b\rangle)$ and ζ is close to $\bar{\zeta}$ then zeros of $g_\zeta(\cdot)$ around $\bar{t}$ are not a Lipschitz function of ζ.

Other sets of interest are parametrized by switching times. The point is proving that the set where the parametrization is not good (not a diffeomorphism) is $\mathcal{H}^{N-1}$-countably rectifiable. Of course, in general switchings are much more than $N-1$.

Future directions: identifying the set of non-Lipschitz points of $\mathcal{T}$ for classes of nonlinear systems through the Hamiltonian flow (with zero Hamiltonian) with the hope of designing a domain decomposition for computing $\mathcal{T}$ under weak controllability assumptions.

THANK YOU!