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Piermarco Cannarsa. Compactness estimates for Hamilton-Jacobi equations. NETCO 2014, Jun 2014, Tours, France. hal-01024237

HAL Id: hal-01024237

<https://inria.hal.science/hal-01024237>

Submitted on 15 Jul 2014

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Compactness estimates for Hamilton-Jacobi equations

Piermarco Cannarsa

University of Rome "Tor Vergata"

New trends in optimal control

TOURS, FRANCE

JUNE 23 – 27, 2014

VINCI INTERNATIONAL CONGRESS CENTER



Outline

- 1 Introduction to compactness estimates
 - Hamilton-Jacobi equations
 - Kolmogorov's ε -entropy
 - Compactness estimates for conservation laws
- 2 Compactness estimates for $u_t + H(\nabla u) = 0$
 - Upper estimate for $u_t + H(\nabla u) = 0$
 - Lower estimate for $u_t + H(\nabla u) = 0$
- 3 Compactness estimates for $u_t + H(x, \nabla u) = 0$



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Hamilton-Jacobi equations

$$\begin{cases} u_t(t, x) + H(x, \nabla u(t, x)) = 0 & (t, x) \in (0, T) \times \mathbb{R}^n \\ u(0, x) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$

where

- $H : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a C^2 smooth function such that

$$(a) \quad \lim_{|p| \rightarrow \infty} \inf_{x \in \mathbb{R}^n} \frac{H(x, p)}{|p|} = +\infty$$

$$(b) \quad D_p^2 H(x, p) \geq \alpha \cdot I_n, \quad \forall (x, p) \in \mathbb{R}^n \times \mathbb{R}^n$$

with $\alpha > 0$

- $u_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ is a Lipschitz function



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The simplest problem in the calculus of variations

Given a C^2 smooth function $L : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$(a) \quad \lim_{|q| \rightarrow \infty} \inf_{x \in \mathbb{R}^n} \frac{L(x, q)}{|q|} = +\infty$$

$$(b) \quad D_q^2 L(x, q) \geq \lambda \cdot \mathbb{I}_n, \quad \forall (x, q) \in \mathbb{R}^n \times \mathbb{R}^n \quad (\lambda > 0)$$

consider the problem of minimizing the functional

$$J(\xi) = \int_0^T L(\xi(t), \xi'(t)) dt$$

over all absolutely continuous arcs $\xi : [0, T] \rightarrow \mathbb{R}^n$ satisfying

$$\xi(0) = x_0 \quad \text{and} \quad \xi(T) = x_T$$

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The semigroup $S_t : \text{Lip}(\mathbb{R}^n) \rightarrow \text{Lip}(\mathbb{R}^n)$

Replacing the initial point constraint with the initial cost u_0 , consider

$$S_t(u_0)(x) = \inf_{\xi(t)=x} \left\{ \int_0^t L(\xi(s), \xi'(s)) dt + u_0(\xi(0)) \right\}$$

$V(t, x) = S_t(u_0)(x)$ viscosity solution



$$\begin{cases} V_t(t, x) + \underbrace{\sup_{q \in \mathbb{R}^n} \{ \langle q, \nabla V(t, x) \rangle - L(x, q) \}}_{H(x, \nabla V(t, x))} = 0 & (t, x) \in (0, T) \times \mathbb{R}^n \\ V(0, x) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$

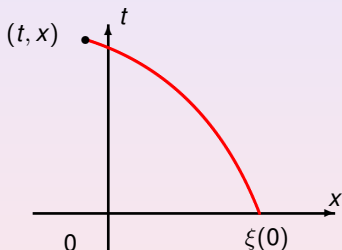


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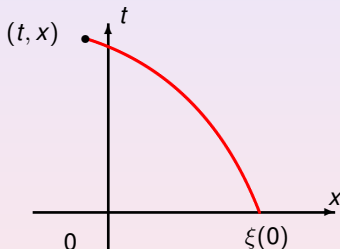


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Overview

Our goal

To study the **irreversibility** of the dynamics of

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Effects of irreversibility:

- **gain of regularity**: semiconcavity

$$S_t(u_0)(x+h) + S_t(u_0)(x-h) - 2S_t(u_0)(x) \leq C_t |h|^2$$

- **loss of regularity**: propagation of singularities

$$x'(t) \in \text{co } D_p H(x(t), \nabla^+ u(t, x(t)))$$

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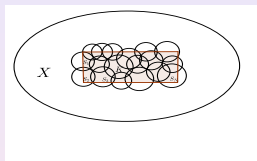
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A compactness measure: Kolmogorov's ε -entropy

Let (X, d) be a metric space and K a totally bounded subset of X



For any $\varepsilon > 0$, let $N_\varepsilon(K)$ be the minimal number of sets in a cover of K by subsets of X having diameter no larger than 2ε

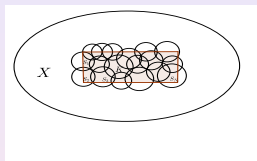
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Our goal

for a suitable class of **initial conditions** \mathcal{C} want to show

$$\mathcal{H}_\varepsilon(S_T(\mathcal{C}) | \mathbf{W}^{1,1}(\mathbb{R}^n)) = \log_2 N_\varepsilon(S_T(\mathcal{C})) \approx \varepsilon^{-n}$$

$N_\varepsilon(S_T(\mathcal{C})) =$ minimal number of subsets of $\mathbf{W}^{1,1}(\mathbb{R}^n)$ with $\text{diam} \leq 2\varepsilon$ covering $S_T(\mathcal{C})$

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Motivations

One relies on Kolmogorov's ε -entropy to:

- provide estimates on the **accuracy and resolution** of numerical methods
 - Lin – Tadmor, 2001: L^1 -Stability and error estimates for approximate Hamilton-Jacobi equations
- analyze **computational complexity** (derive minimum number of needed operations to compute solutions with an error $< \varepsilon$)
- derive **control theoretic properties** (exact/approximate controllability)

for Hamilton-Jacobi equations



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Scalar conservation laws

u is an entropy solution of

$$u_t + f(u)_x = 0 \quad \text{in } [0, +\infty) \times \mathbb{R}$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is (uniformly) strictly convex

$$f''(u) \geq c > 0 \quad \forall u \in \mathbb{R}$$

if

- u distributional solution

$$\iint [u\varphi_t + f(u)\varphi_x] dxdt = 0 \quad \forall \varphi \in C_c^1([0, +\infty) \times \mathbb{R}) \quad (D)$$

- Lax stability condition

$$u(t, x-) \geq u(t, x+) \quad \text{for a.e } t > 0, \quad \forall x \in \mathbb{R} \quad (E)$$



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Theorem (Lax, 1954)

The map $S_t : L^1(\mathbb{R}) \longrightarrow L^1_{\text{loc}}(\mathbb{R})$ is compact for every $t > 0$

A question (by Lax) :

is it possible to give a quantitative estimate of the compactness of S_t ?

1. De Lellis – Golse, 2005, A quantitative compactness for scalar conservation laws
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A special class of Hamilton-Jacobi equations

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with $H \in C^2(\mathbb{R}^n)$ satisfying

(H1) superlinearity: $\lim_{|p| \rightarrow \infty} \frac{H(p)}{|p|} = +\infty$

(H2) uniform convexity: $D^2 H(p) \geq \alpha \cdot \mathbb{I}_n, \quad \forall p \in \mathbb{R}^n$

where $\alpha > 0$ and \mathbb{I}_n is the identity $n \times n$ matrix

Legendre transform of H

$$H^*(q) = \max_{p \in \mathbb{R}^n} \{ \langle p, q \rangle - H(p) \} \quad (q \in \mathbb{R}^n)$$

is in turn superlinear and satisfies

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Hopf-Lax semigroup

For any $u_0 \in \text{Lip}(\mathbb{R}^n)$ the Cauchy problem

$$\begin{cases} u_t(t, x) + H(\nabla u(t, x)) = 0 & (t, x) \in [0, +\infty) \times \mathbb{R}^n \\ u(0, x) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$

admits a unique viscosity solution given by

$$u(t, x) = \min_{y \in \mathbb{R}^n} \left\{ t \cdot H^* \left(\frac{x-y}{t} \right) + u_0(y) \right\}, \quad \forall (t, x) \in (0, +\infty) \times \mathbb{R}^n$$

Our goal: to obtain upper and lower compactness estimates for

Hopf-Lax semigroup

$$\begin{cases} S_t : \text{Lip}(\mathbb{R}^n) \rightarrow \text{Lip}(\mathbb{R}^n) & t \geq 0 \\ S_t(u_0)(x) = \min_{y \in \mathbb{R}^n} \left\{ t \cdot H^* \left(\frac{x-y}{t} \right) + u_0(y) \right\} & x \in \mathbb{R}^n \end{cases}$$



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Our goal: to obtain upper and lower compactness estimates for

Hopf-Lax semigroup

$$\begin{cases} S_t : \text{Lip}(\mathbb{R}^n) \rightarrow \text{Lip}(\mathbb{R}^n) & t \geq 0 \\ S_t(u_0)(x) = \min_{y \in \mathbb{R}^n} \left\{ t \cdot H^* \left(\frac{x-y}{t} \right) + u_0(y) \right\} & x \in \mathbb{R}^n \end{cases}$$



Semiconcavity of the Hopf-Lax semigroup

Given $K, L, M > 0$, define

$$\begin{aligned} \mathcal{C}_{[L,M]} &= \{u \in \text{Lip}(\mathbb{R}^n) : \text{spt}(u) \subset [-L, L]^n, \text{Lip}[u] \leq M\} \\ \mathcal{SC}_{[K,L,M]} &= \{u \in \mathcal{C}_{[L,M]} : u(x+h) + u(x-h) - 2u(x) \leq K|h|^2, \forall x, h \in \mathbb{R}^n\} \end{aligned}$$

Proposition

For any $L, M, T > 0$ and every $u \in \mathcal{C}_{[L,M]}$

- 1 $S_T(u)$ is semiconcave with constant $\frac{1}{\alpha T}$
- 2 $\text{Lip}[S_T(u)] \leq M$
- 3 $\text{spt}(S_T(u) + T \cdot H(0)) \subset [-L_T, L_T]^n$ where $L_T = L + T \cdot \sup_{|p| \leq M} |DH(p)|$

Moreover

$$\begin{cases} u, u^\nu \in \text{Lip}(\mathbb{R}^n), & \text{Lip}[u^\nu] \leq M \\ u^\nu \xrightarrow{\nu \rightarrow \infty} u & \text{in } \mathbf{W}_{loc}^{1,1}(\mathbb{R}^n) \end{cases} \implies S_T u^\nu \xrightarrow{\nu \rightarrow \infty} S_T u \text{ in } \mathbf{W}_{loc}^{1,1}(\mathbb{R}^n)$$

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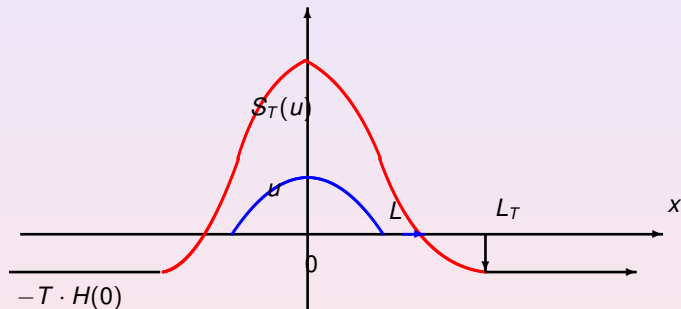
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Flow associated with H-L semigroup

$$S_T(C_{[L,M]}) + T \cdot H(0) \subset SC_{[\frac{1}{\alpha T}, L_T, M]}$$



$$C_{[L,M]} = \{u \in \text{Lip}(\mathbb{R}^n) : \text{spt}(u) \subset [-L, L]^n, \text{Lip}[u] \leq M\}$$

$$SC_{[K,L,M]} = \{u \in C_{[L,M]} : u \text{ semiconcave with constant } K\}$$

Outline

- 1 Introduction to compactness estimates
 - Hamilton-Jacobi equations
 - Kolmogorov's ε -entropy
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- 2 Compactness estimates for $u_t + H(\nabla u) = 0$
 - Upper estimate for $u_t + H(\nabla u) = 0$
 - Lower estimate for $u_t + H(\nabla u) = 0$
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Upper estimate

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Theorem (Ancona – C – Khai T. Nguyen, 2014)

For any $L, M, T > 0$ there exist constant $\varepsilon_0 = \varepsilon_0(L, M, T) > 0$ and $C = C(L, M, T) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,

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Hopf-Lax semigroup

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Main steps of the proof

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Monotone set-valued maps

Let $F : \mathbb{R}^n \rightarrow \mathcal{P}(\mathbb{R}^n)$ be a **set-valued map** ($\mathcal{P}(\mathbb{R}^n)$ = the set of all subsets of \mathbb{R}^n)
 We say that F is **monotone decreasing** if

$$\langle v_2 - v_1, x_2 - x_1 \rangle \leq 0, \quad \forall x_i \in \mathbb{R}^n, v_i \in F(x_i), i = 1, 2$$

Any monotone decreasing set-valued map F is **bounded and a.e. single-valued** in every open set $\Omega \subset \mathbb{R}^n$, relatively compact in the interior of

$$\text{dom}(F) := \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$$

A result by Alberti – Ambrosio, 1999

The restriction of the **monotone set-valued map** F to Ω , viewed as an element in $L^\infty(\Omega, \mathbb{R}^n)$, is in $BV(\Omega, \mathbb{R}^n)$. Moreover,

$$|DF|(\Omega) \leq 2^{\frac{n}{2}} \omega_n [\text{diam}(\Omega) + \text{diam}(F(\Omega))]^n \quad (1)$$

where $F(\Omega) = \cup_{x \in \Omega} F(x)$, $|DF|$ is the total variation of the Radon measure DF , and

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Upper compactness estimate for decreasing SVM's

Proposition

Give any $L, M, C > 0$, consider

$$\mathcal{F}_{[L,M,C]} = \{F : [-L, L]^n \rightarrow \mathcal{P}([-M, M]^n) : F \text{ decreasing, } |DF|([-L, L]^n) \leq C\}$$

Then, for any $\varepsilon > 0$ sufficiently small,

$$\mathcal{H}_\varepsilon\left(\mathcal{F}_{[L,M,C]} \mid \mathbf{L}^1([-L, L]^n, [-M, M]^n)\right) \leq \Gamma^+ \cdot \frac{1}{\varepsilon^n}$$

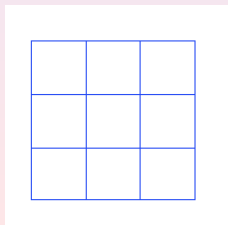
Sketch of proof ($n = 2$):

Given $N \geq 1$, we divide $[-L, L]^2$ into N^2 squares

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Approximation \bar{F} of $F \in \mathcal{F}_{[L,M,C]}$ is constant on each \square_i

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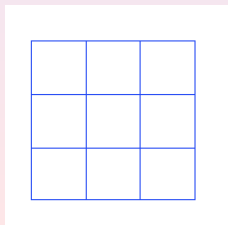
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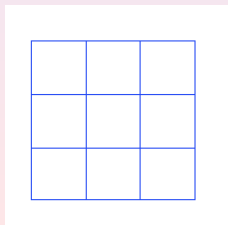
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Main ideas of the proof

- Poincaré inequality for BV functions on a convex domain

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Thus,

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- Construct a finite family of piecewise constant functions, \mathcal{G}_N , such that

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From SVM's to semiconcave functions

Proposition

Given $L, M, K > 0$, for $\varepsilon > 0$ sufficiently small

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use the fact the gradient of a semiconcave function is decreasing up to an additive linear map and apply previous proposition



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Lower estimate

reminder

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Theorem (Ancona, C and Khai T. Nguyen)

Let $M > 0$ be fixed

Then, for all $T > 0$ there exist constants $\Gamma_T > 0$ and $\Lambda_T \geq 0$ such that

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Main ideas of the proof of the lower estimate

- 1 **Controllability type result:** introduce a parameterized class \mathcal{U} of smooth function and show that any element of such a class can be attained, at any given time $T > 0$, by the Hopf-Lax flow $S_T(u)$ for a suitable $u \in C_{[L,M]}$
- 2 **Combinatorial computation:** provide an optimal (w.r.t. parameters) estimate of the maximum number of functions in \mathcal{U} that can be contained in a ball of radius 2ε (with respect to the norm of $\mathbf{W}^{1,1}(\mathbb{R}^n)$)



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Reachability of semiconcave functions

Theorem

Given $T, L, M > 0$, let $K > 0$ be such that

$$K T \leq \frac{1}{2\alpha_M} \quad \text{where} \quad \alpha_M = \sup_{|p| \leq M} \|D^2 H(p)\|$$

Then

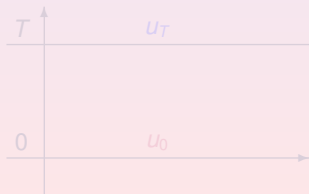
$$SC_{[K,L,M]} - T \cdot H(0) \subset S_T(C_{[L_T,M]})$$

with $L_T = L + T \cdot \sup_{|p| \leq M} |DH(p)|$

Our goal: for any

$u_T \in SC_{[K,L,M]} - T \cdot H(0)$ to find

$u_0 \in C_{[L_T,M]}$ such that $S_T(u_0) = u_T$



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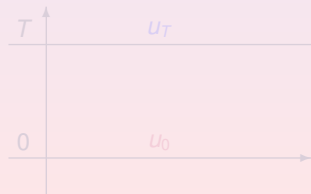
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with $L_T = L + T \cdot \sup_{|p| \leq M} |DH(p)|$

Our goal: for any

$u_T \in SC_{[K,L,M]} - T \cdot H(0)$ to find

$u_0 \in C_{[L_T,M]}$ such that $S_T(u_0) = u_T$



Reachability of semiconcave functions

Theorem

Given $T, L, M > 0$, let $K > 0$ be such that

$$K T \leq \frac{1}{2\alpha_M} \quad \text{where} \quad \alpha_M = \sup_{|p| \leq M} \|D^2 H(p)\|$$

Then

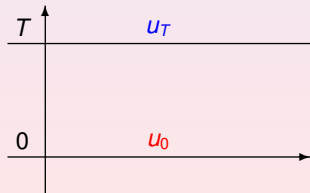
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Backward construction

Solve the equation backwards: set $v(t, x) = S_t(v_0)(x)$ with

$$v_0(x) = -u_T(-x)$$

and define

$$u(t, x) = -v(T - t, -x) \quad (t, x) \in [0, T] \times \mathbb{R}^n$$

Then

- $u(T, \cdot) = u_T$
- $u_0 \doteq u(0, \cdot) \in C_{[L_T, M]}$ by the properties of S_T
- $u_t(t, x) + H(\nabla u(t, x)) = 0$ for a.e. $(t, x) \in [0, T] \times \mathbb{R}^n$

Therefore,

$$u \text{ viscosity solution} \implies u_T = S_T(u_0)$$

The viscosity property follows from the semiconvexity of $v(t, \cdot)$



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The viscosity property follows from the **semiconvexity** of $v(t, \cdot)$



A semiconvexity result

Lemma

Given $M, T > 0$, let u_0 be *semiconvex* with constant $-K$ such that

$$K \leq \frac{1}{2\alpha_M T} \quad \text{where} \quad \alpha_M := \sup_{|p| \leq M} \|D^2 H(p)\| \quad \text{and} \quad \text{Lip}[u_0] \leq M$$

Then

- (i) $x \mapsto S_t u_0(x)$ is *semiconvex* for all $t \in [0, T]$ with a uniform constant
- (ii) $(t, x) \mapsto S_t u_0(x)$ is a C^1 *solution* of $u_t + H(\nabla u) = 0$ on $(0, T] \times \mathbb{R}^N$



Lower bound for $\mathcal{H}_\varepsilon \left(SC_{[K,L,M]} \mid \mathbf{W}^{1,1}(\mathbb{R}^n) \right)$

Proposition

Given $K, L, M > 0$, for any $\varepsilon > 0$

$$\mathcal{H}_\varepsilon \left(SC_{[K,L,M]} \mid \mathbf{W}^{1,1}(\mathbb{R}^n) \right) \geq \frac{\Gamma(K, L, M)}{\varepsilon^n}$$

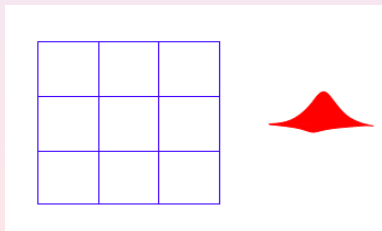
Given $N \geq 1$ integer, divide $[-L, L]^2$ into N^2 squares of side $\frac{2L}{N}$

$$[-L, L]^2 = \bigcup_{i,j=1,\dots,N} \square_{ij}$$

Construct bump functions $b_{ij} : \square_{ij} \rightarrow \mathbb{R}$ such that

- $\|\nabla b_{ij}\|_{L^\infty} \leq \frac{KL}{12N}$, $\|b_{ij}\|_{W^{1,1}} \leq \frac{C}{N^3}$
- ∇b_{ij} Lipschitz with constant K

Sketch of the proof ($n = 2$):



The class \mathcal{U}_N of smooth functions

Let

$$\Delta_N = \left\{ \delta = (\delta_{ij})_{i,j=1}^N : \delta_{ij} \in \{-1, 1\} \right\}$$

Consider the class of smooth functions

$$\mathcal{U}_N = \left\{ u_\delta = \sum_{i,j=1}^N \delta_{ij} \cdot b_{ij} : \delta \in \Delta_N \right\}$$

Then $\#\mathcal{U}_N = 2^{N^2}$. Also, one can show that

- $\mathcal{U}_N \subset SC_{[K,L,M]}$
- $\|u_{\delta'} - u_\delta\|_{W^{1,1}(\mathbb{R}^2)} \leq \varepsilon$ if $\#\{(i,j) : \delta'_{ij} \neq \delta_{ij}\} \leq C_{K,L} N^3 \varepsilon$

Choosing $N \approx \frac{1}{\varepsilon}$, by a combinatorial argument one can show that

$$\mathcal{H}_\varepsilon(\mathcal{U}_N \mid W^{1,1}(\mathbb{R}^2)) \geq \frac{\Gamma}{\varepsilon^2}$$

Therefore,

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End of the proof of the lower estimate ($n \geq 1$)

want to show

Let $M > 0$ be fixed. Then, $\forall T > 0$ there exist constants $\Gamma_T > 0$ and $\Lambda_T \geq 0$ such that

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The case of state dependent Hamiltonians

For the more general problem

$$\begin{cases} u_t(t, x) + H(x, \nabla u(t, x)) = 0 & (t, x) \in (0, T) \times \mathbb{R}^n \\ u(0, x) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$

- **upper estimate**: the previous approach can be extended to give similar results under “natural” assumptions on H
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$$\begin{cases} u_t(t, x) + H(x, \nabla u(t, x)) = 0 & (t, x) \in (0, T) \times \mathbb{R}^n \\ u(0, x) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$

$$\mathcal{C}_{[L, M]} = \{u \in \text{Lip}(\mathbb{R}^n) : \text{spt}(u) \subset [-L, L]^n, \text{Lip}[u] \leq M\}$$

- $\exists L_t > 0$ such that $\forall u_0 \in \mathcal{C}_{[L, M]}$

$$S_t(u_0)(x) = S_t(0)(x) \quad \forall x \in \mathbb{R}^n \setminus [-L_t, L_t]^n$$

- global bounds on the **semiconcavity constant** of $S_t(u_0)$ yield upper estimate

$$\mathcal{H}_\epsilon \left(S_T(\mathcal{C}_{[L, M]}) - S_t(0) \mid \mathbf{W}^{1,1}(\mathbb{R}^n) \right) \leq \frac{C}{\epsilon^n}$$

- semiconvexity/controllability results necessary for **lower estimate**

$$\mathcal{H}_\epsilon \left(S_T(\mathcal{C}_{[L, M]}) - S_t(0) \mid \mathbf{W}^{1,1}(\mathbb{R}^n) \right) \geq \frac{C}{\epsilon^n}$$

only for $T > 0$ **small enough**



*Thank you for your attention
and thanks to*



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Conference on « New Trends in Optimal Control »

23-27 June 2014, Tours, France

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