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Viscosity methods for multiscale financial models with stochastic volatility

Martino Bardi

joint work with

Annalisa Cesaroni and Daria Ghilli and Andrea Scotti

Department of Mathematics
University of Padua, Italy

NetCO 2014 - Conference on new trends in optimal control
Tours, June 23rd 2014

Plan

Introduction on models

- Financial models and stochastic volatility, Gaussian or with jumps
- Fast stochastic volatility

Part 1

- Control systems with random parameters and multiple scales
- The Hamilton-Jacobi-Bellman approach to Singular Perturbations
 - ▶ Tools
 - ▶ Assumptions
 - ▶ A convergence result
- Applications to finance

Part 2

- **Large deviations** for small time to maturity:
see also Daria **Ghilli's poster tomorrow**

Financial models and stochastic volatility

The evolution of the price of a **stock S** is described by

$$d \log S_s = \gamma ds + \sigma dW_s, \quad s = \text{time}, \quad W_s = \text{Wiener proc.},$$

and the classical **Black-Scholes formula** for option pricing and **Merton's optimal portfolio** are derived assuming the **parameters are constants**.

In reality the parameters of such models are **not constants**.

In particular, the **volatility σ** is not a constant, it rather looks like an **ergodic mean-reverting stochastic process**, see next slide.

Therefore it has been modeled as $\sigma = \sigma(y_s)$

with y_s either an Ornstein-Uhlenbeck **diffusion** process,

Refs.: Hull-White 87, Heston 93, Fouque-Papanicolaou-Sircar 2000,...

or by a **non-Gaussian** ergodic mean-reverting process

Refs.: Barndorff-Nielsen and Shephard 2001.

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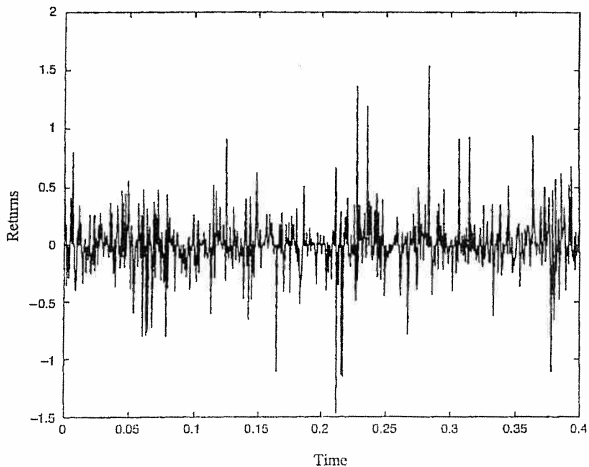


Figure 3.7. 1996 S&P 500 returns computed from half-hourly data.

Diffusion model (Gaussian)

$$dy_s = -y_s ds + \tau d\tilde{W}_s$$

with \tilde{W}_s a Wiener process possibly correlated with W_s
was used for many papers in finance, see the refs. in the book by
Fleming - Soner, 2nd ed., 2006,
for Merton's problem it was studied by Fleming - Hernandez 03.

Non-Gaussian, jump model

$$dy_s = -y_s ds + \tau dZ_s$$

where Z_s is a pure jump Lévy process with positive increments.
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Fast stochastic volatility

It is argued in the book

Fouque, Papanicolaou, Sircar: Derivatives in financial markets with stochastic volatility, 2000,

that the process y_s also evolves on a **faster time scale** than the stock prices: this models better the typical **bursty** behavior of volatility, see previous picture.

The equations for the evolution of a stock S with **fast stochastic volatility** σ proposed in [FPS] are **Gaussian**, with $\varepsilon > 0$,

$$d \log S_s = \gamma ds + \sigma(y_s) dW_s$$

$$dy_s = -\frac{1}{\varepsilon} y_s + \frac{\tau}{\sqrt{\varepsilon}} d\tilde{W}_s$$

and they study the asymptotics $\varepsilon \rightarrow 0$ for many option pricing problems. We'll study also the **non-Gaussian** volatility

$$dy_s = -\frac{1}{\varepsilon} y_{s-\varepsilon} ds + dZ_{s/\varepsilon}$$

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Two-scale control systems with random parameters

We consider control systems with **fast Jump volatility**

$$\begin{aligned} dx_s &= f(x_s, y_{s-}, u_s) ds + \sigma(x_s, y_{s-}, u_s) dW_s & x_s &\in \mathbf{R}^n, \\ dy_s &= -\frac{1}{\varepsilon} y_{s-} ds + dZ_{s/\varepsilon} & y_s &\in \mathbf{R} \end{aligned}$$

Basic assumptions

- f, σ, b, τ Lipschitz in (x, y) (unif. in u) with linear growth
- Z . 1-dim. pure jump Lévy process, independent of W .,
+ conditions (later).

Value function is

$$V^\varepsilon(t, x, y) := \sup_u E[e^{c(t-T)} g(x_T) \mid x_t = x, y_t = y]$$

with $g : \mathbf{R}^n \rightarrow \mathbf{R}$ continuous, $g(x) \leq K(1 + |x|^2)$, $c \geq 0$.

HJB equation

The value V^ε solves the **integro-differential** HJB equation in $(0, T) \times \mathbf{R}^n \times \mathbf{R}$

$$-\frac{\partial V^\varepsilon}{\partial t} + \mathcal{H}(x, y, D_x V^\varepsilon, D_{xx}^2 V^\varepsilon) - \frac{1}{\varepsilon} \mathcal{L}[y, V^\varepsilon] + cV^\varepsilon = 0,$$

$$\mathcal{H}(x, y, p, M) := \min_{u \in U} \left\{ -\text{tr}(\sigma \sigma^T M) / 2 - f \cdot p \right\}$$

$$\mathcal{L}[y, v] := -y v_y(y) + \int_0^{+\infty} (v(z+y) - v(y) - v_y(y)z 1_{z \leq 1}) d\nu(z)$$

is the **generator** of the unscaled volatility process $dy_s = -y_{s-} ds + dZ_s$,

ν is the **Lévy measure** associated to the jump process Z :

$$\nu(B) = E(\#\{s \in [0, 1], Z_s - Z_{s-} \neq 0, Z_s - Z_{s-} \in B\})$$

= expected number of jumps of a certain height
in a unit-time interval.

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PDE approach to the singular limit $\varepsilon \rightarrow 0$

Search an *effective Hamiltonian* \bar{H} such that

$$V^\varepsilon(t, x, y) \rightarrow V(t, x) \quad \text{as } \varepsilon \rightarrow 0,$$

V solution of

$$(\overline{\text{CP}}) \quad \begin{cases} -\frac{\partial V}{\partial t} + \bar{H}(x, D_x V, D_{xx}^2 V) + cV = 0 & \text{in } (0, T) \times \mathbf{R}^n, \\ V(T, x) = g(x) \end{cases}$$

Then, if possible, interpret the effective Hamiltonian \bar{H} as the Bellman Hamiltonian for a new *effective optimal control problem* in \mathbf{R}^n , which is therefore a variational limit of the initial $n + m$ -dimensional problem.

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1. Ergodicity of the unscaled volatility process, or fast subsystem, i.e., of

$$dy_s = -y_{s-} ds + dZ_s$$

Assume conditions such that this process has a unique invariant probability measure μ and it is uniformly ergodic.

By solving an auxiliary (linear) PDE called cell problem we find that the candidate effective Hamiltonian is

$$\bar{H}(x, p, M) = \int_{\mathbf{R}^m} \mathcal{H}(x, y, p, M) d\mu(y).$$

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$$\bar{H}(x, p, M) = \int_{\mathbf{R}^m} \mathcal{H}(x, y, p, M) d\mu(y).$$

2. The generator \mathcal{L} has the Liouville property

(based on the Strong Maximum Principle by Ciomaga 2012), i.e.

any bounded sub- or supersolution of $-\mathcal{L}[y, v] = 0$ is constant.

Then the relaxed semilimits

$$\underline{V}(t, x, y) := \liminf_{\varepsilon \rightarrow 0, t' \rightarrow t, x' \rightarrow x, y' \rightarrow y} V^\varepsilon(t', x', y'),$$

$\overline{V}(t, x, y) := \limsup$ of the same, do not depend on y .

3. Perturbed test function method,

evolving from Evans (periodic homogenisation) and

Alvarez-M.B. (singular perturbations with bounded fast variables),

allows to prove that

$\underline{V}(t, x)$ is supersol., $\overline{V}(t, x)$ is subsol. of limit PDE in $(\overline{\text{CP}})$.

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4. Comparison principle

between a subsolution and a supersolution of the Cauchy problem $(\overline{\text{CP}})$ satisfying

$$|V(t, x)| \leq C(1 + |x|^2),$$

see Da Lio - Ley 2006. It gives

- uniqueness of solution V of $(\overline{\text{CP}})$
- $\underline{V}(t, x) \geq \overline{V}(t, x)$, then $\underline{V} = \overline{V} = V$ and, as $\varepsilon \rightarrow 0$,

$$V^\varepsilon(t, x, y) \rightarrow V(t, x) \quad \text{locally uniformly.}$$

Assumptions

The Lévy measure ν of the jump process Z satisfies

- $\exists C > 0, 0 < p < 2, 0 < \delta \leq 1 : \int_{|z| \leq \delta} |z|^2 \nu(dz) \geq C \delta^{2-p}$
- $\exists q > 0 : \int_{|z| > 1} |z|^q \nu(dz) < +\infty.$

Then the unscaled volatility $dy_s = -y_s ds + dZ_s$ is uniformly ergodic (Kulik 2009).

If, moreover,

- either $p > 1,$
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then the integro-differential generator \mathcal{L} of the process y has the Liouville property.

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Examples

α -stable Lévy processes:

- ν symmetric

$$\nu(dz) = \frac{dz}{|z|^{1+\alpha}}, \quad 0 < \alpha < 2,$$

here $\mathcal{L} = (-\Delta)^{\alpha/2}$ is the fractional Laplacian

- ν not symmetric: no negative jumps

$$\nu(dz) = \frac{dz}{|z|^{1+\alpha}} \mathbf{1}_{\{z \geq 0\}}(z), \quad 1 < \alpha < 2$$

Tempered α -stable Lévy processes:

-

$$\nu(dz) = \frac{e^{-\gamma z} dz}{|z|^{1+\alpha}} \mathbf{1}_{\{z \geq 0\}}(z), \quad 1 < \alpha < 2, \quad \gamma > 0.$$

Convergence Theorem [M.B. - Cesaroni - Scotti 2014]

Theorem

$\lim_{\varepsilon \rightarrow 0} V^\varepsilon(t, x, y) = V(t, x)$ locally uniformly, V solving

$$-\frac{\partial V}{\partial t} + \int_{\mathbf{R}^m} \mathcal{H}(x, y, D_x u, D_{xx}^2 u, 0) d\mu(y) = 0 \quad \text{in } (0, T) \times \mathbf{R}^n$$

with $V(T, x) = g(x)$.

Related earlier results for Gaussian ergodic mean-reverting volatility

$$dy_s = \frac{1}{\varepsilon} b(x_s, y_s) ds + \frac{1}{\sqrt{\varepsilon}} \tau(x_s, y_s) dW_s \quad y_s \in \mathbf{R}^m$$

- τ nondegenerate, b, τ independent of x
[M.B. - Cesaroni - Manca, SIAM J. Financial Math. 2010],
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Financial examples

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$$\tilde{\sigma}^2 := \int \sigma^2(y) \mu(dy) = \text{mean historical volatility,}$$

a linear average of $\sigma^2(\cdot)$.

Merton portfolio optimization problem

Invest u_s in the stock S_s , $1 - u_s$ in a bond with interest rate r . Then the wealth x_s evolves as

$$d x_s = (r + (\gamma - r)u_s)x_s ds + x_s u_s \sigma(y_s) dW_s \quad y_t = y,$$

and want to **maximize the expected utility** at time T , $E[g(x_T)]$,
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This is the HJB equation of a Merton problem with constant volatility $\bar{\sigma}$ = harmonic average of $\sigma(\cdot)$.

$$\bar{\sigma}^2 := \left(\int \frac{1}{\sigma^2(y)} d\mu(y) \right)^{-1} \leq \tilde{\sigma}^2 = \int \sigma^2(y) \mu(dy)$$

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$$-\frac{\partial V^\varepsilon}{\partial t} - rxV_x^\varepsilon + \frac{[(\gamma - r)V_x^\varepsilon]^2}{\sigma^2(y)2V_{xx}^\varepsilon} = \frac{1}{\varepsilon}\mathcal{L}[y, V^\varepsilon]$$

By the Theorem, $V^\varepsilon(t, x, y) \rightarrow V(t, x)$ as $\varepsilon \rightarrow 0$ and V solves

$$-\frac{\partial V}{\partial t} - rxV_x + \frac{(\gamma - r)^2 V_x^2}{2V_{xx}} \int \frac{1}{\sigma^2(y)} d\mu(y) = 0 \quad \text{in } (0, T) \times \mathbf{R}.$$

This is the HJB equation of a **Merton problem** with **constant volatility** $\bar{\sigma}$ = **harmonic average** of $\sigma(\cdot)$.

$$\bar{\sigma}^2 := \left(\int \frac{1}{\sigma^2(y)} d\mu(y) \right)^{-1} \leq \tilde{\sigma}^2 = \int \sigma^2(y) \mu(dy)$$

Then if one uses a constant-parameter model as approximation, the **nonlinear average** $\bar{\sigma}$ is better, it **increases the optimal expected utility**.

Short time and fast volatility: large deviations

For small $\varepsilon > 0$ and $\delta > 0$ look at

$$dX_t = \varepsilon \phi(X_t, Y_t) dt + \sqrt{2\varepsilon} \sigma(X_t, Y_t) dW_t \quad X_0 = x \in \mathbf{R}^n,$$

$$dY_t = \frac{\varepsilon}{\delta} b(Y_t) dt + \sqrt{\frac{2\varepsilon}{\delta}} \tau(Y_t) dW_t \quad Y_0 = y \in \mathbf{R}^m,$$

with ϕ, σ, b, τ periodic in Y (for simplicity). Take

$$\delta = \varepsilon^\alpha, \quad \alpha > 1$$

and $v^\varepsilon(t, x, y) := \varepsilon \log E [e^{h(X_t)/\varepsilon}]$. It satisfies $v^\varepsilon(0, x, y) = h(x)$ and

$$v_t = |\sigma^T D_x v|^2 + \varepsilon [\text{tr}(\sigma \sigma^T D_{xx}^2 v) + \phi \cdot D_x v] + \frac{2}{\varepsilon^{\alpha/2}} (\tau \sigma^T D_x v) \cdot D_y v +$$
$$\frac{1}{\varepsilon^\alpha} |\tau^T D_y v|^2 + \frac{2}{\varepsilon^{\alpha/2-1}} \text{tr}(\sigma \tau^T D_{xy}^2 v) + \frac{1}{\varepsilon^{\alpha-1}} [b \cdot D_y v + \text{tr}(\tau \tau^T D_{yy}^2 v)]$$

Convergence Theorem [M.B. - Cesaroni - Ghilli 2014]

$\forall \alpha > 1 \exists \bar{H}$ continuous such that $\lim_{\varepsilon \rightarrow 0} v^\varepsilon(t, x, y) = v(t, x)$ in the sense of weak semilimits, v solving

$$v_t - \bar{H}(x, Dv) = 0 \quad \text{in } (0, T) \times \mathbf{R}^n$$

with $v(0, x) = h(x)$; \bar{H} depends on the three regimes

- 1 $\alpha > 2$: $\bar{H}(x, p) = \int_{\mathbb{T}^m} |\sigma^T(x, y)p|^2 d\mu(y)$, convergence is uniform
- 2 $\alpha < 2$: \bar{H} has deterministic control formula, convergence is uniform; for $\tau\sigma^T = 0$ $\bar{H}(x, p) = \max_{y \in \mathbf{R}^m} |\sigma^T(x, y)p|^2$
- 3 $\alpha = 2$: \bar{H} has stochastic control formula, convergence uniform if
 - ▶ either $\sigma = \sigma(Y_t)$ independent of X_t , (\bar{H} independent of x)
 - ▶ or $|\sigma^T(x, y)p|^2 \geq \nu|p|^2$, $\nu > 0$, (\bar{H} coercive)
 - ▶ or $\tau\sigma^T = 0$, (independent noise in dX_t and dY_t)

More on this paper in Daria GHILLI's poster tomorrow!

Further results and perspectives

Can treat also

- limit of the optimal feedback in Merton's problem,
- utility depending on y , i.e., $g = g(x, y)$, then the effective terminal condition is $V(T, x) = \int g(x, y) d\mu(y)$,
- problems with two conflicting controllers, i.e., two-person, 0-sum, stochastic **differential games**,
- systems with **more than two scales**.

Developments under investigation:

- more general jump processes for the volatility (without the Liouville property...), e.g., "**inverse Gaussian**",
- jump terms in the stocks dynamics,
- **large deviations** for short maturity asymptotics with **non-Gaussian volatility** and/or in **Merton's problem**.

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Thanks for your attention!