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Hamilton-Jacobi equations on networks as limits of singularly perturbed problems

Yves Achdou

joint work with N. Tchou

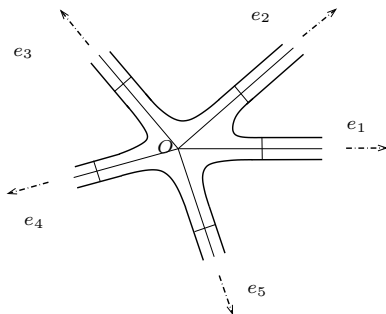
Laboratoire J-L Lions, Université Paris Diderot

Setting: a thick version of a network

Consider a domain $\Omega \subset \mathbb{R}^2$:

- star-shaped w.r.t. the origin O
- $\partial\Omega$ is smooth
- Far enough from the origin O , Ω coincides with the union of N non-intersecting semi-infinite strips directed by the vectors e_i , $i = 1, \dots, N$

The domain Ω



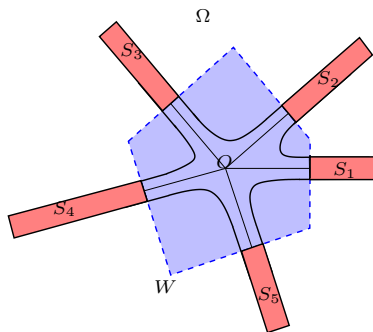
Setting: a more precise definition

- for some $r_0 > 0$, let W be the polygonal set

$$W = \{x \in \mathbb{R}^2 : x \cdot e_i \leq r_0, \forall i = 1, \dots, N\}$$

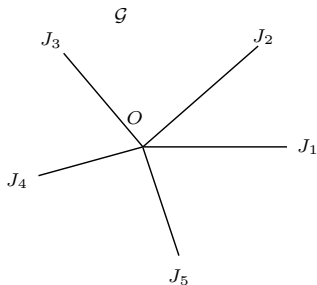
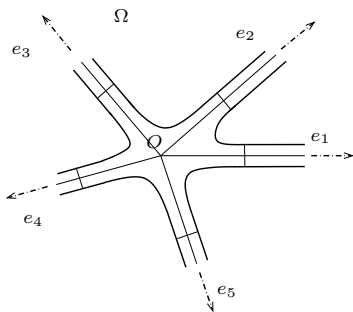
- Then $\Omega \setminus W = \bigcup_{i=1}^N S_i$, where S_i is the half-strip

$$S_i = \{x_i e_i + x_i^\perp e_i^\perp, \quad x_i > r_0, \quad |x_i^\perp| < 1\}$$



$\Omega_\epsilon = \epsilon\Omega$ “tends” to a network \mathcal{G} as $\epsilon \rightarrow 0$

$$\mathcal{G} = \{O\} \cup \bigcup_{i=1}^N J_i,$$
$$J_i = \{x_i e_i, x_i > 0\}.$$



State constrained control problems in $\overline{\Omega_\epsilon}$

$$u_\epsilon(x) = \inf_{\alpha} \int_0^{\infty} \ell_\epsilon(y_\epsilon(t; x, \alpha), \alpha(t)) e^{-\lambda t} dt$$

subject to

$$\begin{cases} \dot{y}_\epsilon(t; x, \alpha) &= \alpha(t), & \text{a.a. } t > 0, \\ y_\epsilon(0; x, \alpha) &= x, \end{cases}$$

with $\alpha : \mathbb{R}_+ \rightarrow A$ measurable, under the state constraint

$$y_\epsilon(t) \in \overline{\Omega_\epsilon}, \quad \forall t \geq 0.$$

Controlability : A is a compact subset of \mathbb{R}^2 such that $B(0, r) \subset A$ for some $r > 0$.

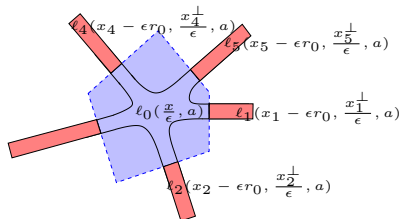
Assumptions on the running costs

- The function $\ell_\epsilon : \Omega_\epsilon \rightarrow \mathbb{R}$ is bounded and continuous
-

$$\ell_\epsilon(x, a) = \begin{cases} \ell_i \left(x_i - \epsilon r_0, \frac{x_i^\perp}{\epsilon}, a \right) & \text{in } \epsilon S_i \\ \ell_0 \left(\frac{x}{\epsilon}, a \right) & \text{in } \epsilon W \cap \Omega_\epsilon \end{cases}$$

$$\ell_i : [0, +\infty) \times [-1, 1] \times A \rightarrow \mathbb{R}$$

$$\ell_0 : (W \cap \Omega) \times A \rightarrow \mathbb{R}$$



ℓ_i and ℓ_0 match properly in order to ensure the continuity of ℓ_ϵ

Assumptions on the running costs

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$$\ell_\epsilon(x, a) = \begin{cases} \ell_i \left(x_i - \epsilon r_0, \frac{x_i^\perp}{\epsilon}, a \right) & \text{in } \epsilon S_i \\ \ell_0 \left(\frac{x}{\epsilon}, a \right) & \text{in } \epsilon W \cap \Omega_\epsilon \end{cases}$$

Remark

In ϵS_i , ℓ_ϵ varies slowly w.r.t. x_i and fast w.r.t. x_i^\perp .

- It is not restrictive to assume that $\ell_0 \geq 0$.

The value function u_ϵ

u_ϵ is bounded uniformly with respect to ϵ , continuous, and is the unique viscosity solution of

$$\begin{aligned}\lambda u_\epsilon(x) + H_\epsilon(x, Du_\epsilon) &\geq 0 && \text{in } \overline{\Omega}_\epsilon, \\ \lambda u_\epsilon(x) + H_\epsilon(x, Du_\epsilon) &\leq 0 && \text{in } \Omega_\epsilon,\end{aligned}$$

where

$$H_\epsilon(x, p) = \max_{a \in A} \left(-p \cdot a - \ell_\epsilon(x, a) \right).$$

Questions

- Asymptotic behavior of u_ϵ as $\epsilon \rightarrow 0$?
- Can we define an effective problem in \mathcal{G} ?

Background and additional new difficulties

- Existing results:
 - Singularly perturbed control problems in thin domains around smooth manifolds: Bensoussan, Alvarez-Bardi, Arztein-Gaitsgory, Gaitsgory-Leisarowitz, Terrone,...
 - Comparison for viscosity solutions on networks: YA-Oudet-Tchou and Imbert-Monneau
 - Very recent results of Lions-Souganidis on homogenization of HJB equations with defects
- Several additional new difficulties:
 - Identification of the effective problem
 - Will the effective problem keep track of ℓ_0 near O ? How?
 - In the perturbed test-function method of Evans, we need to construct correctors in unbounded domains.

Main result

Theorem

Under a further technical assumption, u_ϵ converges locally uniformly to the bounded viscosity solution $u : \mathcal{G} \rightarrow \mathbb{R}$ of

$$\left\{ \begin{array}{l} \lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) = 0 \quad x \in J_i, \\ \lambda u(O) + \max \left(E, \overline{H}(O, \frac{du}{dx_1}(O), \dots, \frac{du}{dx_N}(O)) \right) = 0, \end{array} \right.$$

Main result

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with

$$\overline{H}_i(x_i, p_i) = \sup_{\mu \in \mathcal{Z}_i} \left(\int_{[-1,1] \times A} (-p_i a_i - \ell_i(x_i, y, a)) d\mu(y, a) \right)$$

\mathcal{Z}_i is a compact and convex set of Radon probability measures on $[-1, 1] \times A$

\mathcal{Z}_i : limiting relaxed controls.

Main result

Theorem

Under a further technical assumption, u_ϵ converges locally uniformly to the bounded viscosity solution $u : \mathcal{G} \rightarrow \mathbb{R}$ of

$$\begin{cases} \lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) = 0 & x \in J_i, \\ \lambda u(O) + \max \left(E, \overline{H}(O, \frac{du}{dx_1}(O), \dots, \frac{du}{dx_N}(O)) \right) = 0, \end{cases}$$

with

– E : an effective cost at the junction

Main result

Theorem

Under a further technical assumption, u_ϵ converges locally uniformly to the bounded viscosity solution $u : \mathcal{G} \rightarrow \mathbb{R}$ of

$$\begin{cases} \lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) = 0 & x \in J_i, \\ \lambda u(O) + \max \left(E, \overline{H}(O, \frac{du}{dx_1}(0), \dots, \frac{du}{dx_N}(0)) \right) = 0, \end{cases}$$

with $\overline{H}(O, p_1, \dots, p_N) := \max_{i=1, \dots, N} \overline{H}_i^+(0, p_i)$,

$$\overline{H}_i^+(0, p_i) = \sup_{\mu \in \mathcal{Z}_i^+} \left(\int_{[-1,1] \times A} (-p_i a_i - \ell_i(0, y, a)) d\mu(y, a) \right),$$

$$\mathcal{Z}_i^+ = \left\{ \mu \in \mathcal{Z}_i \text{ s.t. } \int_{[-1,1] \times A} a_i d\mu(y, a) \geq 0 \right\}.$$

What is the meaning of

$$\begin{cases} \lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) = 0 & x \in J_i, \\ \lambda u(O) + \max \left(E, \overline{H}(O, \frac{du}{dx_1}(0), \dots, \frac{du}{dx_N}(0)) \right) = 0 \end{cases} ? \quad (1)$$

See YA-Camilli-Cutri-Tchou(2013), Imbert-Monneau-Zidani(2013), YA-Oudet-Tchou(2014), and Imbert-Monneau(2014).

Definition (Test functions)

- $\phi : \mathcal{G} \rightarrow \mathbb{R}$ is an admissible test-function if
 - ϕ is continuous on \mathcal{G}
 - for any $j \in 1, \dots, N$, $\phi|_{\overline{J}_j} \in \mathcal{C}^1(\overline{J}_j)$
- $\mathcal{R}(\mathcal{G})$: set of the admissible test-functions

- An usc function $u : \mathcal{G} \rightarrow \mathbb{R}$ is a sub-solution of (1) if for any $x \in \mathcal{G}$ and $\phi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \phi$ has a local maximum in x , then

$$\begin{aligned} \lambda u(x) + \bar{H}_i(x, \frac{d\phi}{dx_i}(x)) &\leq 0 && \text{if } x \in J_i, \\ \lambda u(O) + \max \left(E, \bar{H} \left(\frac{d\phi}{dx_1}(O), \dots, \frac{d\phi}{dx_N}(O) \right) \right) &\leq 0. \end{aligned}$$

- A lsc function $u : \mathcal{G} \rightarrow \mathbb{R}$ is a super-solution of (1) if for any $x \in \mathcal{G}$ and $\phi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \phi$ has a local minimum in x , then

$$\begin{aligned} \lambda u(x) + \bar{H}_i(x, \frac{d\phi}{dx_i}(x)) &\geq 0 && \text{if } x \in J_i, \\ \lambda u(O) + \max \left(E, \bar{H} \left(\frac{d\phi}{dx_1}(O), \dots, \frac{d\phi}{dx_N}(O) \right) \right) &\geq 0. \end{aligned}$$

The effective Hamiltonians away from the junctions

Notation: for $x \geq 0, y \in [-1, 1], p \in \mathbb{R}^2$,

$$H_i(x, y, p) = \max_{a \in A} (-p \cdot a - \ell_i(x, y, a))$$

Theorem (Alvarez-Bardi 2000)

$\forall x_i \geq 0, \forall p_i \in \mathbb{R}$, there is a unique $\bar{H}_i(x_i, p_i)$ s.t. the cell problem

$$\begin{aligned} H_i(x_i, y, p_i e_i + D_y \chi_i(y) e_i^\perp) &\geq \bar{H}_i(x_i, p_i) & y \in [-1, 1], \\ H_i(x_i, y, p_i e_i + D_y \chi_i(y) e_i^\perp) &\leq \bar{H}_i(x_i, p_i) & y \in (-1, 1) \end{aligned}$$

has a viscosity solution $\chi_i \in \text{Lip}([-1, 1])$.

Limiting relaxed controls and limit control problem

For $s > 0$, the *occupational measure* μ_s generated by $(y(t), \alpha(t))$ is the Radon probability measure defined on $[-1, 1] \times A$ by

$$\mu_s = \frac{1}{s} \int_0^s \delta_{(y(t), \alpha(t))} dt$$

where $\delta_{(y(t), \alpha(t))}$ is the Dirac mass concentrated at $(y(t), \alpha(t))$.

$\mathcal{Z}(s; i, y_0)$: set of the occupational measures generated by the trajectories $(y(t), \alpha(t))$ up to time s .

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$\mathcal{Z}(s; i, y_0)$: set of the occupational measures generated by the trajectories $(y(t), \alpha(t))$ up to time s .

Theorem (Gaitsgory and Leisarovitz (1999))

There exists a set $\mathcal{Z}_i \subset \mathcal{P}([-1, 1] \times A)$ s.t

$$\forall y_0 \in [-1, 1], \quad \lim_{s \rightarrow \infty} \pi_H(\mathcal{Z}(s; i, y_0), \mathcal{Z}_i) = 0,$$

(π_H : Prokhoroff distance).

\mathcal{Z}_i is convex and compact for the weak- topology.*

By using results from Terrone-2011 and Alvarez-Bardi-2000

-

$$\mathcal{Z}_i \subset \left\{ \mu \in \mathcal{P}([-1, 1] \times A), \int_{[-1, 1] \times A} a_i^\perp d\mu(y, a) = 0 \right\}$$

- \mathcal{Z}_i coincides with the set of limiting relaxed controls, i.e.

$$\begin{aligned} & \bar{H}_i(x_i, p_i) \\ &= \sup_{\mu \in \mathcal{Z}_i} \left(-p_i \int_{[-1, 1] \times A} a_i d\mu(y, a) - \int_{[-1, 1] \times A} \ell_i(x_i, y, a) d\mu(y, a) \right) \end{aligned}$$

Interpretation

Away from the junction, the effective Hamiltonian corresponds to a control problem with controls in \mathcal{Z}_i .

The effective Hamiltonian at the junction

Definition

The effective Hamiltonian at the junction is $\max(E, \overline{H}(O, \cdot))$ where E is a suitable constant to be defined and

$$\overline{H}(O, p_1, \dots, p_N) := \max_{i=1, \dots, N} \overline{H}_i^+(0, p_i)$$

$$\begin{aligned} & \overline{H}_i^+(0, p_i) \\ &= \sup_{\mu \in \mathcal{Z}_i^+} \left(-p_i \int_{[-1,1] \times A} a_i d\mu(y, a) - \int_{[-1,1] \times A} \ell_i(0, y, a) d\mu(y, a) \right) \end{aligned}$$

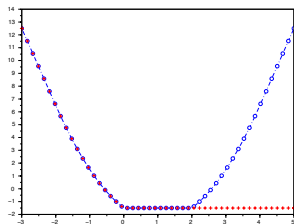
and

$$\mathcal{Z}_i^+ = \left\{ \mu \in \mathcal{Z}_i \text{ s.t. } \int_{[-1,1] \times A} a_i d\mu(y, a) \geq 0 \right\}$$

Lemma: links between $\overline{H}_i(0, p)$ and $\overline{H}_i^+(0, p)$

$$\mathcal{Z}_i^0 := \left\{ \mu \in \mathcal{Z}_i \text{ s.t. } \int_{[-1,1] \times A} a_i d\mu(y, a) = 0 \right\}$$

$$\operatorname{argmin} \overline{H}_i(0, \cdot) = \left[\underline{p}_i, \bar{p}_i \right]$$



The graphs of
 $p \mapsto \overline{H}_i(0, p)$ and
of $p \mapsto \overline{H}_i^+(0, p)$

- $p < \underline{p}_i \Rightarrow \overline{H}_i(0, p) = \overline{H}_i^+(0, p)$ is achieved for $\mu \in \mathcal{Z}_i^+ \setminus \mathcal{Z}_i^0$
- $\underline{p}_i < p < \bar{p}_i \Rightarrow \overline{H}_i(0, p) = \overline{H}_i^+(0, p)$ is achieved for $\mu \in \mathcal{Z}_i^0$
- $p > \bar{p}_i \Rightarrow \overline{H}_i(0, p) > \overline{H}_i^+(0, p) = \min \overline{H}_i(0, \cdot)$

The constant E

Zoom near the junction point O

We extend the function ℓ_0 to the whole domain $\bar{\Omega}$ by setting

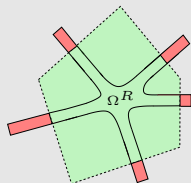
$$\ell_0(z, a) = \ell_i(0, z_i^\perp, a), \quad \text{if } z_i \geq r_0, |z_i^\perp| \leq 1.$$

Ergodic constants in bounded subdomains

Define $\Omega^R := \Omega \cap W_R$ and the ergodic constant E^R , which is the unique number s.t.

$$\begin{aligned} H_0(z, Dw^R(z)) &\geq E^R && \text{in } \bar{\Omega}^R, \\ H_0(z, Dw^R(z)) &\leq E^R && \text{in } \Omega^R, \end{aligned}$$

has a viscosity solution w^R .



The constant E

Lemma

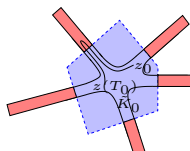
$\exists C > 0$ s.t.

$$\begin{aligned} |E^R| &\leq C && \forall R, \\ \|Dw^R\|_\infty &\leq C && \forall R, \\ R \leq R' &\Rightarrow E^R \leq E^{R'}. \end{aligned}$$

Definition

$$E := \lim_{R \rightarrow \infty} E^R.$$

The constant E



Theorem

$\exists C > 0$ s.t. $\forall z_0 \in \tilde{K}_0, \forall \alpha \in \mathcal{A}_{z_0}, \forall T > 0,$

if $z(s) := z_0 + \int_0^s \alpha(\theta) d\theta$ satisfies $z(T) \in \tilde{K}_0$, then

$$\int_0^T \ell_0(z(s), \alpha(s)) ds \geq -ET - C.$$

Lemma

Under the assumptions, $\forall i = 1, \dots, N,$ $\min_{p_i} \bar{H}_i(0, p_i) \leq E.$

Proof of the convergence result

The main step is to prove the following theorem

Theorem

Consider the relaxed semilimits, of u_ϵ : for all $x \in \mathcal{G}$,

$$\underline{u}(x) = \liminf_{\epsilon \rightarrow 0+, x' \rightarrow x, x' \in \Omega_\epsilon} u_\epsilon(x'); \quad \bar{u}(x) = \limsup_{\epsilon \rightarrow 0+, x' \rightarrow x, x' \in \Omega_\epsilon} u_\epsilon(x').$$

Then, under the assumptions, \underline{u} is a bounded supersolution of

$$\begin{aligned} \lambda \underline{u}(x) + \overline{H}(x, D\underline{u}(x)) &\geq 0, & \text{if } x \in \mathcal{G} \setminus \{O\}, \\ \lambda \underline{u}(O) + \max(E, \overline{H}(O, D\underline{u}(O))) &\geq 0, & \text{if } x = O, \end{aligned}$$

and \bar{u} is a bounded subsolution of

$$\begin{aligned} \lambda \bar{u}(x) + \overline{H}(x, D\bar{u}(x)) &\leq 0, & \text{if } x \in \mathcal{G} \setminus \{O\}, \\ \lambda \bar{u}(O) + \max(E, \overline{H}(O, D\bar{u}(O))) &\leq 0, & \text{if } x = O. \end{aligned}$$

then use comparison results on \mathcal{G} , [AOT2014] or [IM2014]

Some ideas of the proof

- Not standard only at the junction
 - Try to use Evans' method of perturbed test-functions
 - The test-functions ϕ have N slopes at $O : p = (p_1, \dots, p_N)$
 - Evans' method requires the construction of bounded correctors in unbounded domains, (ideally Ω)
 - Bounded correctors may not exist in the full domain Ω , because, given a set of slopes at O , i.e. $p = (p_1, \dots, p_N)$,

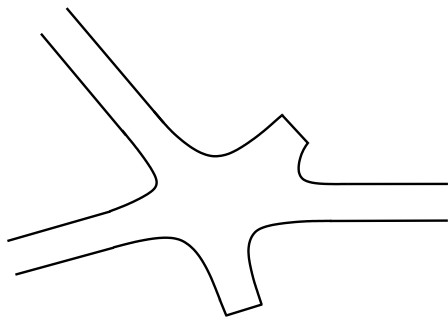
$$\overline{H}(O, p_1, \dots, p_n) = \max_{i=1, \dots, N} \overline{H}_i^+(0, p_i)$$

is achieved by $i \in \mathcal{I}(p) \subset \{1, \dots, N\}$. If $j \notin \mathcal{I}(p)$, then the optimal trajectories starting from $z \in S_j$ should leave S_j : this leads to the fact that the corrector should not be bounded when $|z| \rightarrow \infty$, $z \in S_j$.

- Hence, we construct the correctors in subdomain obtained by truncating the half-strips S_j if $j \notin \mathcal{I}(p)$.

The corrector (when $\overline{H}(O, p_1, \dots, p_N) > E$)

The truncated domain Ω_p



Theorem

For $p \in \mathbb{R}^N : \overline{H}(O, p) > E$. Let ψ_p be a smooth function such that $D\psi_p = p_i e_i$ in S_i . Under the assumptions, $\exists \chi_p$, a bounded and Lipschitz viscosity solution of

$$H_0(z, D\psi_p + D\chi_p) - \overline{H}(O, p) \geq 0 \quad \text{in } \overline{\Omega}_p,$$

$$H_0(z, D\psi_p + D\chi_p) - \overline{H}(O, p) \leq 0 \quad \text{in } \Omega_p.$$

The further assumption

Assumption

1) For any real number p_i such that $p_i < \underline{p}_i$,

there exist two constants $L_i \geq 0$ and $C_i > 0$ such that

$\forall y_0 \in [-1, 1], \forall t > 0$, there exists a control law $\tilde{\alpha} \in \mathcal{A}_{i, y_0}$ with

$$\int_0^s \tilde{\alpha}_i(\tau) d\tau \geq -L_i, \quad \forall 0 \leq s \leq t,$$

$$\int_0^t (p_i \tilde{\alpha}_i(s) + \ell_i(0, y(s), \tilde{\alpha}(s))) ds \leq v_i(0, p_i, y_0, t) + C_i,$$

where $y(t) = y_0 + \int_0^t \tilde{\alpha}_i^\perp(s) ds$.

Recall that

$$v_i(x_i, p_i, y_0, t) = \inf_{\alpha \in \mathcal{A}_{i, y_0}} \left\{ \int_0^t p_i \alpha_i(s) + \ell_i(x_i, y(s), \alpha(s)) ds \right\}$$