



HAL
open science

Hamilton-Jacobi equations on networks as limits of singularly perturbed problems

Yves Achdou

► **To cite this version:**

Yves Achdou. Hamilton-Jacobi equations on networks as limits of singularly perturbed problems. NETCO 2014, 2014, Tours, France. hal-01023969

HAL Id: hal-01023969

<https://inria.hal.science/hal-01023969>

Submitted on 15 Jul 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Hamilton-Jacobi equations on networks as limits of singularly perturbed problems

Yves Achdou

joint work with N. Tchou

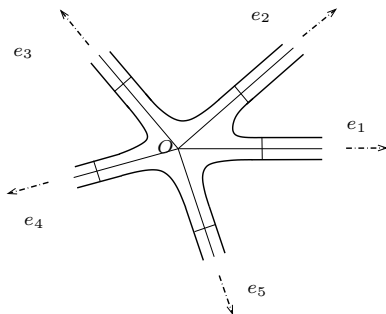
Laboratoire J-L Lions, Université Paris Diderot

Setting: a thick version of a network

Consider a domain $\Omega \subset \mathbb{R}^2$:

- star-shaped w.r.t. the origin O
- $\partial\Omega$ is smooth
- Far enough from the origin O , Ω coincides with the union of N non-intersecting semi-infinite strips directed by the vectors e_i , $i = 1, \dots, N$

The domain Ω



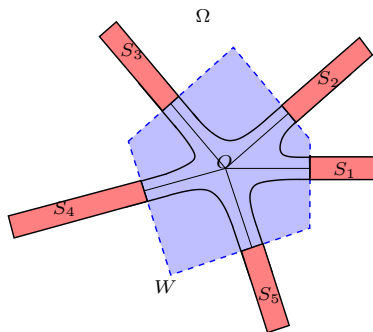
Setting: a more precise definition

- for some $r_0 > 0$, let W be the polygonal set

$$W = \{x \in \mathbb{R}^2 : x \cdot e_i \leq r_0, \forall i = 1, \dots, N\}$$

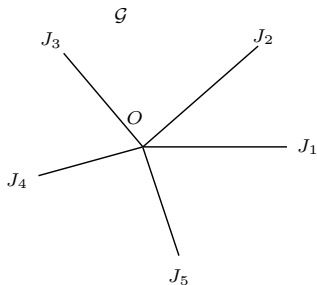
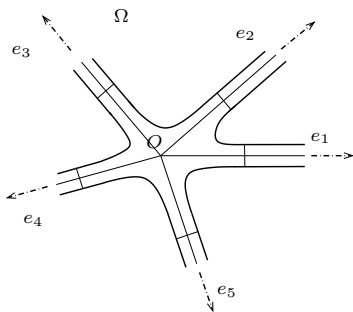
- Then $\Omega \setminus W = \bigcup_{i=1}^N S_i$, where S_i is the half-strip

$$S_i = \left\{ x_i e_i + x_i^\perp e_i^\perp, \quad x_i > r_0, \quad |x_i^\perp| < 1 \right\}$$



$\Omega_\epsilon = \epsilon\Omega$ “tends” to a network \mathcal{G} as $\epsilon \rightarrow 0$

$$\mathcal{G} = \{O\} \cup \bigcup_{i=1}^N J_i,$$
$$J_i = \{x_i e_i, x_i > 0\}.$$



State constrained control problems in $\overline{\Omega_\epsilon}$

$$u_\epsilon(x) = \inf_{\alpha} \int_0^{\infty} \ell_\epsilon(y_\epsilon(t; x, \alpha), \alpha(t)) e^{-\lambda t} dt$$

subject to

$$\begin{cases} \dot{y}_\epsilon(t; x, \alpha) &= \alpha(t), & \text{a.a. } t > 0, \\ y_\epsilon(0; x, \alpha) &= x, \end{cases}$$

with $\alpha : \mathbb{R}_+ \rightarrow A$ measurable, under the state constraint

$$y_\epsilon(t) \in \overline{\Omega_\epsilon}, \quad \forall t \geq 0.$$

Controlability : A is a compact subset of \mathbb{R}^2 such that $B(0, r) \subset A$ for some $r > 0$.

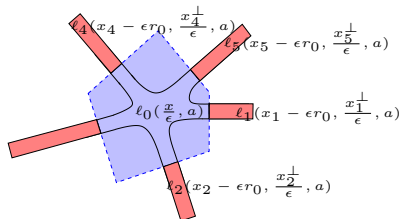
Assumptions on the running costs

- The function $\ell_\epsilon : \Omega_\epsilon \rightarrow \mathbb{R}$ is bounded and continuous
-

$$\ell_\epsilon(x, a) = \begin{cases} \ell_i \left(x_i - \epsilon r_0, \frac{x_i^\perp}{\epsilon}, a \right) & \text{in } \epsilon S_i \\ \ell_0 \left(\frac{x}{\epsilon}, a \right) & \text{in } \epsilon W \cap \Omega_\epsilon \end{cases}$$

$$\ell_i : [0, +\infty) \times [-1, 1] \times A \rightarrow \mathbb{R}$$

$$\ell_0 : (W \cap \Omega) \times A \rightarrow \mathbb{R}$$



ℓ_i and ℓ_0 match properly in order to ensure the continuity of ℓ_ϵ

Assumptions on the running costs

- The function $\ell_\epsilon : \Omega_\epsilon \rightarrow \mathbb{R}$ is bounded and continuous
-

$$\ell_\epsilon(x, a) = \begin{cases} \ell_i \left(x_i - \epsilon r_0, \frac{x_i^\perp}{\epsilon}, a \right) & \text{in } \epsilon S_i \\ \ell_0 \left(\frac{x}{\epsilon}, a \right) & \text{in } \epsilon W \cap \Omega_\epsilon \end{cases}$$

Remark

In ϵS_i , ℓ_ϵ varies slowly w.r.t. x_i and fast w.r.t. x_i^\perp .

- It is not restrictive to assume that $\ell_0 \geq 0$.

The value function u_ϵ

u_ϵ is bounded uniformly with respect to ϵ , continuous, and is the unique viscosity solution of

$$\begin{aligned}\lambda u_\epsilon(x) + H_\epsilon(x, Du_\epsilon) &\geq 0 && \text{in } \overline{\Omega}_\epsilon, \\ \lambda u_\epsilon(x) + H_\epsilon(x, Du_\epsilon) &\leq 0 && \text{in } \Omega_\epsilon,\end{aligned}$$

where

$$H_\epsilon(x, p) = \max_{a \in A} \left(-p \cdot a - \ell_\epsilon(x, a) \right).$$

Questions

- Asymptotic behavior of u_ϵ as $\epsilon \rightarrow 0$?
- Can we define an effective problem in \mathcal{G} ?

Background and additional new difficulties

- Existing results:
 - Singularly perturbed control problems in thin domains around smooth manifolds: Bensoussan, Alvarez-Bardi, Arztein-Gaitsgory, Gaitsgory-Leisarowitz, Terrone,...
 - Comparison for viscosity solutions on networks: YA-Oudet-Tchou and Imbert-Monneau
 - Very recent results of Lions-Souganidis on homogenization of HJB equations with defects

- Several additional new difficulties:
 - Identification of the effective problem
 - Will the effective problem keep track of ℓ_0 near O ? How?
 - In the perturbed test-function method of Evans, we need to construct correctors in unbounded domains.

Main result

Theorem

Under a further technical assumption, u_ϵ converges locally uniformly to the bounded viscosity solution $u : \mathcal{G} \rightarrow \mathbb{R}$ of

$$\left\{ \begin{array}{l} \lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) = 0 \quad x \in J_i, \\ \lambda u(O) + \max \left(E, \overline{H}(O, \frac{du}{dx_1}(O), \dots, \frac{du}{dx_N}(O)) \right) = 0, \end{array} \right.$$

Main result

Theorem

Under a further technical assumption, u_ϵ converges locally uniformly to the bounded viscosity solution $u : \mathcal{G} \rightarrow \mathbb{R}$ of

$$\begin{cases} \lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) = 0 & x \in J_i, \\ \lambda u(O) + \max \left(E, \overline{H}(O, \frac{du}{dx_1}(O), \dots, \frac{du}{dx_N}(O)) \right) = 0, \end{cases}$$

with

$$\overline{H}_i(x_i, p_i) = \sup_{\mu \in \mathcal{Z}_i} \left(\int_{[-1,1] \times A} (-p_i a_i - \ell_i(x_i, y, a)) d\mu(y, a) \right)$$

\mathcal{Z}_i is a compact and convex set of Radon probability measures on $[-1, 1] \times A$

\mathcal{Z}_i : limiting relaxed controls.

Main result

Theorem

Under a further technical assumption, u_ϵ converges locally uniformly to the bounded viscosity solution $u : \mathcal{G} \rightarrow \mathbb{R}$ of

$$\left\{ \begin{array}{l} \lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) = 0 \quad x \in J_i, \\ \lambda u(O) + \max \left(E, \overline{H}(O, \frac{du}{dx_1}(O), \dots, \frac{du}{dx_N}(O)) \right) = 0, \end{array} \right.$$

with

– E : an effective cost at the junction

Main result

Theorem

Under a further technical assumption, u_ϵ converges locally uniformly to the bounded viscosity solution $u : \mathcal{G} \rightarrow \mathbb{R}$ of

$$\begin{cases} \lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) = 0 & x \in J_i, \\ \lambda u(O) + \max \left(E, \overline{H}(O, \frac{du}{dx_1}(0), \dots, \frac{du}{dx_N}(0)) \right) = 0, \end{cases}$$

with $\overline{H}(O, p_1, \dots, p_N) := \max_{i=1, \dots, N} \overline{H}_i^+(0, p_i)$,

$$\overline{H}_i^+(0, p_i) = \sup_{\mu \in \mathcal{Z}_i^+} \left(\int_{[-1,1] \times A} \left(-p_i a_i - \ell_i(0, y, a) \right) d\mu(y, a) \right),$$

$$\mathcal{Z}_i^+ = \left\{ \mu \in \mathcal{Z}_i \text{ s.t. } \int_{[-1,1] \times A} a_i d\mu(y, a) \geq 0 \right\}.$$

What is the meaning of

$$\begin{cases} \lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) = 0 & x \in J_i, \\ \lambda u(O) + \max \left(E, \overline{H}(O, \frac{du}{dx_1}(0), \dots, \frac{du}{dx_N}(0)) \right) = 0 \end{cases} ? \quad (1)$$

See YA-Camilli-Cutri-Tchou(2013), Imbert-Monneau-Zidani(2013), YA-Oudet-Tchou(2014), and Imbert-Monneau(2014).

Definition (Test functions)

- $\phi : \mathcal{G} \rightarrow \mathbb{R}$ is an admissible test-function if
 - ϕ is continuous on \mathcal{G}
 - for any $j \in 1, \dots, N$, $\phi|_{\overline{J}_j} \in \mathcal{C}^1(\overline{J}_j)$
- $\mathcal{R}(\mathcal{G})$: set of the admissible test-functions

- An usc function $u : \mathcal{G} \rightarrow \mathbb{R}$ is a sub-solution of (1) if for any $x \in \mathcal{G}$ and $\phi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \phi$ has a local maximum in x , then

$$\begin{aligned} \lambda u(x) + \bar{H}_i(x, \frac{d\phi}{dx_i}(x)) &\leq 0 && \text{if } x \in J_i, \\ \lambda u(O) + \max \left(E, \bar{H} \left(\frac{d\phi}{dx_1}(O), \dots, \frac{d\phi}{dx_N}(O) \right) \right) &\leq 0. \end{aligned}$$

- A lsc function $u : \mathcal{G} \rightarrow \mathbb{R}$ is a super-solution of (1) if for any $x \in \mathcal{G}$ and $\phi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \phi$ has a local minimum in x , then

$$\begin{aligned} \lambda u(x) + \bar{H}_i(x, \frac{d\phi}{dx_i}(x)) &\geq 0 && \text{if } x \in J_i, \\ \lambda u(O) + \max \left(E, \bar{H} \left(\frac{d\phi}{dx_1}(O), \dots, \frac{d\phi}{dx_N}(O) \right) \right) &\geq 0. \end{aligned}$$

The effective Hamiltonians away from the junctions

Notation: for $x \geq 0, y \in [-1, 1], p \in \mathbb{R}^2$,

$$H_i(x, y, p) = \max_{a \in A} (-p \cdot a - \ell_i(x, y, a))$$

Theorem (Alvarez-Bardi 2000)

$\forall x_i \geq 0, \forall p_i \in \mathbb{R}$, there is a unique $\bar{H}_i(x_i, p_i)$ s.t. the cell problem

$$H_i(x_i, y, p_i e_i + D_y \chi_i(y) e_i^\perp) \geq \bar{H}_i(x_i, p_i) \quad y \in [-1, 1],$$

$$H_i(x_i, y, p_i e_i + D_y \chi_i(y) e_i^\perp) \leq \bar{H}_i(x_i, p_i) \quad y \in (-1, 1)$$

has a viscosity solution $\chi_i \in \text{Lip}([-1, 1])$.

Limiting relaxed controls and limit control problem

For $s > 0$, the *occupational measure* μ_s generated by $(y(t), \alpha(t))$ is the Radon probability measure defined on $[-1, 1] \times A$ by

$$\mu_s = \frac{1}{s} \int_0^s \delta_{(y(t), \alpha(t))} dt$$

where $\delta_{(y(t), \alpha(t))}$ is the Dirac mass concentrated at $(y(t), \alpha(t))$.

$\mathcal{Z}(s; i, y_0)$: set of the occupational measures generated by the trajectories $(y(t), \alpha(t))$ up to time s .

Limiting relaxed controls and limit control problem

For $s > 0$, the *occupational measure* μ_s generated by $(y(t), \alpha(t))$ is the Radon probability measure defined on $[-1, 1] \times A$ by

$$\mu_s = \frac{1}{s} \int_0^s \delta_{(y(t), \alpha(t))} dt$$

where $\delta_{(y(t), \alpha(t))}$ is the Dirac mass concentrated at $(y(t), \alpha(t))$.

$\mathcal{Z}(s; i, y_0)$: set of the occupational measures generated by the trajectories $(y(t), \alpha(t))$ up to time s .

Theorem (Gaitsgory and Leisarovitz (1999))

There exists a set $\mathcal{Z}_i \subset \mathcal{P}([-1, 1] \times A)$ s.t

$$\forall y_0 \in [-1, 1], \quad \lim_{s \rightarrow \infty} \pi_H(\mathcal{Z}(s; i, y_0), \mathcal{Z}_i) = 0,$$

(π_H : Prokhoroff distance).

\mathcal{Z}_i is convex and compact for the weak- topology.*

By using results from Terrone-2011 and Alvarez-Bardi-2000

•

$$\mathcal{Z}_i \subset \left\{ \mu \in \mathcal{P}([-1, 1] \times A), \int_{[-1, 1] \times A} a_i^\perp d\mu(y, a) = 0 \right\}$$

• \mathcal{Z}_i coincides with the set of limiting relaxed controls, i.e.

$$\begin{aligned} & \bar{H}_i(x_i, p_i) \\ &= \sup_{\mu \in \mathcal{Z}_i} \left(-p_i \int_{[-1, 1] \times A} a_i d\mu(y, a) - \int_{[-1, 1] \times A} \ell_i(x_i, y, a) d\mu(y, a) \right) \end{aligned}$$

Interpretation

Away from the junction, the effective Hamiltonian corresponds to a control problem with controls in \mathcal{Z}_i .

The effective Hamiltonian at the junction

Definition

The effective Hamiltonian at the junction is $\max(E, \overline{H}(O, \cdot))$ where E is a suitable constant to be defined and

$$\overline{H}(O, p_1, \dots, p_N) := \max_{i=1, \dots, N} \overline{H}_i^+(0, p_i)$$

$$\begin{aligned} & \overline{H}_i^+(0, p_i) \\ &= \sup_{\mu \in \mathcal{Z}_i^+} \left(-p_i \int_{[-1,1] \times A} a_i d\mu(y, a) - \int_{[-1,1] \times A} \ell_i(0, y, a) d\mu(y, a) \right) \end{aligned}$$

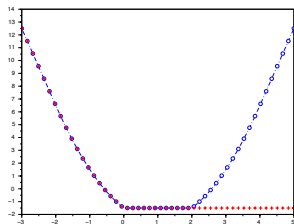
and

$$\mathcal{Z}_i^+ = \left\{ \mu \in \mathcal{Z}_i \text{ s.t. } \int_{[-1,1] \times A} a_i d\mu(y, a) \geq 0 \right\}$$

Lemma: links between $\overline{H}_i(0, p)$ and $\overline{H}_i^+(0, p)$

$$\mathcal{Z}_i^0 := \left\{ \mu \in \mathcal{Z}_i \text{ s.t. } \int_{[-1,1] \times A} a_i d\mu(y, a) = 0 \right\}$$

$$\operatorname{argmin} \overline{H}_i(0, \cdot) = \left[\underline{p}_i, \bar{p}_i \right]$$



The graphs of
 $p \mapsto \overline{H}_i(0, p)$ and
of $p \mapsto \overline{H}_i^+(0, p)$

- $p < \underline{p}_i \Rightarrow \overline{H}_i(0, p) = \overline{H}_i^+(0, p)$ is achieved for $\mu \in \mathcal{Z}_i^+ \setminus \mathcal{Z}_i^0$
- $\underline{p}_i < p < \bar{p}_i \Rightarrow \overline{H}_i(0, p) = \overline{H}_i^+(0, p)$ is achieved for $\mu \in \mathcal{Z}_i^0$
- $p > \bar{p}_i \Rightarrow \overline{H}_i(0, p) > \overline{H}_i^+(0, p) = \min \overline{H}_i(0, \cdot)$

The constant E

Zoom near the junction point O

We extend the function ℓ_0 to the whole domain $\bar{\Omega}$ by setting

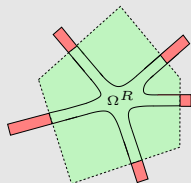
$$\ell_0(z, a) = \ell_i(0, z_i^\perp, a), \quad \text{if } z_i \geq r_0, |z_i^\perp| \leq 1.$$

Ergodic constants in bounded subdomains

Define $\Omega^R := \Omega \cap W_R$ and the ergodic constant E^R , which is the unique number s.t.

$$\begin{aligned} H_0(z, Dw^R(z)) &\geq E^R && \text{in } \bar{\Omega}^R, \\ H_0(z, Dw^R(z)) &\leq E^R && \text{in } \Omega^R, \end{aligned}$$

has a viscosity solution w^R .



The constant E

Lemma

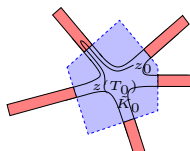
$\exists C > 0$ s.t.

$$\begin{aligned} |E^R| &\leq C && \forall R, \\ \|Dw^R\|_\infty &\leq C && \forall R, \\ R \leq R' &\Rightarrow && E^R \leq E^{R'}. \end{aligned}$$

Definition

$$E := \lim_{R \rightarrow \infty} E^R.$$

The constant E



Theorem

$\exists C > 0$ s.t. $\forall z_0 \in \tilde{K}_0, \forall \alpha \in \mathcal{A}_{z_0}, \forall T > 0,$

if $z(s) := z_0 + \int_0^s \alpha(\theta) d\theta$ satisfies $z(T) \in \tilde{K}_0$, then

$$\int_0^T \ell_0(z(s), \alpha(s)) ds \geq -ET - C.$$

Lemma

Under the assumptions, $\forall i = 1, \dots, N,$ $\min_{p_i} \bar{H}_i(0, p_i) \leq E.$

Proof of the convergence result

The main step is to prove the following theorem

Theorem

Consider the relaxed semilimits, of u_ϵ : for all $x \in \mathcal{G}$,

$$\underline{u}(x) = \liminf_{\epsilon \rightarrow 0+, x' \rightarrow x, x' \in \Omega_\epsilon} u_\epsilon(x'); \quad \bar{u}(x) = \limsup_{\epsilon \rightarrow 0+, x' \rightarrow x, x' \in \Omega_\epsilon} u_\epsilon(x').$$

Then, under the assumptions, \underline{u} is a bounded supersolution of

$$\begin{aligned} \lambda \underline{u}(x) + \overline{H}(x, D\underline{u}(x)) &\geq 0, & \text{if } x \in \mathcal{G} \setminus \{O\}, \\ \lambda \underline{u}(O) + \max(E, \overline{H}(O, D\underline{u}(O))) &\geq 0, & \text{if } x = O, \end{aligned}$$

and \bar{u} is a bounded subsolution of

$$\begin{aligned} \lambda \bar{u}(x) + \overline{H}(x, D\bar{u}(x)) &\leq 0, & \text{if } x \in \mathcal{G} \setminus \{O\}, \\ \lambda \bar{u}(O) + \max(E, \overline{H}(O, D\bar{u}(O))) &\leq 0, & \text{if } x = O. \end{aligned}$$

then use comparison results on \mathcal{G} , [AOT2014] or [IM2014]

Some ideas of the proof

- Not standard only at the junction
 - Try to use Evans' method of perturbed test-functions
 - The test-functions ϕ have N slopes at $O : p = (p_1, \dots, p_N)$
 - Evans' method requires the construction of bounded correctors in unbounded domains, (ideally Ω)
 - Bounded correctors may not exist in the full domain Ω , because, given a set of slopes at O , i.e. $p = (p_1, \dots, p_N)$,

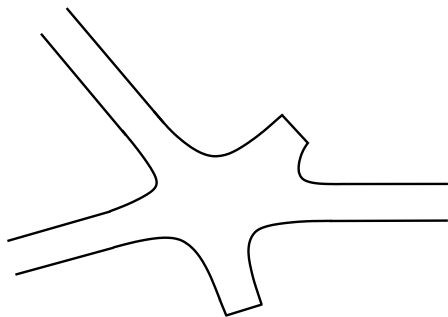
$$\overline{H}(O, p_1, \dots, p_n) = \max_{i=1, \dots, N} \overline{H}_i^+(0, p_i)$$

is achieved by $i \in \mathcal{I}(p) \subset \{1, \dots, N\}$. If $j \notin \mathcal{I}(p)$, then the optimal trajectories starting from $z \in S_j$ should leave S_j : this leads to the fact that the corrector should not be bounded when $|z| \rightarrow \infty$, $z \in S_j$.

- Hence, we construct the correctors in subdomain obtained by truncating the half-strips S_j if $j \notin \mathcal{I}(p)$.

The corrector (when $\overline{H}(O, p_1, \dots, p_N) > E$)

The truncated domain Ω_p



Theorem

For $p \in \mathbb{R}^N : \overline{H}(O, p) > E$. Let ψ_p be a smooth function such that $D\psi_p = p_i e_i$ in S_i . Under the assumptions, $\exists \chi_p$, a bounded and Lipschitz viscosity solution of

$$H_0(z, D\psi_p + D\chi_p) - \overline{H}(O, p) \geq 0 \quad \text{in } \overline{\Omega}_p,$$

$$H_0(z, D\psi_p + D\chi_p) - \overline{H}(O, p) \leq 0 \quad \text{in } \Omega_p.$$

The further assumption

Assumption

1) For any real number p_i such that $p_i < \underline{p}_i$,

there exist two constants $L_i \geq 0$ and $C_i > 0$ such that

$\forall y_0 \in [-1, 1], \forall t > 0$, there exists a control law $\tilde{\alpha} \in \mathcal{A}_{i, y_0}$ with

$$\int_0^s \tilde{\alpha}_i(\tau) d\tau \geq -L_i, \quad \forall 0 \leq s \leq t,$$
$$\int_0^t (p_i \tilde{\alpha}_i(s) + \ell_i(0, y(s), \tilde{\alpha}(s))) ds \leq v_i(0, p_i, y_0, t) + C_i,$$

where $y(t) = y_0 + \int_0^t \tilde{\alpha}_i^\perp(s) ds$.

Recall that

$$v_i(x_i, p_i, y_0, t) = \inf_{\alpha \in \mathcal{A}_{i, y_0}} \left\{ \int_0^t p_i \alpha_i(s) + \ell_i(x_i, y(s), \alpha(s)) ds \right\}$$