Hamilton-Jacobi equations on networks as limits of singularly perturbed problems

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joint work with N. Tchou

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Setting: a thick version of a network

Consider a domain $\Omega \subset \mathbb{R}^2$:

- star-shaped w.r.t. the origin $O$
- $\partial \Omega$ is smooth
- Far enough from the origin $O$, $\Omega$ coincides with the union of $N$ non-intersecting semi-infinite strips directed by the vectors $e_i$, $i = 1, \ldots, N$
Setting: a more precise definition

- for some $r_0 > 0$, let $W$ be the polygonal set
  \[ W = \{ x \in \mathbb{R}^2 : \ x \cdot e_i \leq r_0, \ \forall i = 1, \ldots, N \} \]
- Then \( \Omega \setminus W = \bigcup_{i=1}^{N} S_i \), where \( S_i \) is the half-strip
  \[ S_i = \left\{ x_i e_i + x_i^\perp e_i^\perp, \ x_i > r_0, \ |x_i^\perp| < 1 \right\} \]
$\Omega_\epsilon = \epsilon \Omega$ “tends” to a network $\mathcal{G}$ as $\epsilon \to 0$

$$\mathcal{G} = \{O\} \cup \bigcup_{i=1}^{N} J_i,$$

$$J_i = \{x_i e_i, \ x_i > 0\}.$$
State constrained control problems in $\Omega_\epsilon$

$$u_\epsilon(x) = \inf_\alpha \int_0^\infty \ell_\epsilon(y_\epsilon(t; x, \alpha), \alpha(t)) e^{-\lambda t} dt$$

subject to

$$\left\{ \begin{array}{l}
\dot{y}_\epsilon(t; x, \alpha) = \alpha(t), \quad \text{a.a. } t > 0, \\
y_\epsilon(0; x, \alpha) = x,
\end{array} \right.$$ 

with $\alpha : \mathbb{R}_+ \to A$ measurable, under the state constraint

$$y_\epsilon(t) \in \Omega_\epsilon, \quad \forall t \geq 0.$$

Controlability: $A$ is a compact subset of $\mathbb{R}^2$ such that $B(0, r) \subset A$ for some $r > 0$. 

Y. Achdou \quad Dimension reduction
Assumptions on the running costs

- The function \( \ell_\epsilon : \Omega_\epsilon \to \mathbb{R} \) is bounded and continuous.

\[
\ell_\epsilon(x, a) = \begin{cases} 
\ell_i \left( x_i - \epsilon r_0, \frac{x_i}{\epsilon}, a \right) & \text{in } \epsilon S_i \\
\ell_0 \left( \frac{x}{\epsilon}, a \right) & \text{in } \epsilon W \cap \Omega_\epsilon 
\end{cases}
\]

\( \ell_i : [0, +\infty) \times [-1, 1] \times A \to \mathbb{R} \)

\( \ell_0 : (W \cap \Omega) \times A \to \mathbb{R} \)

\( \ell_i \) and \( \ell_0 \) match properly in order to ensure the continuity of \( \ell_\epsilon \)
Assumptions on the running costs

- The function \( \ell_\epsilon : \Omega_\epsilon \rightarrow \mathbb{R} \) is bounded and continuous

\[
\ell_\epsilon(x, a) = \begin{cases} 
\ell_i \left( x_i - \epsilon r_0, \frac{x_i^\perp}{\epsilon}, a \right) & \text{in } \epsilon S_i \\
\ell_0 \left( \frac{x}{\epsilon}, a \right) & \text{in } \epsilon W \cap \Omega_\epsilon 
\end{cases}
\]

**Remark**

In \( \epsilon S_i \), \( \ell_\epsilon \) varies slowly w.r.t. \( x_i \) and fast w.r.t. \( x_i^\perp \).

- It is not restrictive to assume that \( \ell_0 \geq 0 \).
The value function \( u_\epsilon \)

\( u_\epsilon \) is bounded uniformly with respect to \( \epsilon \), continuous, and is the unique viscosity solution of

\[
\begin{align*}
\lambda u_\epsilon(x) + H_\epsilon(x, Du_\epsilon) &\geq 0 \quad \text{in } \overline{\Omega}_\epsilon, \\
\lambda u_\epsilon(x) + H_\epsilon(x, Du_\epsilon) &\leq 0 \quad \text{in } \Omega_\epsilon,
\end{align*}
\]

where

\[
H_\epsilon(x, p) = \max_{a \in A} \left( -p \cdot a - \ell_\epsilon(x, a) \right).
\]

Questions

- Asymptotic behavior of \( u_\epsilon \) as \( \epsilon \to 0 \)?
- Can we define an effective problem in \( \mathcal{G} \)?
Existing results:
- Singularly perturbed control problems in thin domains around smooth manifolds: Bensoussan, Alvarez-Bardi, Arztein-Gaitsgory, Gaitsgory-Leisarowitz, Terrone,...
- Comparison for viscosity solutions on networks: YA-Oudet-Tchou and Imbert-Monneau
- Very recent results of Lions-Souganidis on homogenization of HJB equations with defects

Several additional new difficulties:
- Identification of the effective problem
- Will the effective problem keep track of $\ell_0$ near $O$? How?
- In the perturbed test-function method of Evans, we need to construct correctors in unbounded domains.
Main result

Theorem

Under a further technical assumption, \( u_\varepsilon \) converges locally uniformly to the bounded viscosity solution \( u : \mathcal{G} \to \mathbb{R} \) of

\[
\begin{cases}
\lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) = 0 & x \in J_i, \\
\lambda u(O) + \max \left( E, \overline{H}(O, \frac{du}{dx_1}(0), \ldots, \frac{du}{dx_N}(0)) \right) = 0,
\end{cases}
\]

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Dimension reduction
Main result

**Theorem**

Under a further technical assumption, $u_\varepsilon$ converges locally uniformly to the bounded viscosity solution $u : \mathcal{G} \rightarrow \mathbb{R}$ of

\[
\begin{cases}
\lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) &= 0 \quad x \in J_i, \\
\lambda u(O) + \max\left(\mathcal{E}, \overline{H}(O, \frac{du}{dx_1}(0), \ldots, \frac{du}{dx_N}(0))\right) &= 0,
\end{cases}
\]

with

\[
\overline{H}_i(x_i, p_i) = \sup_{\mu \in \mathcal{Z}_i} \left(\int_{[-1,1] \times A} \left(-p_ia_i - \ell_i(x_i, y, a)\right) d\mu(y, a)\right)
\]

$\mathcal{Z}_i$ is a compact and convex set of Radon probability measures on $[-1, 1] \times A$

$\mathcal{Z}_i$: limiting relaxed controls.
Main result

Theorem

Under a further technical assumption, \( u_\varepsilon \) converges locally uniformly to the bounded viscosity solution \( u : \mathcal{G} \to \mathbb{R} \) of

\[
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\lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) &= 0 \quad x \in J_i, \\
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\end{align*}
\]

with

\(-E : \) an effective cost at the junction.

Y. Achdou — Dimension reduction
Main result

**Theorem**

Under a further technical assumption, $u_\varepsilon$ converges locally uniformly to the bounded viscosity solution $u : \mathcal{G} \to \mathbb{R}$ of

$$
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\end{align*}
$$

with

$$
\overline{H}(O, p_1, \ldots, p_N) := \max_{i=1, \ldots, N} \overline{H}_i^+(0, p_i),
$$

$$
\overline{H}_i^+(0, p_i) = \sup_{\mu \in \mathcal{Z}_i^+} \left( \int_{[-1,1] \times A} \left( -p_ia_i - \ell_i(0, y, a) \right) d\mu(y, a) \right),
$$

$$
\mathcal{Z}_i^+ = \left\{ \mu \in \mathcal{Z}_i \text{ s.t.} \int_{[-1,1] \times A} a_id\mu(y, a) \geq 0 \right\}.
$$
Hamilton-Jacobi on the network   \(^{(1/2)}\)

What is the meaning of

\[
\begin{align*}
\lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) &= 0 \quad x \in J_i, \\
\lambda u(O) + \max \left( E, \overline{H}(O, \frac{du}{dx_1}(0), \ldots, \frac{du}{dx_N}(0)) \right) &= 0
\end{align*}
\]

(1)


**Definition (Test functions)**

- \( \phi : \mathcal{G} \to \mathbb{R} \) is an admissible test-function if
  - \( \phi \) is continuous on \( \mathcal{G} \)
  - for any \( j \in 1, \ldots, N \), \( \phi|_{\overline{J}_j} \in \mathcal{C}^1(\overline{J}_j) \)
- \( \mathcal{R}(\mathcal{G}) \): set of the admissible test-functions
An usc function $u : \mathcal{G} \rightarrow \mathbb{R}$ is a sub-solution of (1) if for any $x \in \mathcal{G}$ and $\phi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \phi$ has a local maximum in $x$, then

$$
\lambda u(x) + \overline{H}_i(x, \frac{d\phi}{dx_i}(x)) \leq 0 \quad \text{if } x \in J_i,
$$

$$
\lambda u(O) + \max \left( E, \overline{H}(\frac{d\phi}{dx_1}(O), \ldots, \frac{d\phi}{dx_N}(O)) \right) \leq 0.
$$

A lsc function $u : \mathcal{G} \rightarrow \mathbb{R}$ is a super-solution of (1) if for any $x \in \mathcal{G}$ and $\phi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \phi$ has a local minimum in $x$, then

$$
\lambda u(x) + \overline{H}_i(x, \frac{d\phi}{dx_i}(x)) \geq 0 \quad \text{if } x \in J_i,
$$

$$
\lambda u(O) + \max \left( E, \overline{H}(\frac{d\phi}{dx_1}(O), \ldots, \frac{d\phi}{dx_N}(O)) \right) \geq 0.
$$
The effective Hamiltonians away from the junctions

Notation: for \( x \geq 0, y \in [-1, 1], p \in \mathbb{R}^2, \)

\[
H_i(x, y, p) = \max_{a \in A} (-p \cdot a - \ell_i(x, y, a))
\]

Theorem (Alvarez-Bardi 2000)

\( \forall x_i \geq 0, \forall p_i \in \mathbb{R}, \) there is a unique \( \overline{H}_i(x_i, p_i) \) s.t. the cell problem

\[
\begin{align*}
H_i(x_i, y, p_ie_i + D_y \chi_i(y)e_i^\perp) &\geq \overline{H}_i(x_i, p_i) & y \in [-1, 1], \\
H_i(x_i, y, p_ie_i + D_y \chi_i(y))e_i^\perp &\leq \overline{H}_i(x_i, p_i) & y \in (-1, 1)
\end{align*}
\]

has a viscosity solution \( \chi_i \in \text{Lip}([-1, 1]). \)
Limiting relaxed controls and limit control problem

For \( s > 0 \), the occupational measure \( \mu_s \) generated by \((y(t), \alpha(t))\) is the Radon probability measure defined on \([-1, 1] \times A\) by

\[
\mu_s = \frac{1}{s} \int_0^s \delta_{(y(t), \alpha(t))} \, dt
\]

where \( \delta_{(y(t), \alpha(t))} \) is the Dirac mass concentrated at \((y(t), \alpha(t))\).

\( \mathcal{Z}(s; i, y_0) \): set of the occupational measures generated by the trajectories \((y(t), \alpha(t))\) up to time \( s \).
Limiting relaxed controls and limit control problem

For \( s > 0 \), the occupational measure \( \mu_s \) generated by \((y(t), \alpha(t))\) is the Radon probability measure defined on \([-1, 1] \times A\) by

\[
\mu_s = \frac{1}{s} \int_{0}^{s} \delta_{(y(t), \alpha(t))} \, dt
\]

where \( \delta_{(y(t), \alpha(t))} \) is the Dirac mass concentrated at \((y(t), \alpha(t))\).

\( \mathcal{Z}(s; i, y_0) \): set of the occupational measures generated by the trajectories \((y(t), \alpha(t))\) up to time \( s \).

**Theorem (Gaitsgory and Leisarowitz (1999))**

There exists a set \( \mathcal{Z}_i \subset \mathcal{P}([-1, 1] \times A) \) s.t.

\[
\forall y_0 \in [-1, 1], \quad \lim_{s \to \infty} \pi_H(\mathcal{Z}(s; i, y_0), \mathcal{Z}_i) = 0,
\]

(\( \pi_H \): Prokhoroff distance).

\( \mathcal{Z}_i \) is convex and compact for the weak-* topology.
By using results from Terrone-2011 and Alvarez-Bardi-2000

\[ \mathcal{Z}_i \subset \left\{ \mu \in \mathcal{P}([-1, 1] \times A), \int_{[-1,1] \times A} a_i^\perp d\mu(y, a) = 0 \right\} \]

\[ \mathcal{Z}_i \text{ coincides with the set of limiting relaxed controls, i.e.} \]

\[ \overline{H}_i(x_i, p_i) = \sup_{\mu \in \mathcal{Z}_i} \left( -p_i \int_{[-1,1] \times A} a_i d\mu(y, a) - \int_{[-1,1] \times A} \ell_i(x_i, y, a) d\mu(y, a) \right) \]

**Interpretation**

Away from the junction, the effective Hamiltonian corresponds to a control problem with controls in \( \mathcal{Z}_i \).
The effective Hamiltonian at the junction

Definition

The effective Hamiltonian at the junction is \( \max(E, \overline{H}(O, \cdot)) \) where \( E \) is a suitable constant to be defined and

\[
\overline{H}(O, p_1, \ldots, p_N) := \max_{i=1,\ldots,N} \overline{H}_i^+(0, p_i)
\]

\[
\overline{H}_i^+(0, p_i)
\]

\[
= \sup_{\mu \in \mathcal{Z}_i^+} \left( -p_i \int_{[-1,1] \times A} a_i d\mu(y, a) - \int_{[-1,1] \times A} \ell_i(0, y, a) d\mu(y, a) \right)
\]

and

\[
\mathcal{Z}_i^+ = \left\{ \mu \in \mathcal{Z}_i \text{ s.t. } \int_{[-1,1] \times A} a_i d\mu(y, a) \geq 0 \right\}
\]
Lemma: links between \( \overline{H}_i(0, p) \) and \( \overline{H}_i^+(0, p) \)

\[
\mathcal{Z}_i^0 := \left\{ \mu \in \mathcal{Z}_i \text{ s.t. } \int_{[-1,1] \times A} a_i d\mu(y, a) = 0 \right\}
\]

\[
\arg\min \overline{H}_i(0, \cdot) = \left[ p_i, \overline{p}_i \right]
\]

The graphs of \( p \mapsto \overline{H}_i(0, p) \) and of \( p \mapsto \overline{H}_i^+(0, p) \)

\[ p < p_i \Rightarrow \overline{H}_i(0, p) = \overline{H}_i^+(0, p) \text{ is achieved for } \mu \in \mathcal{Z}_i^+ \setminus \mathcal{Z}_i^0 \]

\[ p_i < p < \overline{p}_i \Rightarrow \overline{H}_i(0, p) = \overline{H}_i^+(0, p) \text{ is achieved for } \mu \in \mathcal{Z}_i^0 \]

\[ p > \overline{p}_i \Rightarrow \overline{H}_i(0, p) > \overline{H}_i^+(0, p) = \min \overline{H}_i(0, \cdot) \]
The constant $E$

**Zoom near the junction point $O$**

We extend the function $\ell_0$ to the whole domain $\overline{\Omega}$ by setting

$$
\ell_0(z, a) = \ell_i(0, z_i^\perp, a), \quad \text{if } z_i \geq r_0, \quad |z_i^\perp| \leq 1.
$$

**Ergodic constants in bounded subdomains**

Define $\Omega^R := \Omega \cap W_R$ and the ergodic constant $E^R$, which is the unique number s.t.

$$
H_0(z, Dw^R(z)) \geq E^R \quad \text{in } \overline{\Omega}^R,
$$
$$
H_0(z, Dw^R(z)) \leq E^R \quad \text{in } \Omega^R,
$$

has a viscosity solution $w^R$. 

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The constant $E$

**Lemma**

$\exists C > 0$ s.t.

\[
|E^R| \leq C \quad \forall R, \\
\|Dw^R\|_\infty \leq C \quad \forall R, \\
R \leq R' \quad \Rightarrow \quad E^R \leq E^{R'}.
\]

**Definition**

\[
E := \lim_{R \to \infty} E^R.
\]

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Dimension reduction
The constant $E$

**Theorem**

$\exists C > 0$ s.t. $\forall z_0 \in \tilde{K}_0$, $\forall \alpha \in A_{z_0}$, $\forall T > 0$,

if $z(s) := z_0 + \int_0^s \alpha(\theta)d\theta$ satisfies $z(T) \in \tilde{K}_0$, then

$$\int_0^T \ell_0(z(s), \alpha(s))ds \geq -ET - C.$$

**Lemma**

Under the assumptions, $\forall i = 1, \ldots, N$, $\min_{p_i} \overline{H}_i(0, p_i) \leq E$. 
Proof of the convergence result

The main step is to prove the following theorem

**Theorem**

Consider the relaxed semilimits, of $u_\varepsilon$: for all $x \in \mathcal{G}$,

$$
\underline{u}(x) = \liminf_{\varepsilon \to 0^+, x' \to x, x' \in \Omega_\varepsilon} u_\varepsilon(x'); \quad \overline{u}(x) = \limsup_{\varepsilon \to 0^+, x' \to x, x' \in \Omega_\varepsilon} u_\varepsilon(x').
$$

Then, under the assumptions, $\underline{u}$ is a bounded supersolution of

$$
\lambda \underline{u}(x) + \overline{H}(x, Du(x)) \geq 0, \quad \text{if } x \in \mathcal{G}\setminus\{O\},
$$

$$
\lambda \underline{u}(O) + \max(E, \overline{H}(O, Du(O))) \geq 0, \quad \text{if } x = O,
$$

and $\overline{u}$ is a bounded subsolution of

$$
\lambda \overline{u}(x) + \overline{H}(x, Du(x)) \leq 0, \quad \text{if } x \in \mathcal{G}\setminus\{O\},
$$

$$
\lambda \overline{u}(O) + \max(E, \overline{H}(O, Du(O))) \leq 0, \quad \text{if } x = O.
$$

then use comparison results on $\mathcal{G}$, [AOT2014] or [IM2014]
Some ideas of the proof

- Not standard only at the junction
  - Try to use Evans’ method of perturbed test-functions
  - The test-functions $\phi$ have $N$ slopes at $O: p = (p_1, \ldots, p_N)$
  - Evans’ method requires the construction of bounded correctors in unbounded domains, (ideally $\Omega$)
  - Bounded correctors may not exist in the full domain $\Omega$, because, given a set of slopes at $O$, i.e. $p = (p_1, \ldots, p_N)$,

\[
\overline{H}(O, p_1, \ldots, p_n) = \max_{i=1,\ldots,N} \overline{H}_i^+(0, p_i)
\]

is achieved by $i \in I(p) \subset \{1, \ldots, N\}$. If $j \notin I(p)$, then the optimal trajectories starting from $z \in S_j$ should leave $S_j$: this leads to the fact that the corrector should not be bounded when $|z| \to \infty$, $z \in S_j$.

- Hence, we construct the correctors in subdomain obtained by truncating the half-strips $S_j$ if $j \notin I(p)$. 

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Dimension reduction
The corrector (when $\overline{H}(O, p_1, \ldots, p_N) > E$)

The truncated domain $\Omega_p$

**Theorem**

For $p \in \mathbb{R}^N: \overline{H}(O, p) > E$. Let $\psi_p$ be a smooth function such that $D\psi_p = p_i e_i$ in $S_i$. Under the assumptions, $\exists \chi_p$, a bounded and Lipschitz viscosity solution of

$$ H_0(z, D\psi_p + D\chi_p) - \overline{H}(O, p) \geq 0 \quad \text{in } \overline{\Omega}_p, $$

$$ H_0(z, D\psi_p + D\chi_p) - \overline{H}(O, p) \leq 0 \quad \text{in } \Omega_p. $$
The further assumption

Assumption

1) For any real number \( p_i \) such that \( p_i < \underline{p_i} \),

there exist two constants \( L_i \geq 0 \) and \( C_i > 0 \) such that

\[ \forall y_0 \in [-1, 1], \forall t > 0, \text{there exists a control law } \tilde{\alpha} \in \mathcal{A}_{i,y_0} \text{ with} \]

\[ \int_0^s \tilde{\alpha}_i(\tau)d\tau \geq -L_i, \quad \forall 0 \leq s \leq t, \]

\[ \int_0^t \left( p_i\tilde{\alpha}_i(s) + \ell_i(0, y(s), \tilde{\alpha}(s)) \right)ds \leq v_i(0, p_i, y_0, t) + C_i, \]

where \( y(t) = y_0 + \int_0^t \tilde{\alpha}_i^\perp(s)ds \).

Recall that

\[ v_i(x_i, p_i, y_0, t) = \inf_{\alpha \in \mathcal{A}_{i,y_0}} \left\{ \int_0^t p_i\alpha_i(s) + \ell_i(x_i, y(s), \alpha(s))ds \right\} \]