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Hamilton-Jacobi equations on networks as limits of singularly perturbed problems

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joint work with N. Tchou

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Setting: a thick version of a network

Consider a domain $\Omega \subset \mathbb{R}^2$:

- star-shaped w.r.t. the origin $O$
- $\partial \Omega$ is smooth
- Far enough from the origin $O$, $\Omega$ coincides with the union of $N$ non-intersecting semi-infinite strips directed by the vectors $e_i$, $i = 1, \ldots, N$
Setting: a more precise definition

- for some $r_0 > 0$, let $W$ be the polygonal set
  \[ W = \{ x \in \mathbb{R}^2 : x \cdot e_i \leq r_0, \forall i = 1, \ldots, N \} \]

- Then $\Omega \setminus W = \bigcup_{i=1}^{N} S_i$, where $S_i$ is the half-strip
  \[ S_i = \left\{ x_i e_i + x_i^\perp e_i^\perp, \quad x_i > r_0, \ |x_i^\perp| < 1 \right\} \]
$\Omega_\epsilon = \epsilon \Omega$ “tends” to a network $\mathcal{G}$ as $\epsilon \to 0$

$$\mathcal{G} = \{O\} \cup \bigcup_{i=1}^{N} J_i, \quad J_i = \{x_i e_i, \ x_i > 0\}.$$
State constrained control problems in $\bar{\Omega}_\epsilon$

$$u_\epsilon(x) = \inf_{\alpha} \int_0^\infty \ell_\epsilon\left(y_\epsilon(t; x, \alpha), \alpha(t)\right) e^{-\lambda t} dt$$

subject to

\[
\begin{align*}
\dot{y}_\epsilon(t; x, \alpha) &= \alpha(t), \quad \text{a.a. } t > 0, \\
y_\epsilon(0; x, \alpha) &= x,
\end{align*}
\]

with $\alpha : \mathbb{R}_+ \to A$ measurable, under the state constraint

$$y_\epsilon(t) \in \bar{\Omega}_\epsilon, \quad \forall t \geq 0.$$

Controlability : $A$ is a compact subset of $\mathbb{R}^2$ such that $B(0, r) \subset A$ for some $r > 0$. 

Y. Achdou  \hspace{1cm} Dimension reduction
Assumptions on the running costs

- The function $\ell_\epsilon : \Omega_\epsilon \to \mathbb{R}$ is bounded and continuous

$$
\ell_\epsilon(x, a) = \begin{cases} 
\ell_i \left( x_i - \epsilon r_0, \frac{x_i^\perp}{\epsilon}, a \right) & \text{in } \epsilon S_i \\
\ell_0 \left( \frac{x}{\epsilon}, a \right) & \text{in } \epsilon W \cap \Omega_\epsilon
\end{cases}
$$

$\ell_i : [0, +\infty) \times [-1, 1] \times A \to \mathbb{R}$

$\ell_0 : (W \cap \Omega) \times A \to \mathbb{R}$

$\ell_i$ and $\ell_0$ match properly in order to ensure the continuity of $\ell_\epsilon$
Assumptions on the running costs

- The function $\ell_\epsilon : \Omega_\epsilon \to \mathbb{R}$ is bounded and continuous.

$$
\ell_\epsilon (x, a) = \begin{cases} 
\ell_i \left( x_i - \epsilon r_0, \frac{x_i^\perp}{\epsilon}, a \right) & \text{in } \epsilon S_i \\
\ell_0 \left( \frac{x}{\epsilon}, a \right) & \text{in } \epsilon W \cap \Omega_\epsilon 
\end{cases}
$$

Remark

In $\epsilon S_i$, $\ell_\epsilon$ varies slowly w.r.t. $x_i$ and fast w.r.t. $x_i^\perp$.

- It is not restrictive to assume that $\ell_0 \geq 0$. 

Y. Achdou Dimension reduction
The value function $u_\varepsilon$

$u_\varepsilon$ is bounded uniformly with respect to $\varepsilon$, continuous, and is the unique viscosity solution of

$$
\lambda u_\varepsilon(x) + H_\varepsilon(x, Du_\varepsilon) \geq 0 \quad \text{in } \overline{\Omega}_\varepsilon,
$$

$$
\lambda u_\varepsilon(x) + H_\varepsilon(x, Du_\varepsilon) \leq 0 \quad \text{in } \Omega_\varepsilon,
$$

where

$$
H_\varepsilon(x, p) = \max_{a \in A} \left( -p \cdot a - \ell_\varepsilon(x, a) \right).
$$

Questions

- Asymptotic behavior of $u_\varepsilon$ as $\varepsilon \to 0$?
- Can we define an effective problem in $\mathcal{G}$?
Background and additional new difficulties

Existing results:
- Singularly perturbed control problems in thin domains around smooth manifolds: Bensoussan, Alvarez-Bardi, Arztein-Gaitsgory, Gaitsgory-Leisarowitz, Terrone, ...
- Comparison for viscosity solutions on networks: YA-Oudet-Tchou and Imbert-Monneau
- Very recent results of Lions-Souganidis on homogenization of HJB equations with defects

Several additional new difficulties:
- Identification of the effective problem
- Will the effective problem keep track of $\ell_0$ near $O$? How?
- In the perturbed test-function method of Evans, we need to construct correctors in unbounded domains.
Main result

Theorem

Under a further technical assumption, $u_\varepsilon$ converges locally uniformly to the bounded viscosity solution $u : \mathcal{G} \to \mathbb{R}$ of

$$
\begin{cases}
\lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) = 0 & x \in J_i,
\end{cases}
$$

$$
\begin{cases}
\lambda u(O) + \max \left( E, \overline{H}(O, \frac{du}{dx_1}(0), \ldots, \frac{du}{dx_N}(0)) \right) = 0,
\end{cases}
$$
Main result

Theorem

Under a further technical assumption, $u_\varepsilon$ converges locally uniformly to the bounded viscosity solution $u : \mathcal{G} \to \mathbb{R}$ of

$$\begin{cases}
\lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) = 0 & x \in J_i, \\
\lambda u(O) + \max \left( E, \overline{H}(O, \frac{du}{dx_1}(0), \ldots, \frac{du}{dx_N}(0)) \right) = 0,
\end{cases}$$

with

$$\overline{H}_i(x_i, p_i) = \sup_{\mu \in \mathcal{Z}_i} \left( \int_{[-1,1] \times A} \left( -p_i a_i - \ell_i(x_i, y, a) \right) d\mu(y, a) \right)$$

$\mathcal{Z}_i$ is a compact and convex set of Radon probability measures on $[-1, 1] \times A$

$\mathcal{Z}_i$: limiting relaxed controls.
Main result

Theorem

Under a further technical assumption, \( u_\varepsilon \) converges locally uniformly to the bounded viscosity solution \( u : \mathcal{G} \to \mathbb{R} \) of

\[
\begin{aligned}
\lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) &= 0 \quad x \in J_i, \\
\lambda u(O) + \max \left( E, \overline{H}(O, \frac{du}{dx_1}(0), \ldots, \frac{du}{dx_N}(0)) \right) &= 0,
\end{aligned}
\]

with

\[-E : \text{an effective cost at the junction}\]
Main result

Theorem

Under a further technical assumption, $u_\varepsilon$ converges locally uniformly to the bounded viscosity solution $u : \mathcal{G} \to \mathbb{R}$ of

$$\begin{cases}
\lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) = 0 & x \in J_i, \\
\lambda u(O) + \max \left( E, \overline{H}(O, \frac{du}{dx_1}(0), \ldots, \frac{du}{dx_N}(0)) \right) = 0,
\end{cases}$$

with

$$\overline{H}(O, p_1, \ldots, p_N) := \max_{i=1,\ldots,N} \overline{H}^+_i(0, p_i),$$

$$\overline{H}^+_i(0, p_i) = \sup_{\mu \in \mathcal{Z}^+_i} \left( \int_{[-1,1] \times A} \left( -p_ia_i - \ell_i(0, y, a) \right) d\mu(y, a) \right),$$

$$\mathcal{Z}^+_i = \left\{ \mu \in \mathcal{Z}_i \text{ s.t. } \int_{[-1,1] \times A} a_id\mu(y, a) \geq 0 \right\}.$$
What is the meaning of

\[
\begin{cases}
\lambda u(x) + \overline{H}_i(x, \frac{du}{dx_i}(x)) = 0 & x \in J_i, \\
\lambda u(O) + \max \left( E, \overline{H}(O, \frac{du}{dx_1}(0), \ldots, \frac{du}{dx_N}(0)) \right) = 0
\end{cases}
\]

(1)


**Definition (Test functions)**

- \( \phi : \mathcal{G} \to \mathbb{R} \) is an admissible test-function if
  - \( \phi \) is continuous on \( \mathcal{G} \)
  - for any \( j \in 1, \ldots, N \), \( \phi|_{J_j} \in \mathcal{C}^1(J_j) \)
- \( \mathcal{R}({\mathcal{G}}) \): set of the admissible test-functions
Hamilton-Jacobi on the network  

- An usc function $u : \mathcal{G} \to \mathbb{R}$ is a sub-solution of (1) if for any $x \in \mathcal{G}$ and $\phi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \phi$ has a local maximum in $x$, then

$$\lambda u(x) + \overline{H}_i(x, \frac{d\phi}{dx_i}(x)) \leq 0 \quad \text{if } x \in J_i,$$

$$\lambda u(O) + \max \left( E, \overline{H}(\frac{d\phi}{dx_1}(O), \ldots, \frac{d\phi}{dx_N}(O)) \right) \leq 0.$$

- A lsc function $u : \mathcal{G} \to \mathbb{R}$ is a super-solution of (1) if for any $x \in \mathcal{G}$ and $\phi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \phi$ has a local minimum in $x$, then

$$\lambda u(x) + \overline{H}_i(x, \frac{d\phi}{dx_i}(x)) \geq 0 \quad \text{if } x \in J_i,$$

$$\lambda u(O) + \max \left( E, \overline{H}(\frac{d\phi}{dx_1}(O), \ldots, \frac{d\phi}{dx_N}(O)) \right) \geq 0.$$
The effective Hamiltonians away from the junctions

Notation: for $x \geq 0, y \in [-1, 1], p \in \mathbb{R}^2$,

$$H_i(x, y, p) = \max_{a \in A} (-p \cdot a - \ell_i(x, y, a))$$

Theorem (Alvarez-Bardi 2000)

$\forall x_i \geq 0, \forall p_i \in \mathbb{R}$, there is a unique $\overline{H}_i(x_i, p_i)$ s.t. the cell problem

$$H_i(x_i, y, p_i e_i + D_y \chi_i(y) e_i^\perp) \geq \overline{H}_i(x_i, p_i) \quad y \in [-1, 1],$$
$$H_i(x_i, y, p_i e_i + D_y \chi_i(y) e_i^\perp) \leq \overline{H}_i(x_i, p_i) \quad y \in (-1, 1)$$

has a viscosity solution $\chi_i \in \text{Lip}([-1, 1])$. 

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Dimension reduction
For $s > 0$, the *occupational measure* $\mu_s$ generated by $(y(t), \alpha(t))$ is the Radon probability measure defined on $[-1, 1] \times A$ by

$$\mu_s = \frac{1}{s} \int_0^s \delta_{(y(t), \alpha(t))} \, dt$$

where $\delta_{(y(t), \alpha(t))}$ is the Dirac mass concentrated at $(y(t), \alpha(t))$.

$\mathcal{Z}(s; i, y_0)$: set of the occupational measures generated by the trajectories $(y(t), \alpha(t))$ up to time $s$. 
Limiting relaxed controls and limit control problem

For $s > 0$, the occupational measure $\mu_s$ generated by $(y(t), \alpha(t))$ is the Radon probability measure defined on $[-1, 1] \times A$ by

$$
\mu_s = \frac{1}{s} \int_0^s \delta(y(t), \alpha(t)) \, dt
$$

where $\delta(y(t), \alpha(t))$ is the Dirac mass concentrated at $(y(t), \alpha(t))$.

$\mathcal{Z}(s; i, y_0)$: set of the occupational measures generated by the trajectories $(y(t), \alpha(t))$ up to time $s$.

Theorem (Gaitsgory and Leisarowitz (1999))

There exists a set $\mathcal{Z}_i \subset \mathcal{P}([-1, 1] \times A)$ s.t.

$$
\forall y_0 \in [-1, 1], \quad \lim_{s \to \infty} \pi_H(\mathcal{Z}(s; i, y_0), \mathcal{Z}_i) = 0,
$$

($\pi_H$ : Prokhoroff distance).

$\mathcal{Z}_i$ is convex and compact for the weak-* topology.
By using results from Terrone-2011 and Alvarez-Bardi-2000

\[ Z_i \subset \left\{ \mu \in \mathcal{P}([-1, 1] \times A), \int_{[-1,1] \times A} a_i^\perp d\mu(y, a) = 0 \right\} \]

\( \mathcal{Z}_i \) coincides with the set of limiting relaxed controls, i.e.

\[
\overline{H}_i(x_i, p_i) = \sup_{\mu \in \mathcal{Z}_i} \left( -p_i \int_{[-1,1] \times A} a_i d\mu(y, a) - \int_{[-1,1] \times A} \ell_i(x_i, y, a) d\mu(y, a) \right)
\]

**Interpretation**

Away from the junction, the effective Hamiltonian corresponds to a control problem with controls in \( \mathcal{Z}_i \).
The effective Hamiltonian at the junction

Definition

The effective Hamiltonian at the junction is \( \max(E, \overline{H}(O, \cdot)) \)
where \( E \) is a suitable constant to be defined and

\[
\overline{H}(O, p_1, \ldots, p_N) := \max_{i=1,\ldots,N} \overline{H}_i^+(0, p_i)
\]

\[
\overline{H}_i^+(0, p_i) = \sup_{\mu \in \mathcal{Z}_i^+} \left( -p_i \int_{[-1,1] \times A} a_i d\mu(y, a) - \int_{[-1,1] \times A} \ell_i(0, y, a) d\mu(y, a) \right)
\]

and

\[
\mathcal{Z}_i^+ = \left\{ \mu \in \mathcal{Z}_i \text{ s.t. } \int_{[-1,1] \times A} a_i d\mu(y, a) \geq 0 \right\}
\]
Lemma: links between $\overline{H}_i(0, p)$ and $\overline{H}_i^+(0, p)$

$$\mathcal{Z}_i^0 := \left\{ \mu \in \mathcal{Z}_i \text{ s.t. } \int_{[-1,1] \times A} a_i d\mu(y, a) = 0 \right\}$$

$$\text{argmin} \overline{H}_i(0, \cdot) = [p_i, \overline{p}_i]$$

The graphs of
\[ p \mapsto \overline{H}_i(0, p) \] and
\[ p \mapsto \overline{H}_i^+(0, p) \]

- $p < p_i \Rightarrow \overline{H}_i(0, p) = \overline{H}_i^+(0, p)$ is achieved for $\mu \in \mathcal{Z}_i^+ \setminus \mathcal{Z}_i^0$
- $p_i < p < \overline{p}_i \Rightarrow \overline{H}_i(0, p) = \overline{H}_i^+(0, p)$ is achieved for $\mu \in \mathcal{Z}_i^0$
- $p > \overline{p}_i \Rightarrow \overline{H}_i(0, p) > \overline{H}_i^+(0, p) = \text{min} \overline{H}_i(0, \cdot)$
The constant $E$

Zoom near the junction point $O$

We extend the function $\ell_0$ to the whole domain $\bar{\Omega}$ by setting

$$\ell_0(z, a) = \ell_i(0, z_i^\perp, a), \quad \text{if} \quad z_i \geq r_0, \quad |z_i^\perp| \leq 1.$$ 

Ergodic constants in bounded subdomains

Define $\Omega^R := \Omega \cap W_R$ and the ergodic constant $E^R$, which is the unique number s.t.

$$H_0(z, Dw^R(z)) \geq E^R \quad \text{in} \quad \bar{\Omega}^R,$$

$$H_0(z, Dw^R(z)) \leq E^R \quad \text{in} \quad \Omega^R,$$

has a viscosity solution $w^R$. 

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Dimension reduction
**The constant $E$**

**Lemma**

$\exists C > 0$ s.t.

\[
|E^R| \leq C \quad \forall R, \\
\|Dw^R\|_\infty \leq C \quad \forall R, \\
R \leq R' \implies E^R \leq E^{R'}.
\]

**Definition**

\[
E := \lim_{R \to \infty} E^R.
\]
The constant $E$

**Theorem**

$\exists C > 0 \text{ s.t. } \forall z_0 \in \tilde{K}_0, \forall \alpha \in A_{z_0}, \forall T > 0,$

if $z(s) := z_0 + \int_0^s \alpha(\theta)d\theta$ satisfies $z(T) \in \tilde{K}_0$, then

$$\int_0^T \ell_0(z(s), \alpha(s))ds \geq -ET - C.$$ 

**Lemma**

*Under the assumptions, $\forall i = 1, \ldots, N$, $\min_{p_i} \overline{H}_i(0, p_i) \leq E.$*
Proof of the convergence result

The main step is to prove the following theorem

**Theorem**

Consider the relaxed semilimits, of \( u_\varepsilon \): for all \( x \in \mathcal{G} \),

\[
\underline{u}(x) = \liminf_{\varepsilon \to 0^+, x' \to x, x' \in \Omega_\varepsilon} u_\varepsilon(x'); \quad \overline{u}(x) = \limsup_{\varepsilon \to 0^+, x' \to x, x' \in \Omega_\varepsilon} u_\varepsilon(x').
\]

Then, under the assumptions, \( \underline{u} \) is a bounded supersolution of

\[
\lambda \underline{u}(x) + \overline{H}(x, Du(x)) \geq 0, \quad \text{if } x \in \mathcal{G}\setminus\{O\},
\]

\[
\lambda \underline{u}(O) + \max(E, \overline{H}(O, Du(O))) \geq 0, \quad \text{if } x = O,
\]

and \( \overline{u} \) is a bounded subsolution of

\[
\lambda \overline{u}(x) + \overline{H}(x, Du(x)) \leq 0, \quad \text{if } x \in \mathcal{G}\setminus\{O\},
\]

\[
\lambda \overline{u}(O) + \max(E, \overline{H}(O, Du(O))) \leq 0, \quad \text{if } x = O.
\]

then use comparison results on \( \mathcal{G} \), [AOT2014] or [IM2014]
Some ideas of the proof

- Not standard only at the junction
  - Try to use Evans’ method of perturbed test-functions
  - The test-functions $\phi$ have $N$ slopes at $O$: $p = (p_1, \ldots, p_N)$
  - Evans’ method requires the construction of bounded correctors in unbounded domains, (ideally $\Omega$)
  - Bounded correctors may not exist in the full domain $\Omega$, because, given a set of slopes at $O$, i.e. $p = (p_1, \ldots, p_N)$,

\[
\overline{H}(O, p_1, \ldots, p_n) = \max_{i=1,\ldots,N} \overline{H}_i^+(0, p_i)
\]

is achieved by $i \in \mathcal{I}(p) \subset \{1, \ldots, N\}$. If $j \notin \mathcal{I}(p)$, then the optimal trajectories starting from $z \in S_j$ should leave $S_j$: this leads to the fact that the corrector should not be bounded when $|z| \to \infty$, $z \in S_j$.

- Hence, we construct the correctors in subdomain obtained by truncating the half-strips $S_j$ if $j \notin \mathcal{I}(p)$.
The corrector (when $H(O, p_1, \ldots, p_N) > E$)

The truncated domain $\Omega_p$

**Theorem**

For $p \in \mathbb{R}^N: H(O, p) > E$. Let $\psi_p$ be a smooth function such that $D\psi_p = p_i e_i$ in $S_i$. Under the assumptions, $\exists \chi_p$, a bounded and Lipschitz viscosity solution of

$$
H_0(z, D\psi_p + D\chi_p) - H(O, p) \geq 0 \quad \text{in } \overline{\Omega}_p,
$$

$$
H_0(z, D\psi_p + D\chi_p) - H(O, p) \leq 0 \quad \text{in } \Omega_p.
$$
The further assumption

Assumption

1) For any real number $p_i$ such that $p_i < p_i^\perp$, there exist two constants $L_i \geq 0$ and $C_i > 0$ such that

$\forall y_0 \in [-1, 1], \forall t > 0$, there exists a control law $\tilde{\alpha} \in A_{i,y_0}$ with

$$\int_0^s \tilde{\alpha}_i(\tau)d\tau \geq -L_i, \ \forall 0 \leq s \leq t,$$

$$\int_0^t (p_i\tilde{\alpha}_i(s) + \ell_i(0, y(s), \tilde{\alpha}(s)))ds \leq v_i(0, p_i, y_0, t) + C_i,$$

where $y(t) = y_0 + \int_0^t \tilde{\alpha}_i^\perp(s)ds$.

Recall that

$$v_i(x_i, p_i, y_0, t) = \inf_{\alpha \in A_{i,y_0}} \left\{ \int_0^t p_i\alpha_i(s) + \ell_i(x_i, y(s), \alpha(s))ds \right\}$$