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## Traffic flow on networks I, II, III

Alberto Bressan

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# Traffic flow on networks - I

Alberto Bressan

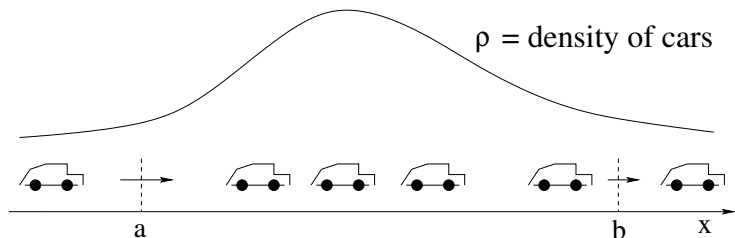
Department of Mathematics, Penn State University

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- Review of scalar conservation laws, the Lax formula
- Flow on a network of roads: boundary conditions at intersections
- Well posedness of the Cauchy problem for  $L^\infty$  data
  
- Optimal scheduling, with departure and arrival costs:  
optima and equilibria on a single road
- Global optima and Nash equilibria on a network of roads

# A conservation law describing traffic flow

(Lighthill-Whitham-Richards, 1955-56)



$$\rho_t + [\rho v(\rho)]_x = 0$$

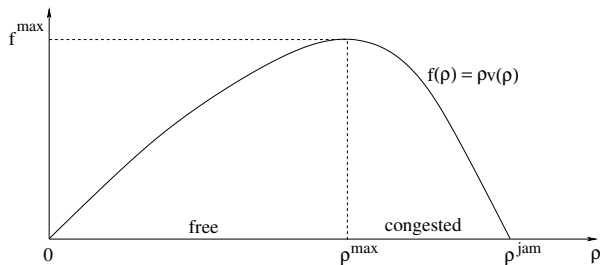
$v(\rho) = \text{velocity of cars (depending only on the density)}$

$f(\rho) = \rho v(\rho) = \text{flux}$

# The flux function

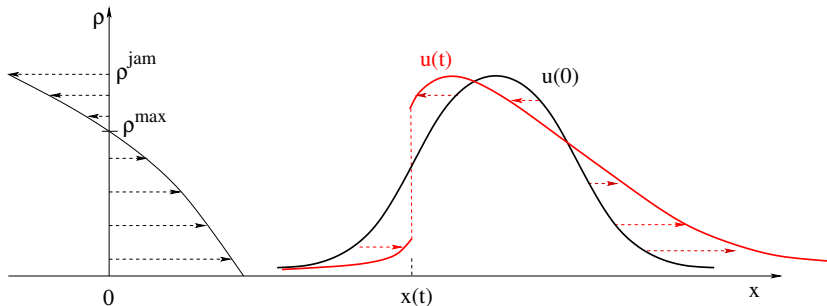
$$\rho_t + f(\rho)_x = 0, \quad f(\rho) = \rho v(\rho)$$

$$f \in \mathcal{C}^2, \quad f''(\rho) < 0, \quad f(0) = f(\rho^{\text{jam}}) = 0$$



# The Cauchy problem

Characteristic speed:  $f'(\rho)$  decreases with  $\rho$

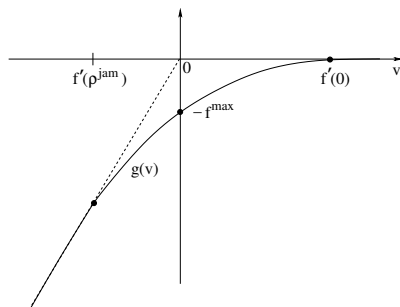
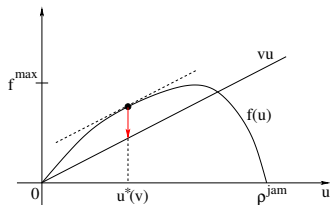


Shocks travel with Rankine-Hugoniot speed  $\dot{x} = \frac{f(\rho^+) - f(\rho^-)}{\rho^+ - \rho^-}$

They are admissible if the Lax condition  $\rho^- < \rho^+$  holds

# The Lax formula

Legendre transform:  $g(v) \doteq \inf_{u \in [0, \rho^{jam}]} \{vu - f(u)\}$



$$g(v) = u^*(v)v - f(u^*(v))$$

$$\text{where } f'(u^*(v)) = v$$

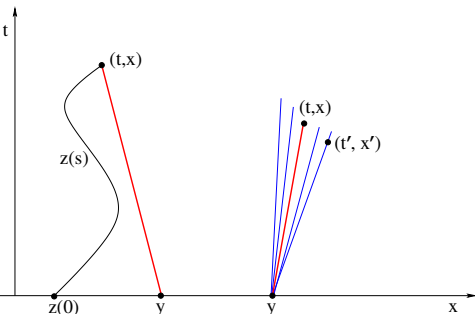
$$\begin{cases} v \leq f'(\rho^{jam}) \\ v \geq f'(0) \end{cases} \begin{matrix} \implies \\ \implies \end{matrix} \begin{cases} g(v) = \rho^{jam} v \\ g(v) = 0 \end{cases}$$

# An optimization problem

$$u_t + f(u)_x = 0 \quad u(0, x) = \bar{u}(x) \in [0, \rho^{jam}]$$

Given the initial data  $\bar{V}(x) \doteq \int_{-\infty}^x \bar{u}(y) dy$  and a terminal point  $(t, x)$ ,

maximize:  $\bar{V}(z(0)) + \int_0^t g(\dot{z}(s)) ds$ , subject to  $z(t) = x$



- optimal trajectory is affine, with  $\dot{x} \in [f'(\rho^{jam}), f'(0)]$

- Value function is

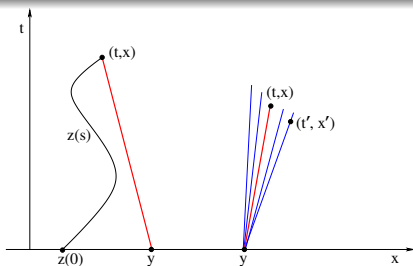
$$V(t, x) = \max_y \bar{V}(y) + t g\left(\frac{x - y}{t}\right)$$



# Properties of the Legendre transform

$$f'(\rho) = v \iff g'(v) = \rho$$

In this case,  $g(v) = \rho v - f(\rho)$



$$V(t', x') \geq V^-(t', x') = \bar{V}(y) + t' g\left(\frac{x' - y}{t'}\right)$$

Differentiating w.r.t.  $(t', x')$ , choosing  $\rho$  such that  $f'(\rho) = \frac{x-y}{t}$  one obtains

$$\begin{aligned}(\partial_t V^-, \partial_x V^-)(t, x) &= \left( g\left(\frac{x-y}{t}\right) - \frac{x-y}{t} g'\left(\frac{x-y}{t}\right), g'\left(\frac{x-y}{t}\right) \right) \\ &= \left( \rho \cdot \frac{x-y}{t} - f(\rho) - \frac{x-y}{t} \cdot \rho, \rho \right) = (-f(\rho), \rho)\end{aligned}$$

# A Hamilton-Jacobi equation

If  $V(t, x) = \bar{V}(y) + t' g\left(\frac{x-y}{t}\right)$ , choose  $\rho$  so that  $f'(\rho) = \frac{x-y}{t}$ .

Then  $(\partial_t V, \partial_x V) = (-f(\rho), \rho)$  lies in the sub-differential of  $V$  at  $(t, x)$ .

Since  $V$  is Lipschitz continuous,

$$V_t + f(V_x) = 0 \quad \text{for a.e. } (t, x)$$

Moreover  $\rho = V_x$  yields a solution of conservation law

$$\rho_t + f(\rho)_x = 0$$

satisfying the Oleinik admissibility conditions: only upward jumps

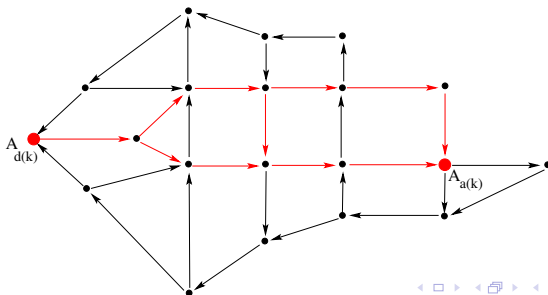
# Traffic Flow on a Network of Roads

Vehicle density  $\rho$  is described by a scalar conservation law on each road

$$\rho_t + f_k(\rho)_x = 0$$

+ boundary conditions at road intersections

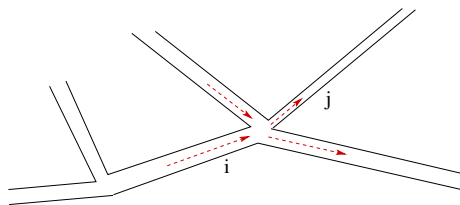
- Describe the evolution of traffic density
- Study optimization problems (for a central planner)
- Study Nash equilibrium problems



# Boundary conditions at junctions

incoming roads:  $i \in \mathcal{I}$

outgoing roads:  $j \in \mathcal{O}$



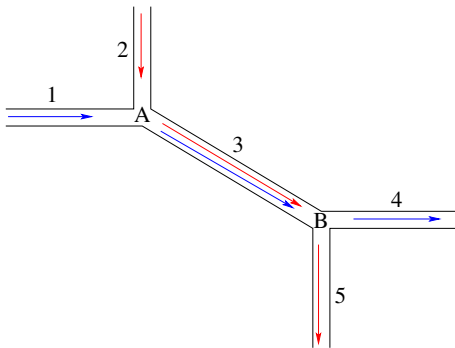
Boundary conditions account for:

- $\theta_{ij}$  = fraction of drivers from road  $i$  that turn into road  $j$ .
- $c_i$  = relative priority of drivers from road  $i$   
(i.e., percentage of time drivers from road  $i$  get green light)

$$\sum_j \theta_{ij} = 1$$

$$\sum_i c_i = 1$$

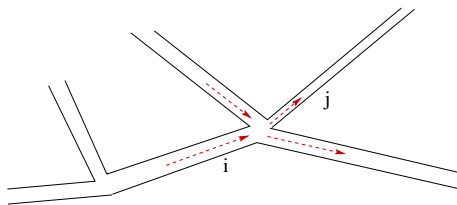
Drivers' turning preferences  $\theta_{ij}$  must be determined as part of the solution



# of vehicles on road  $i$  that wish to turn into road  $j$  is conserved:

$$(\rho\theta_{ij})_t + (\rho v_i(\rho)\theta_{ij})_x = 0$$

# Traffic flow on a network of roads



On the  $i$ -th incoming road, car flow is described by

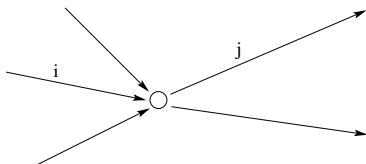
$$\begin{cases} \rho_t + f_i(\rho)_x = 0 & \text{conservation law} \\ \theta_{ij,t} + v_i(\rho)\theta_{ij,x} = 0 & \text{linear transport equation} \end{cases}$$

$\theta_{ij}$  are *passive scalars*, relevant only at intersection

# Boundary conditions at junctions

incoming roads:  $i \in \mathcal{I}$

outgoing roads:  $j \in \mathcal{O}$



Boundary conditions should relate the pointwise limits

$$\begin{cases} \rho_i(t, 0-) \\ \rho_j(t, 0+) \end{cases} \quad \text{depending on } \theta_{ij}(t, 0-)$$

Conservation equations:

$$\sum_i f_i(\rho_i(t, 0-))\theta_{ij}(t, 0-) = f_j(\rho_j(t, 0+)) \quad j \in \mathcal{O}$$

- H.Holden, N.H.Risebro, A mathematical model of traffic flow on a network of unidirectional roads, *SIAM J. Math. Anal.* **26**, 1995.
- G.M.Coclite, M.Garavello, B.Piccoli, Traffic flow on a road network, *SIAM J. Math. Anal.* **36**, 2005.
- M.Herty, S.Moutari, M.Rasche, Optimization criteria for modeling intersections of vehicular traffic flow, *Netw. Heterog. Media* **1**, 2006.
- M.Garavello, B.Piccoli, Conservation laws on complex networks, *Ann.I.H.Poincaré* **26** 2009.
- M.Garavello, B.Piccoli, *Traffic Flow on Networks*, AIMS, 2006.



- Assign boundary conditions by describing how Riemann problems should be solved
- By taking limits of front tracking approximations, this uniquely determines how Cauchy problems should be solved

This approach works well if  $\theta_{ij}$  are constants

What happens if  $\theta_{ij} = \theta_{ij}(t, x)$  ??

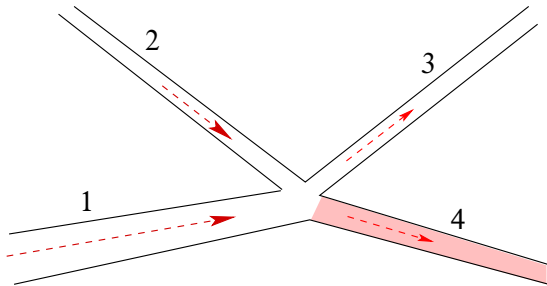
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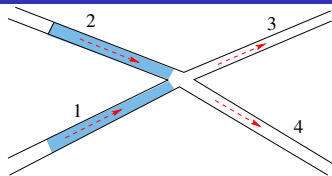
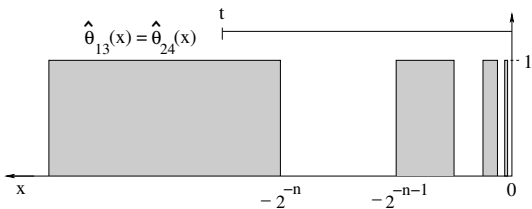
What happens if  $\theta_{ij} = \theta_{ij}(t, x)$  ??

## Modeling assumptions

- If all cars arriving at the intersection can immediately move to outgoing roads, no queue is formed.
- If outgoing roads are congested, the inflow of cars from road 1 is twice as large as the inflow from road 2.



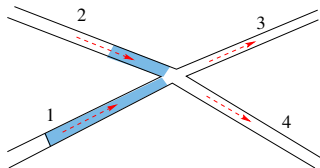
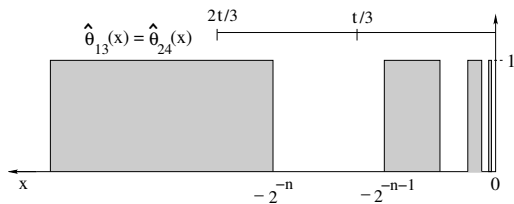
# Example 1: $\theta_{ij}$ with unbounded variation, two solutions



$$f_k(\rho) = 2\rho - \rho^2 \quad \text{maximum flux: } f_k^{max} = 1$$

$$\text{Initial data: } \hat{\rho}_k(x) = 1, \quad k = 1, 2, 3, 4$$

$$\hat{\theta}_{13}(x) = \hat{\theta}_{24}(x) = \begin{cases} 1 & \text{if } -2^{-n} < x < -2^{-n-1}, \quad n \text{ even} \\ 0 & \text{if } -2^{-n} < x < -2^{-n-1}, \quad n \text{ odd} \end{cases}$$



Solution 1. Incoming fluxes:  $f_1(t, 0) = 1$ ,  $f_2(t, 0) = 1$

Solution 2. Incoming fluxes:  $f_1(t, 0) = \frac{2}{3}$ ,  $f_2(t, 0) = \frac{1}{3}$

# Multiple solutions for a hyperbolic system

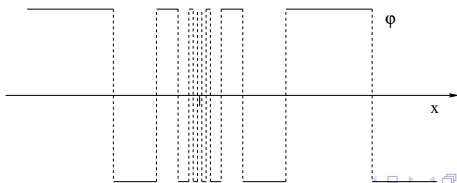
(A.B. & Wen Shen, *Nonlinear Analysis, T.M.A., 1998*)

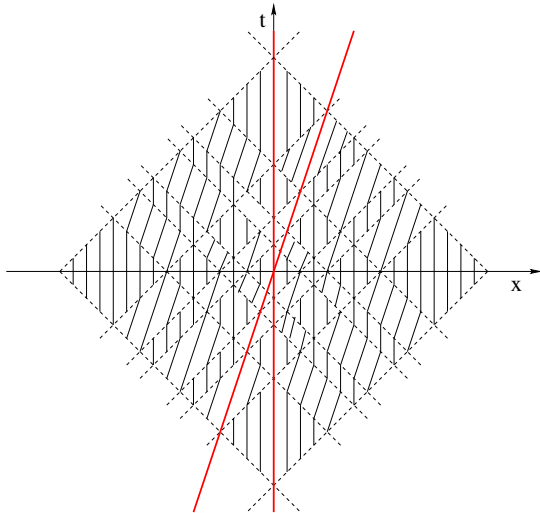
$$\left\{ \begin{array}{l} u_t + u_x = 0, \\ v_t - v_x = 0, \\ w_t + \left[ \frac{1-uv}{6} \cdot w \right]_x = 0, \end{array} \right. \quad \left\{ \begin{array}{l} u(0, x) = \bar{u}(x), \\ v(0, x) = \bar{v}(x), \\ w(0, x) = \bar{w}(x). \end{array} \right.$$

strictly hyperbolic, linearly degenerate, with characteristic speeds

$-1 < \frac{1-uv}{6} < 1$ . Initial data  $\bar{u}, \bar{v}, \bar{w} \in BV \implies$  solution is unique.

$$\bar{u}(x) = \bar{v}(x) = \varphi(x) \doteq \begin{cases} 1 & \text{if } 2^{-2n-1} < |x| < 2^{-2n} \\ -1 & \text{if } 2^{-2n} < |x| < 2^{-2n+1} \end{cases}$$

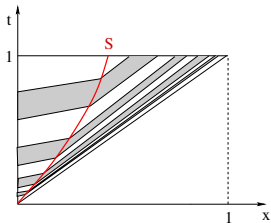
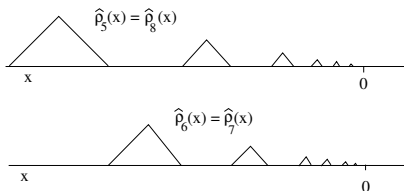
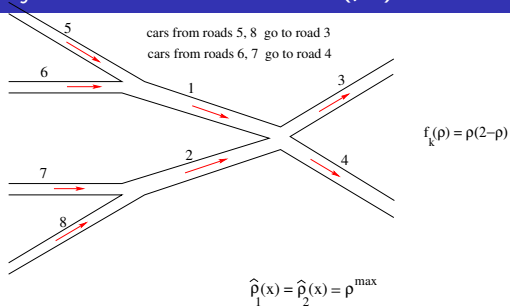




$\dot{x} = \frac{1-u(t,x)v(t,x)}{6}$  has infinitely many solutions through the origin

$\implies$  Cauchy problem has infinitely many solutions

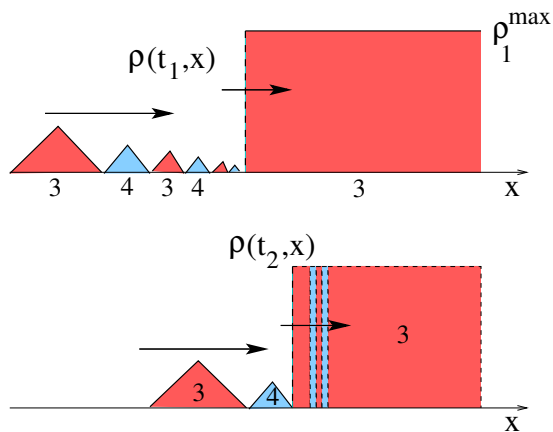
# Example 2: $\theta_{ij}$ constant, $Tot.Var.(\rho_i)$ small, two solutions



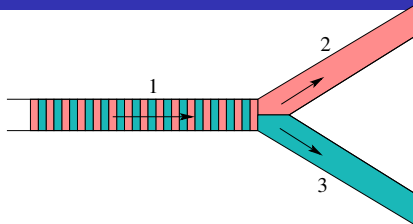
At some time  $T > 0$ , the same initial data as in Example 1 is created at the junction of roads 1 and 2



# Instantaneous blow up of $Tot.Var.\{\theta_{13}\}$ on road 1



# Example 3: lack of continuity w.r.t. weak convergence



$$f_1^{max} = 2, \quad f_2^{max} = f_3^{max} = 1.$$

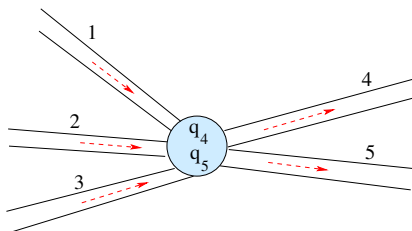
$$(\theta_{12}^n, \theta_{13}^n)(x) = \begin{cases} (1, 0) & \text{if } x \in ](k-1)/n, k/n], \quad k \text{ even} \\ (0, 1) & \text{if } x \in ](k-1)/n, k/n], \quad k \text{ odd.} \end{cases}$$

Flux exiting road 1:  $f_1^{(n)}(t, 0) \equiv 1$

As  $n \rightarrow \infty$ , the weak limit is  $\theta_{12} = \theta_{13} = \frac{1}{2}$

Flux out of road 1:  $f_1(t, 0) = 2 \neq 1 = \lim_{n \rightarrow \infty} f_1^{(n)}(t, 0)$

# An intersection model with buffers



- the intersection contains a buffer with finite size
- $t \mapsto q_j(t)$  = queues in front of outgoing roads  $j \in \mathcal{O}$ , within the buffer
- incoming drivers are admitted to the intersection at a rate depending on the size of these queues
- drivers already inside the intersection flow out to the road of their choice at the largest possible rate

- M. Herty, J. P. Lebacque, and S. Moutari, A novel model for intersections of vehicular traffic flow. *Netw. Heterog. Media* 2009.
- M. Garavello and P. Goatin, The Cauchy problem at a node with buffer. *Discrete Contin. Dyn. Syst.* 2012.
- M. Garavello and B. Piccoli, A multibuffer model for LWR road networks, in *Advances in Dynamic Network Modeling in Complex Transportation Systems*, 2013.

Toward the analysis of global optima and Nash equilibria, we need

- well posedness for  $L^\infty$  initial data  $\hat{\rho}_k, \hat{\theta}_{ij}$
- continuity w.r.t. weak convergence

$$\begin{cases} \rho_{k,t} + f_k(\rho_k)_x = 0 & \text{conservation laws} \\ \theta_{ij,t} + v_i(\rho_i)\theta_{ij,x} = 0 & \text{linear transport equations} \end{cases}$$

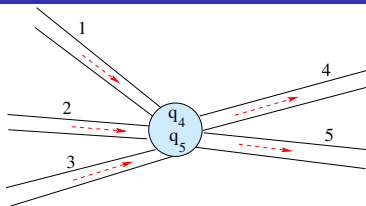
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$$\left\{ \begin{array}{ll} \rho_{k,t} + f_k(\rho_k)_x = 0 & \text{conservation laws} \\ \theta_{ij,t} + v_i(\rho_i)\theta_{ij,x} = 0 & \text{linear transport equations} \end{array} \right.$$

# The basic model



Incoming roads:  $i \in \mathcal{I}$

Outgoing roads:  $j \in \mathcal{O}$

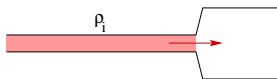
Boundary values at the junction:

$$\left\{ \begin{array}{ll} \bar{\theta}_{ij}(t) \doteq \lim_{x \rightarrow 0^-} \theta_{ij}(t, x), & i \in \mathcal{I}, j \in \mathcal{O} \\ \bar{\rho}_i(t) \doteq \lim_{x \rightarrow 0^-} \rho_i(t, x), & i \in \mathcal{I} \\ \bar{\rho}_j(t) \doteq \lim_{x \rightarrow 0^+} \rho_j(t, x), & j \in \mathcal{O} \\ \bar{f}_i(t) \doteq f_i(\bar{\rho}_i(t)) = \lim_{x \rightarrow 0^-} f_i(\rho_i(t, x)), & i \in \mathcal{I} \\ \bar{f}_j(t) \doteq f_j(\bar{\rho}_j(t)) = \lim_{x \rightarrow 0^+} f_j(\rho_j(t, x)), & j \in \mathcal{O} \end{array} \right.$$

$q_j(t) = \#$  of cars within the intersection, waiting to access road  $j \in \mathcal{O}$ .

Conservation of the total number of cars implies

$$\dot{q}_j = \frac{d}{dt}q_j(t) = \sum_{i \in \mathcal{I}} \bar{f}_i \bar{\theta}_{ij} - \bar{f}_j \quad j \in \mathcal{O}$$



$$\omega_i = \omega_i(\bar{\rho}_i) \doteq \begin{cases} f_i(\bar{\rho}_i) & \text{if } \bar{\rho}_i \text{ is a free state} \\ f_i^{\max} & \text{if } \bar{\rho}_i \text{ is a congested state} \end{cases} \quad i \in \mathcal{I},$$

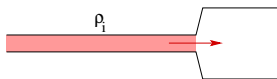
= maximum possible flux at the end of an incoming road

$\omega_i$  is the largest flux  $f_i(\rho)$  among all states  $\rho$  that can be connected to  $\bar{\rho}_i$  with a wave of negative speed

$q_j(t) = \#$  of cars within the intersection, waiting to access road  $j \in \mathcal{O}$ .

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= maximum possible flux at the end of an incoming road

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# (SBJ) - Single Buffer Junction

$M > 0$  = maximum number of cars that can occupy the intersection at any given time

$c_i > 0$ ,  $i \in \mathcal{I}$ , priorities given to different incoming roads

**(SBJ)** Incoming fluxes  $\bar{f}_i$  satisfy

$$\bar{f}_i = \min \left\{ \omega_i, c_i \left( M - \sum_{j \in \mathcal{O}} q_j \right) \right\}, \quad i \in \mathcal{I}$$

Outgoing fluxes  $\bar{f}_j$  satisfy

$$\begin{cases} \text{if } q_j > 0, \text{ then } \bar{f}_j = \omega_j \\ \text{if } q_j = 0, \text{ then } \bar{f}_j = \min \left\{ \omega_j, \sum_{i \in \mathcal{I}} \bar{f}_i \bar{\theta}_{ij} \right\} \end{cases} \quad j \in \mathcal{O}$$

# (MBJ) - Multiple Buffer Junction

$M_j > 0$  = maximum number of cars that can occupy the intersection, waiting to enter road  $j$

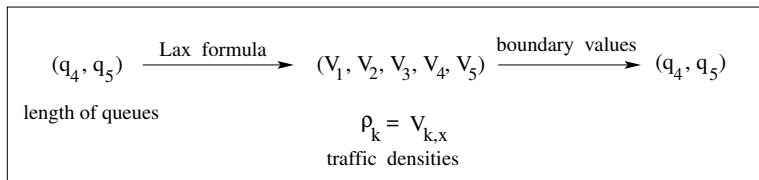
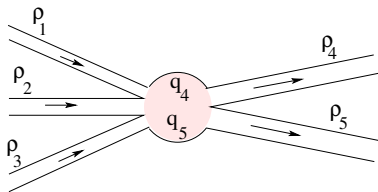
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**(MBJ)** Incoming fluxes  $\bar{f}_i$  satisfy

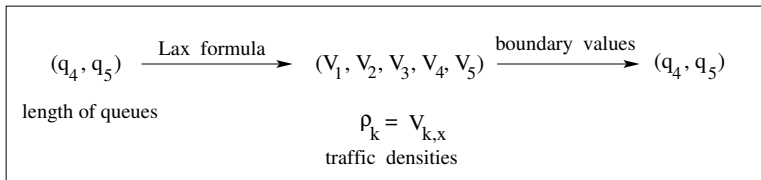
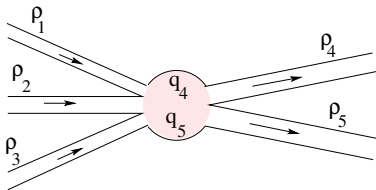
$$\bar{f}_i = \min \left\{ \omega_i, \frac{c_i(M_j - q_j)}{\theta_{ij}}, j \in \mathcal{O} \right\}, \quad i \in \mathcal{I}$$

Outgoing fluxes  $\bar{f}_j$  satisfy

$$\begin{cases} \text{if } q_j > 0, \text{ then } \bar{f}_j = \omega_j, \\ \text{if } q_j = 0, \text{ then } \bar{f}_j = \min \left\{ \omega_j, \sum_{i \in \mathcal{I}} \bar{f}_i \bar{\theta}_{ij} \right\} \end{cases} \quad j \in \mathcal{O}$$



- If the queue sizes  $q_j(t)$  within the buffer are known, then the initial-boundary value problems can be independently solved along each incoming road. These solutions can be computed by solving suitable variational problems. From the value functions  $V_k$ , the traffic density  $\rho_k = V_{k,x}$  along each incoming or outgoing road is recovered by a Lax type formula.



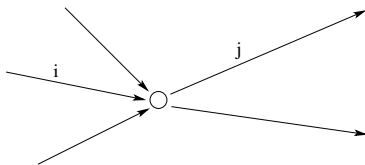
- Conversely, if these value functions  $V_k$  are known, then the queue sizes  $q_j$  can be determined by balancing the boundary fluxes of all incoming and outgoing roads
- The solution of the Cauchy problem is obtained as the unique fixed point of a contractive transformation
- The present model accounts for backward propagation of queues along roads leading to a crowded intersection, it achieves well-posedness for general  $L^\infty$  data, and continuity w.r.t. weak convergence

- P. Le Floch, Explicit formula for scalar non-linear conservation laws with boundary condition. *Math. Methods Appl. Sciences* (1988)

$$\rho_t + f(\rho)_x = 0 \quad \rho(0, x) = a(x), \quad \rho(t, 0) = b(t)$$

A variational inequality determines if the boundary value can be pointwise attained or not.

- C. Imbert, R. Monneau, and H. Zidani, A Hamilton-Jacobi approach to junction problems and application to traffic flows. *ESAIM Control Optim. Calc. Var.* (2013).



A single optimization problem determines the solution simultaneously on all roads

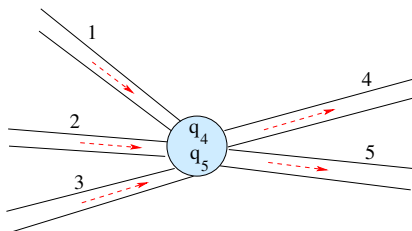
# Traffic flow on networks - II

Alberto Bressan

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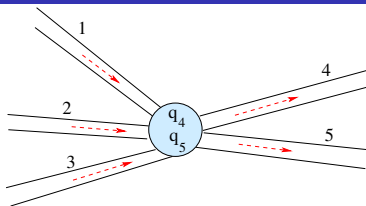
bressan@math.psu.edu

# An intersection model with buffers



- the intersection contains a buffer with finite size
- $t \mapsto q_j(t)$  = queues in front of outgoing roads  $j \in \mathcal{O}$  , within the buffer
- incoming drivers are admitted to the intersection at a rate depending on the size of these queues
- drivers already inside the intersection flow out to the road of their choice at the largest possible rate

# The basic model



Incoming roads:  $i \in \mathcal{I}$

Outgoing roads:  $j \in \mathcal{O}$

Boundary values at the junction:

$$\left\{ \begin{array}{ll} \bar{\theta}_{ij}(t) \doteq \lim_{x \rightarrow 0^-} \theta_{ij}(t, x), & i \in \mathcal{I}, j \in \mathcal{O} \\ \bar{\rho}_i(t) \doteq \lim_{x \rightarrow 0^-} \rho_i(t, x), & i \in \mathcal{I} \\ \bar{\rho}_j(t) \doteq \lim_{x \rightarrow 0^+} \rho_j(t, x), & j \in \mathcal{O} \\ \bar{f}_i(t) \doteq f_i(\bar{\rho}_i(t)) = \lim_{x \rightarrow 0^-} f_i(\rho_i(t, x)), & i \in \mathcal{I} \\ \bar{f}_j(t) \doteq f_j(\bar{\rho}_j(t)) = \lim_{x \rightarrow 0^+} f_j(\rho_j(t, x)), & j \in \mathcal{O} \end{array} \right.$$



# Size of queues at intersection

$q_j(t) = \#$  of cars within the intersection, waiting to access road  $j \in \mathcal{O}$ .

Conservation of the total number of cars implies

$$\dot{q}_j = \frac{d}{dt}q_j(t) = \sum_{i \in \mathcal{I}} \bar{f}_i \bar{\theta}_{ij} - \bar{f}_j \quad j \in \mathcal{O}$$

# Upper bounds on the exiting or entering fluxes

$$\omega_i(\bar{\rho}_i) \doteq \begin{cases} f_i(\bar{\rho}_i) & \text{if } \bar{\rho}_i \text{ is a free state,} \\ f_i^{max} & \text{if } \bar{\rho}_i \text{ is a congested state,} \end{cases} \quad i \in \mathcal{I},$$

= maximum possible flux at the exit of an incoming road.

$\omega_i$  is the largest flux  $f_i(\rho)$  among all states  $\rho$  that can be connected to  $\bar{\rho}_i$  with a wave of negative speed



$$\omega_j(\bar{\rho}_j) \doteq \begin{cases} f_j^{max} & \text{if } \bar{\rho}_j \text{ is a free state,} \\ f_j(\bar{\rho}_j) & \text{if } \bar{\rho}_j \text{ is a congested state,} \end{cases} \quad j \in \mathcal{O},$$

= maximum possible flux at the entrance of an outgoing road.

# (SBJ) - Single Buffer Junction

$M > 0$  = maximum number of cars that can occupy the intersection

$c_i > 0$ ,  $i \in \mathcal{I}$ , priorities given to different incoming roads

**(SBJ)** Incoming fluxes  $\bar{f}_i$  satisfy

$$\bar{f}_i = \min \left\{ \omega_i(\bar{\rho}_i), c_i \left( M - \sum_{j \in \mathcal{O}} q_j \right) \right\}, \quad i \in \mathcal{I}$$

Outgoing fluxes  $\bar{f}_j$  satisfy

$$\begin{cases} \text{if } q_j > 0, \text{ then } \bar{f}_j = \omega_j(\bar{\rho}_j), \\ \text{if } q_j = 0, \text{ then } \bar{f}_j = \min \left\{ \omega_j(\bar{\rho}_j), \sum_{i \in \mathcal{I}} \bar{f}_i \bar{\theta}_{ij} \right\} \end{cases} \quad j \in \mathcal{O}$$

# (MBJ) - Multiple Buffer Junction

$M_j > 0$  = maximum number of cars that can occupy the intersection, waiting to enter road  $j$

$c_i > 0$ ,  $i \in \mathcal{I}$ , priorities given to different incoming roads

**(MBJ)** Incoming fluxes  $\bar{f}_i$  satisfy

$$\bar{f}_i = \min \left\{ \omega_i(\bar{\rho}_i), \frac{c_i(M_j - q_j)}{\theta_{ij}}, j \in \mathcal{O} \right\}, \quad i \in \mathcal{I}$$

Outgoing fluxes  $\bar{f}_j$  satisfy

$$\begin{cases} \text{if } q_j > 0, \text{ then } \bar{f}_j = \omega_j(\bar{\rho}_j), \\ \text{if } q_j = 0, \text{ then } \bar{f}_j = \min \left\{ \omega_j(\bar{\rho}_j), \sum_{i \in \mathcal{I}} \bar{f}_i \bar{\theta}_{ij} \right\} \end{cases} \quad j \in \mathcal{O}$$

# The Cauchy problem

$$\left\{ \begin{array}{l} \rho_{k,t} + f_k(\rho_k)_x = 0 \\ \theta_{ij,t} + v_i(\rho_i)\theta_{ij,x} = 0 \\ \dot{q}_j = \sum_{i \in \mathcal{I}} \bar{f}_i \bar{\theta}_{ij} - \bar{f}_j \end{array} \right. \quad \left\{ \begin{array}{l} \rho_k(0, x) = \hat{\rho}_k(x) \\ \theta_{ij}(0, x) = \hat{\theta}_{ij}(x) \\ q_j(0) = \hat{q}_j \end{array} \right. \quad \begin{array}{l} k \in \mathcal{I} \cup \mathcal{O} \\ i \in \mathcal{I}, j \in \mathcal{O} \\ j \in \mathcal{O} \end{array}$$

+ boundary conditions at the intersection

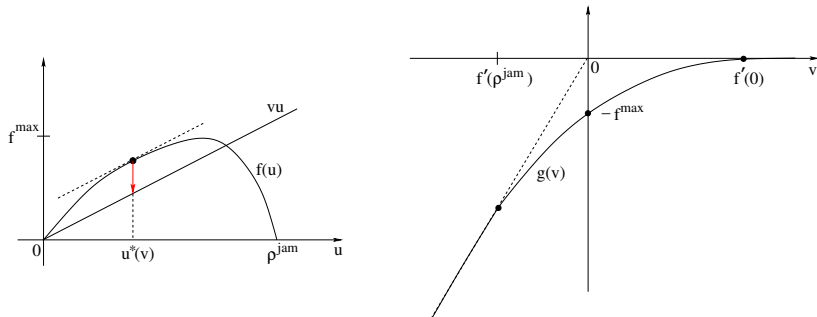
# Construction of solutions

Assume: size of queues  $q_j(t)$  is known

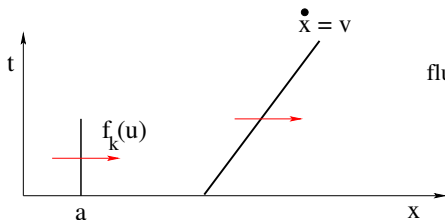
- Introduce an optimization problem whose value function  $V_i$  yields the density of cars on each incoming road:  $\rho_i = V_{i,x}$ , for model **(SBJ)**
- Same for model **(MBJ)**
- Introduce a variational problem whose value function  $V_j$  yields the density of cars on each outgoing road:  $\rho_j = V_{j,x}$ .
- Compute an updated value for the queues:  $\tilde{q}_j = \Lambda_j(q)$
- Construct the solution by finding a fixed point of  $\Lambda$

# The Legendre transform of $f_k$

Legendre transform: 
$$g_k(v) \doteq \inf_{u \in [0, \rho_k^{\text{jam}}]} \{uv - f_k(u)\}, \quad k \in \mathcal{I} \cup \mathcal{O}$$



# A variational problem on each road



flux across the characteristic

$$= f_k(u) - vu$$

$$g_k(v) = - [\text{flux of cars from left to right, across the characteristic}]$$

For boundary conditions **(SBJ)**, define the **maximum permissible flux at the end of road  $i$**

$$h_i(\mathbf{q}) \doteq \min \left\{ f_i^{max}, c_i \cdot \left( M - \sum_{j \in \mathcal{O}} q_j \right) \right\} \quad i \in \mathcal{I}$$



# Optimization Problem 1 - incoming roads with boundary condition (SBJ)

For any  $i \in \mathcal{I}$ , given a terminal point  $(\bar{t}, \bar{x})$ , and boundary data

$$\bar{V}_i(x) \doteq \int_{-\infty}^x \hat{\rho}_i(y) dy \quad q_j(t), \quad j \in \mathcal{O}$$

consider the optimization problem:

$$\text{maximize:} \quad \bar{V}_i(x(0)) + \int_0^{\bar{t}} L_i(x(t), \dot{x}(t)) dt$$

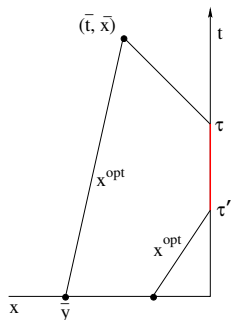
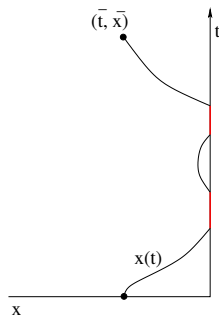
$$\text{payoff:} \quad L_i(x(t), \dot{x}(t)) = \begin{cases} g_i(\dot{x}(t)) & \text{if } x(t) < 0 \\ -h_i(\mathbf{q}(t)) & \text{if } x(t) = 0 \end{cases}$$

# Optimization Problem 1

$$\text{maximize: } \bar{V}_i(x(0)) + \int_0^{\bar{t}} L_i(x(t), \dot{x}(t)) dt$$

among all absolutely continuous functions  $x : [0, \bar{t}] \mapsto \mathbb{R}$  such that

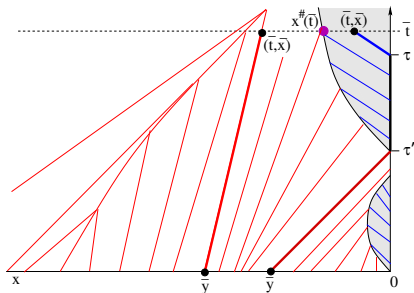
$$x(\bar{t}) = \bar{x}, \quad x(t) \leq 0 \quad \text{for all } t \in [0, \bar{t}]$$



# The Value Function $V_i$

- an optimal solution  $x^{opt}$  exists, and is the concatenation of at most three affine functions
- $\dot{x}^{opt} \in [f'_i(0), f'_i(\rho_i^{jam})]$
- the traffic density  $\rho_i(t, x) = V_{i,x}(t, x)$  is an entropy solution of the conservation law, satisfying initial + boundary conditions
- Lax formula: for a.e.  $(\bar{t}, \bar{x})$ , the optimal solution is unique and

$$f'_i(\rho_k(\bar{t}, \bar{x})) = \dot{x}^{opt}(\bar{t}) = \begin{cases} \frac{\bar{x} - \bar{y}}{\bar{t}} \\ \bar{x} \\ \frac{\bar{x}}{\bar{t} - \tau} \end{cases} \quad \text{or}$$



$$V_i(t, x) \doteq \max \left\{ \max_{y \leq 0} \left[ \bar{V}_i(y) + t g_i \left( \frac{x - y}{t} \right) \right], \right. \\ \left. \max_{0 \leq \tau' \leq \tau \leq t, y \leq 0} \left[ \bar{V}_i(y) + \tau' g_i \left( \frac{-y}{\tau'} \right) - \int_{\tau'}^{\tau} h_i(\mathbf{q}(s)) ds + (t - \tau) g_i \left( \frac{x}{t - \tau} \right) \right] \right\}. \quad (\text{LF1})$$

$V_i(t, x)$  = total amount of cars which at time  $t$  are still inside the half line  $]-\infty, x]$

$\bar{V}_i(0) - V_i(t, 0)$  = total amount of cars which have exited from road  $i$  during the time interval  $[0, t]$

## Properties of the Legendre transform

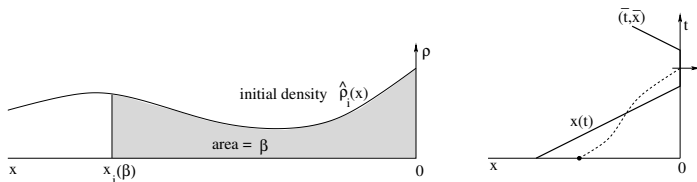
$$f'(\rho) = v \iff g'(v) = \rho$$

In this case,  $g(v) = \rho v - f(\rho)$

$$\begin{aligned} 0 &= \frac{d}{d\tau} \left[ - \int_{\tau'}^{\tau} h_i(\mathbf{q}(s)) ds + (t - \tau) g_i \left( \frac{x}{t - \tau} \right) \right] \\ &= - h_i(\mathbf{q}(\tau)) - g_i \left( \frac{x}{t - \tau} \right) + \frac{x}{t - \tau} g_i' \left( \frac{x}{t - \tau} \right) \\ &= - h_i(\mathbf{q}(\tau)) - \left[ \rho \frac{x}{t - \tau} - f_i(\rho) \right] + \frac{x}{t - \tau} \rho \\ &= - h_i(\mathbf{q}(\tau)) + f_i(\rho) \end{aligned}$$

# Optimization Problem 2 - incoming roads with boundary condition (MBJ)

$$h_i(\mathbf{q}, \theta) \doteq \min \left\{ f_i^{max}, c_i \cdot \frac{M_j - q_j}{\theta_{ij}}; j \in \mathcal{O} \right\} \quad (\text{boundary flux})$$



Define  $\beta \mapsto x_i(\beta)$  implicitly by  $\int_{x(\beta)}^0 \hat{\rho}_i(x) dx = \beta$ , then set  $\theta_{ij}(\beta) \doteq \hat{\theta}_{ij}(x_i(\beta))$

Let  $t \mapsto \beta(t)$  be the solution to the ODE

$$\dot{\beta}(t) = \begin{cases} g_i(\dot{x}(t)) & \text{if } x(t) < 0 \\ h_i(\mathbf{q}(t), \theta(\beta(t))) & \text{if } x(t) = 0 \end{cases} \quad \text{with } \beta(0) = \bar{V}_i(x(0))$$

## Optimization Problem 2

For any  $i \in \mathcal{I}$ , given a terminal point  $(\bar{t}, \bar{x})$ , and initial-boundary data

$$\bar{V}_i(x) \doteq \int_{-\infty}^x \hat{\rho}_i(y) dy \quad q_j(t), \quad j \in \mathcal{O}$$

consider the optimization problem

$$\text{maximize:} \quad \bar{V}_i(x(0)) + \int_0^{\bar{t}} L_i(x(t), \dot{x}(t)) dt$$

$$\text{payoff:} \quad L_i(x(t), \dot{x}(t)) \doteq \begin{cases} g_i(\dot{x}(t)) & \text{if } x(t) < 0 \\ -h_i(\mathbf{q}(t), \theta(\beta(t))) & \text{if } x(t) = 0 \end{cases}$$

among all absolutely continuous functions  $x : [0, \bar{t}] \mapsto \mathbb{R}$  such that

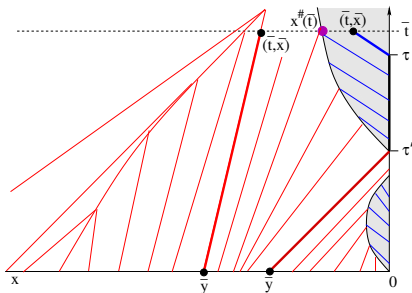
$$x(\bar{t}) = \bar{x}, \quad x(t) \leq 0 \quad \text{for all } t \in [0, \bar{t}]$$

# The Value Function $V_i$

- an optimal solution  $x^{opt}$  exists, and is the concatenation of at most three affine functions
- $\dot{x}^{opt} \in [f'_i(0), f'_i(\rho_i^{jam})]$
- the traffic density  $\rho_i(t, x) = V_{i,x}(t, x)$  is an entropy solution of the conservation law, satisfying initial + boundary conditions
- Lax formula: for a.e.  $(\bar{t}, \bar{x})$ , the optimal solution is unique and

$$f'_i(\rho_i(\bar{t}, \bar{x})) = \dot{x}^{opt}(\bar{t}) = \begin{cases} \frac{\bar{x} - \bar{y}}{\bar{t}} \\ \bar{x} \\ \frac{\bar{x}}{\bar{t} - \tau} \end{cases} \quad \text{or}$$



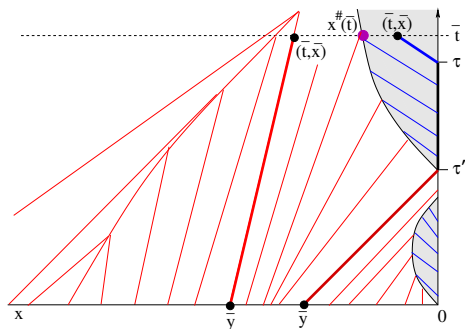


$$V_i(t, x) \doteq \max \left\{ \max_{y \leq 0} \left[ \widehat{V}_i(y) + t g_i \left( \frac{x-y}{t} \right) \right], \right. \\ \left. \max_{0 \leq \tau' \leq \tau \leq t, y \leq 0} \left[ G_i(\tau, \tau'; y) + (t - \tau) g_i \left( \frac{x}{t - \tau} \right) \right] \right\}, \quad (\text{LF2})$$

$G_i(\tau, \tau'; y) \doteq -\beta(\tau)$ , where

$$\frac{d}{ds} \beta(s) = h_i(\mathbf{q}(s), \theta_{ij}(\beta(s))), \quad s \in [\tau', \tau], \quad \beta(\tau') = \widehat{V}_i(y) + \tau' g_i \left( \frac{-y}{\tau'} \right)$$

# Motivation



Choosing  $\rho$  such that  $f'_i(\rho) = \frac{x}{t - \tau}$ , the optimality condition

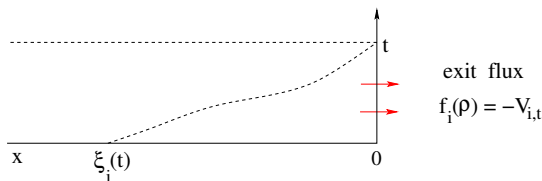
$$\frac{d}{d\tau} \left[ G_i(\tau, \tau', y) + (t - \tau) g_i \left( \frac{x}{t - \tau} \right) \right] = 0$$

implies  $f_i(\rho) = h_i(\mathbf{q}(\tau), \theta_{ij}(\beta(\tau)))$

# Optimization Problem 3 - outgoing roads

The initial position of the car on road  $i$  that reaches the intersection at time  $t$  is

$$\xi_i(t) \doteq \max \left\{ z \in ]-\infty, 0]; \int_z^0 \hat{\rho}_i(y) dy = \bar{V}_i(0) - V_i(t, 0) \right\}$$



The total number of drivers that have reached the intersection before time  $t$  and wanted to turn into road  $j$  is

$$F_j(t) = \hat{q}_j + \sum_{i \in \mathcal{I}} \int_{\xi_i(t)}^0 \hat{\rho}_i(y) \hat{\theta}_{ij}(y) dy$$

# Optimization Problem 3

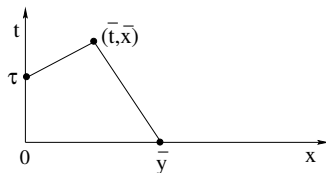
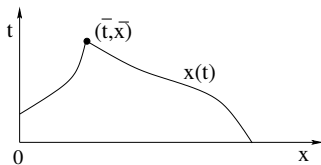
For any  $j \in \mathcal{O}$ , given a terminal point  $(\bar{t}, \bar{x})$ , and initial-boundary data

$$\bar{V}_j(x) \doteq \int_0^x \hat{\rho}_j(y) dy, \quad F_j(t)$$

consider the problem of maximizing the functional

$$\max \left\{ \bar{V}_j(x(0)) + \int_0^{\bar{t}} g_j(\dot{x}(t)) dt, \quad \max_{\tau \geq 0, x(\tau)=0} \left( -F_j(\tau) + \int_{\tau}^{\bar{t}} g_j(\dot{x}(t)) dt \right) \right\}$$

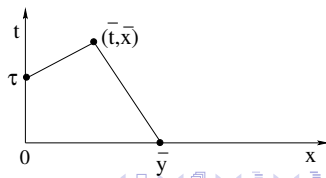
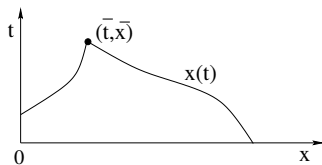
subject to  $x(\bar{t}) = \bar{x}$ ,  $x(t) \geq 0$  for all  $t \in [0, \bar{t}]$

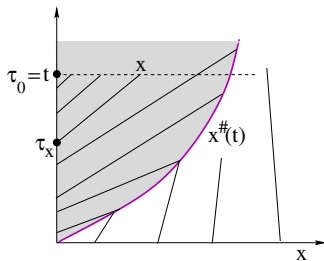
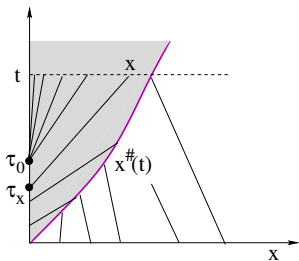
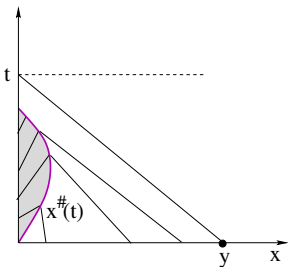


# The Value Function $V_j$

- an optimal solution  $x^{opt}$  exists, and is affine
- $\dot{x}^{opt} \in [f'_j(0), f'_j(\rho_j^{jam})]$
- the traffic density  $\rho_j(t, x) = V_{j,x}(t, x)$  is an entropy solution of the conservation law, satisfying initial + boundary conditions
- Lax formula: for a.e.  $(\bar{t}, \bar{x})$ , the optimal solution is unique and

$$f'_j(\rho_j(\bar{t}, \bar{x})) = \dot{x}^{opt}(\bar{t}) = \begin{cases} \frac{\bar{x} - \bar{y}}{\bar{t}} \\ \frac{\bar{x}}{\bar{t} - \tau} \end{cases} \quad \text{or}$$





$$V_j(t, x) \doteq \max \left\{ \max_{y \geq 0} \left[ \widehat{V}_j(y) + t g_j \left( \frac{x-y}{t} \right) \right], \max_{0 \leq \tau \leq t} \left[ -F_j(\tau) + (t-\tau) g_j \left( \frac{x}{t-\tau} \right) \right] \right\} \quad (\text{LF3})$$

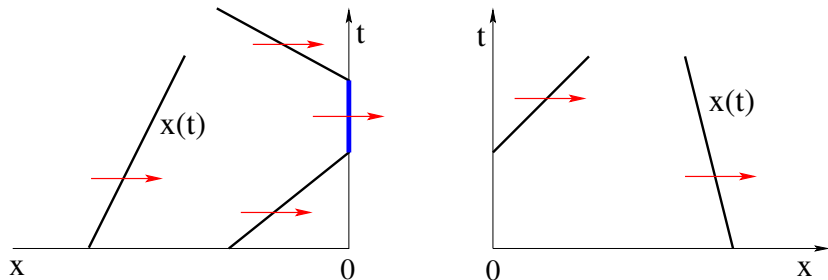
$\widehat{V}_j(x) - V_j(t, x) = \#$  of cars that have crossed the point  $x$  during the time interval  $[0, t]$ .

$-V_j(t, 0) = \#$  of cars that have entered road  $j$  (possibly after waiting in a queue) during the time interval  $[0, t]$ .

## Remark on the payoff functionals

In all cases, for optimal trajectories the integrand functional is

$L(x, \dot{x}) = -$  [flux of cars from left to right, across the line  $x(t)$ ]



$$f'_k(\rho) = \dot{x} \quad \Longrightarrow \quad g_k(\dot{x}) = - \left( f_k(\rho) - \dot{x} \rho \right)$$

# Length of the queues

$\hat{q}_j$  = initial size of the queue in front of road  $j$

$F_j(t)$  = total number of cars that have reached the intersection before time  $t$  and wish to turn into road  $j$

$-V_j(t, 0)$  = total number of cars that have entered road  $j$  during the interval  $[0, t]$

Size of the queue at the entrance of road  $j$

$$q_j(t) = F_j(t) + V_j(t, 0) \quad (\mathbf{Q})$$



# The Cauchy problem near an intersection

$$\left\{ \begin{array}{l} \rho_{k,t} + f_k(\rho_k)_x = 0 \\ \theta_{ij,t} + v_i(\rho_i)\theta_{ij,x} = 0 \\ \dot{q}_j = \sum_{i \in \mathcal{I}} \bar{f}_i \bar{\theta}_{ij} - \bar{f}_j \end{array} \right. \quad \left\{ \begin{array}{l} \rho_k(0, x) = \hat{\rho}_k(x) \\ \theta_{ij}(0, x) = \hat{\theta}_{ij}(x) \\ q_j(0) = \hat{q}_j \end{array} \right. \quad \begin{array}{l} k \in \mathcal{I} \cup \mathcal{O} \\ i \in \mathcal{I}, j \in \mathcal{O} \\ j \in \mathcal{O} \end{array} \quad \text{(CP)}$$

+ boundary conditions at the intersection

# Admissible solutions to the Cauchy problem near an intersection

**Definition.** We say that the functions  $\rho_k = \rho_k(t, x)$  and  $q_j = q_j(t)$  provide an **admissible solution** to the Cauchy problem **(CP)** with junction conditions **(SBJ)** if there exist Lipschitz continuous functions  $V_k = V_k(t, x)$  such that

- (i)  $\rho_k = V_{k,x}$
- (ii) For  $i \in \mathcal{I}$ , the functions  $V_i$  satisfy **(LF1)**
- (iii) For  $j \in \mathcal{O}$ , the functions  $V_j$  satisfy **(LF3)**
- (iv) For  $j \in \mathcal{O}$ , the functions  $q_j$  satisfy **(Q)**

In case of the junction conditions **(MBJ)**, instead of **(LF1)** the functions  $V_i$  are required to satisfy **(LF2)**

# Well posedness of the Cauchy problem with $L^\infty$ data

## Theorem (A.B. & Khai Nguyen, 2014).

Consider any measurable initial data  $\hat{\rho}_k(x) \in [0, \rho_k^{jam}]$ ,  $\hat{\theta}_{ij}(x) \in [0, 1]$ , and  $\hat{q}_j$ , with

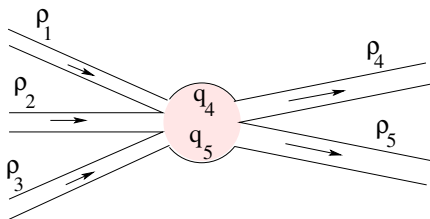
$$\sum_{j \in \mathcal{O}} \hat{q}_j < M \quad \text{or} \quad \hat{q}_j < M_j \text{ for all } j \in \mathcal{O}$$

respectively in the cases **(SBJ)** or **(MBJ)**.

Then the Cauchy problem **(CP)** has a unique admissible solution, globally defined for all  $t \geq 0$ .

One then checks that the functions  $\rho_k = V_{k,x}$  are weak solutions to the conservation laws, satisfy the usual (Lax, Oleinik, ...) entropy admissibility conditions, together with the required boundary conditions **(SBJ)** or **(MBJ)**.

# Sketch of the proof



$$(q_4, q_5) \xrightarrow{\text{Lax formula}} (V_1, V_2, V_3, V_4, V_5) \xrightarrow{\text{boundary values}} (q_4, q_5)$$

length of queues

$$\rho_k = V_{k,x}$$

traffic densities

find a fixed point !

- $q_j(t)$  = size of queue in front of  $j$ -th road
- $V_i(t, x)$  = value function on  $i$ -th incoming road
- $F_j(t)$  = # of cars that have entered the intersection up to time  $t$  and wish to go into road  $j$
- $V_j(t, x)$  = value function on  $j$ -th outgoing road
- $\Lambda_j(\mathbf{q})(t) = F_j(t) + V_j(t, 0)$  = updated size of the queue

The sequence of maps

$$\mathbf{q} = (q_j)_{j \in \mathcal{O}} \mapsto (V_i)_{i \in \mathcal{I}} \mapsto (F_j)_{j \in \mathcal{O}} \mapsto (V_j)_{j \in \mathcal{O}} \mapsto (\Lambda_j(\mathbf{q}))_{j \in \mathcal{O}}$$

defines a strict contraction, on a suitably small time interval  $[0, T]$ .  
Hence it has a unique fixed point, which provides the required solution.

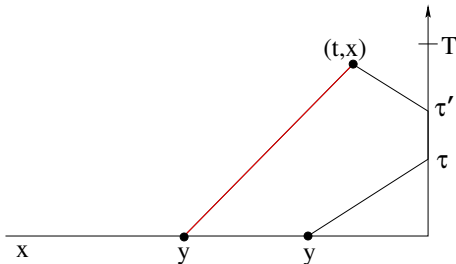
The argument can then be iterated on  $[0, T_1]$ ,  $[T_1, T_2]$ ,  $\dots$ ,  
with  $T_\nu \rightarrow \infty$ .

# Contraction property

Assume:  $\sup_{t \in [0, T], j \in \mathcal{O}} |q_j(t) - \tilde{q}_j(t)| = \delta$

$$V_i(t, x) \doteq \max \left\{ \max_{y \leq 0} \left[ \bar{V}_i(y) + t g_i \left( \frac{x-y}{t} \right) \right], \right. \\ \left. \max_{0 \leq \tau' \leq \tau \leq t, y \leq 0} \left[ \bar{V}_i(y) + \tau' g_i \left( \frac{-y}{\tau'} \right) - \int_{\tau'}^{\tau} h_i(\mathbf{q}(s)) ds + (t - \tau) g_i \left( \frac{x}{t - \tau} \right) \right] \right\}. \quad (\text{LF1})$$

$$|V_i(t, x) - \tilde{V}_i(t, x)| \leq C T \delta.$$



# Continuity w.r.t. weak convergence

## Theorem (A.B. & Khai Nguyen, 2014).

Consider a sequence of initial data  $(\hat{\rho}_k^\nu, \hat{\theta}_{ij}^\nu, \hat{q}_j^\nu)_{\nu \geq 1}$  such that, as  $\nu \rightarrow \infty$ ,

$$\hat{q}_j^\nu \rightarrow \hat{q}_j, \quad j \in \mathcal{O},$$

together with the weak convergence

$$\begin{cases} \hat{\rho}_i^\nu \hat{\theta}_{ij}^\nu \rightharpoonup \hat{\rho}_i \hat{\theta}_{ij}, & i \in \mathcal{I}, j \in \mathcal{O}, \\ \hat{\rho}_j^\nu \rightharpoonup \hat{\rho}_j, & j \in \mathcal{O}. \end{cases}$$

Calling  $\rho_k^\nu = V_{k,x}^\nu$  and  $q_j^\nu$  the corresponding solutions, for every  $t > 0$  one has

- uniform convergence:  $q_j^\nu(t) \rightarrow q_j(t)$  for  $t$  on bounded sets,
- strong convergence in  $\mathbf{L}_{loc}^1$ :  $\rho_k^\nu(t, \cdot) \rightarrow \rho_k(t, \cdot) \quad k \in \mathcal{I} \cup \mathcal{O}$

The result holds both in the case **(SBJ)** of a single buffer and in the case **(MBJ)** of multiple buffers.

# Traffic flow on networks - III

Alberto Bressan

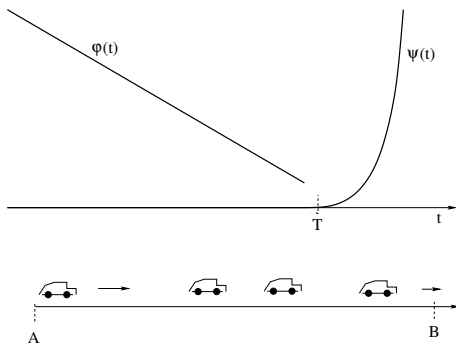
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# An Optimal Scheduling Problem

- Car drivers starting from a location  $A$  (a residential neighborhood) need to reach a destination  $B$  (a working place) at a given time  $T$ .
- There is a cost  $\varphi(\tau_d)$  for departing early and a cost  $\psi(\tau_a)$  for arriving late.



# Elementary solution

Assume: the travel time  $\Delta t$  is constant, hence  $\tau_a = \tau_d + \Delta t$

Then the optimal departure time is

$$\tau_d^{\text{opt}} = \operatorname{argmin}_t \left\{ \varphi(t) + \psi(t + \Delta t) \right\}$$

If everyone departs exactly at the same optimal time,  
a traffic jam is created and this strategy is not optimal anymore

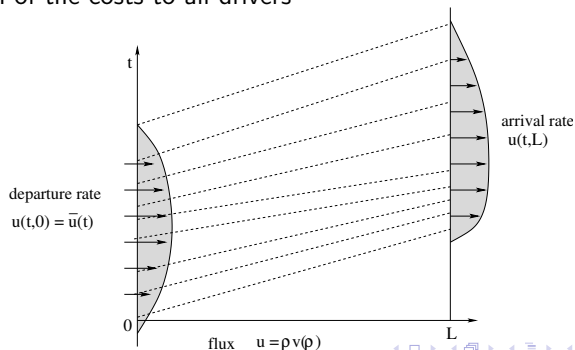
Speed of cars depends on traffic density !

# An optimization problem for traffic flow

Given **departure cost**  $\varphi(t)$  and **arrival cost**  $\psi(t)$ ,  
choose a departure rate  $\bar{u}(t)$  so that the solution of the conservation law

$$\begin{cases} \rho_t + [\rho v(\rho)]_x = 0 & x \in [0, L] \\ \rho(t, 0)v(\rho(t, 0)) = \bar{u}(t) & t \in \mathbb{R} \end{cases}$$

minimizes the sum of the costs to all drivers



$$u(t, x) \doteq \rho(t, x) v(\rho(t, x)) = \text{flux of cars}$$

Minimize total cost: 
$$J(\bar{u}) \doteq \int \varphi(t) u(t, 0) dt + \int \psi(t) u(t, L) dt$$

Choose the optimal departure rate  $\bar{u}(t)$ , subject to the constraints

$$\bar{u}(t) \geq 0, \quad \int \bar{u}(t) dt = \kappa = [\text{total number of drivers}]$$

# Equivalent formulations

## Boundary value problem for the density $\rho$ :

$$\text{conservation law: } \rho_t + [\rho v(\rho)]_x = 0, \quad (t, x) \in \mathbb{R} \times [0, L]$$

$$\text{control (on the boundary data): } \rho(t, 0)v(\rho(t, 0)) = \bar{u}(t)$$

## Cauchy problem for the flux $u$ :

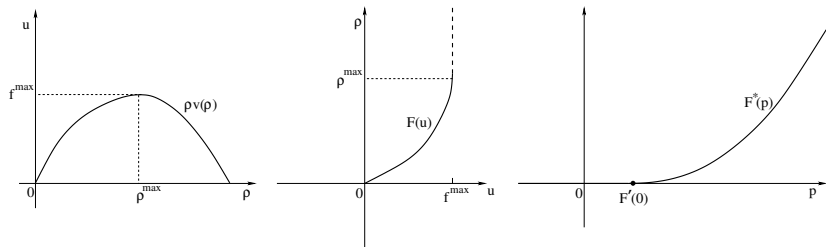
$$\text{conservation law: } u_x + F(u)_t = 0, \quad u = \rho v(\rho), \quad F(u) = \rho$$

$$\text{control (on the initial data): } u(t, 0) = \bar{u}(t)$$

$$\text{Cost: } J(u) = \int_{-\infty}^{+\infty} \varphi(t)u(t, 0) dt + \int_{-\infty}^{+\infty} \psi(t)u(t, L) dt$$

$$\text{Constraint: } \int_{-\infty}^{+\infty} \bar{u}(t) dt = \kappa$$

# The flux function and its Legendre transform



$$u = \rho v(\rho), \quad \rho = F(u)$$

Legendre transform: 
$$F^*(p) \doteq \max_u \{ pu - F(u) \}$$

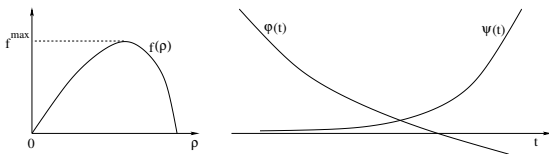
# Existence of a globally optimal solution

(A1) The flux function  $\rho \mapsto f(\rho) = \rho v(\rho)$  is strictly concave down.

$$f(0) = f(\rho^{jam}) = 0, \quad f'' < 0.$$

(A2) The cost functions  $\varphi, \psi$  satisfy

$$\varphi' < 0, \quad \psi, \psi' \geq 0, \quad \lim_{|t| \rightarrow \infty} (\varphi(t) + \psi(t)) = +\infty$$



**Theorem (A.B. & Ke Han, *SIAM J. Math. Anal.*, 2012).**

Let (A1)-(A2) hold. Then, for any  $\kappa > 0$ , there exists a unique admissible initial data  $\bar{u}$  minimizing the total cost  $J(\cdot)$ .

# Sketch of the proof

- Consider a minimizing sequence of departure rates  $\bar{u}_n$
- It is not restrictive to assume  $\bar{u}_n(t) \in [0, f^{max}]$
- The supports of  $\bar{u}_n$  are uniformly bounded
- Take a weakly convergent subsequence  $\bar{u}_n \rightharpoonup \bar{u}$
- The functional is continuous w.r.t. weak convergence, hence  $\bar{u}$  is optimal

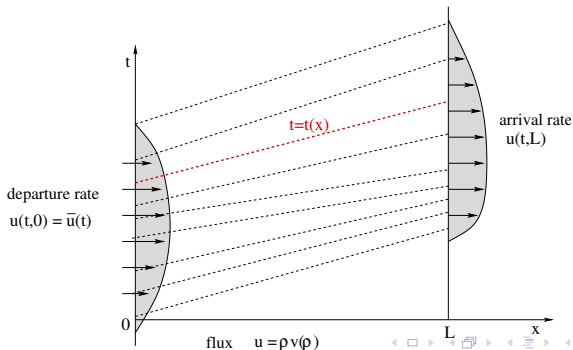


# Characterization of the globally optimal solution

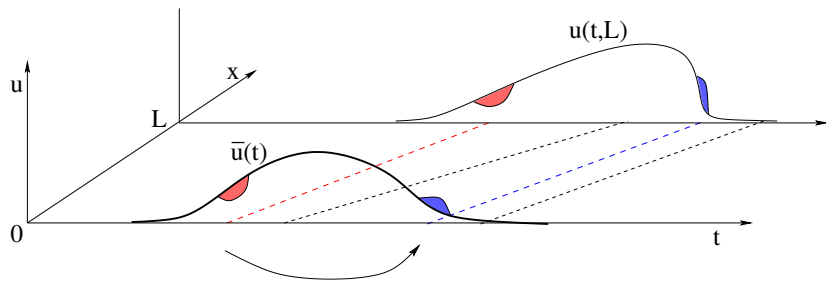
If  $\rho = \rho(t, x)$  is the density in a globally optimal solution, then

- $\rho$  is continuous (no shocks appear)
- there exists a constant  $C$  such that, for any **characteristic** line  $t = t(x)$

$$\begin{cases} \varphi(t(0)) + \psi(t(L)) = C & \text{if } t(0) \in \text{Supp}(\bar{u}) \\ \varphi(t(0)) + \psi(t(L)) \geq C & \text{if } t(0) \notin \text{Supp}(\bar{u}) \end{cases}$$

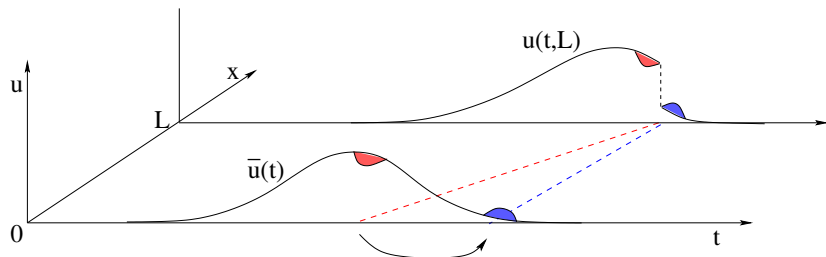


# Derivation of necessary conditions



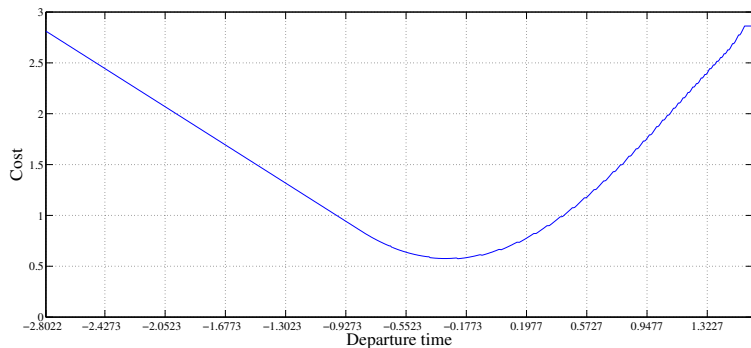
shift mass to a characteristic with lower total cost

# Ruling out shocks



shift mass to a characteristic with lower total cost

# Does everyone pay the same cost?



Cost vs. departure time in a globally optimal solution

# The Nash equilibrium solution

A solution  $u = u(t, x)$  is a **Nash equilibrium** if no driver can reduce his/her own cost by choosing a different departure time.

This implies that **all drivers pay the same cost**.

To find a Nash equilibrium, introduce the integrated variable

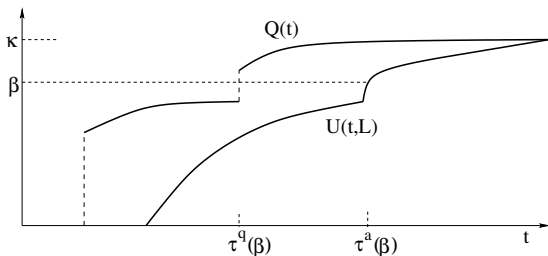
$$U(t, x) \doteq \int_{-\infty}^t \rho(s, x) v(\rho(s, x)) ds = [\text{number of drivers}$$

that have crossed the point  $x$  on the road within time  $t$ ]

This solves a Hamilton-Jacobi equation

$$U_x + F(U_t) = 0$$

Note: a **queue** can form at the entrance of the highway



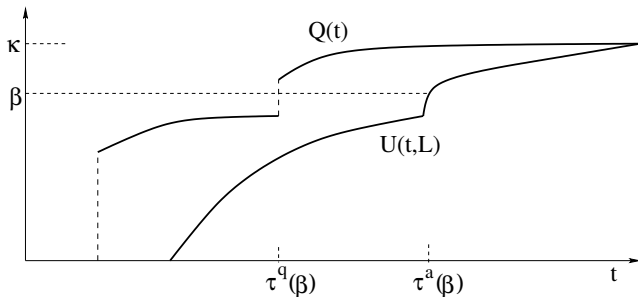
$Q(t)$  = number of drivers who have started their journey before time  $t$   
(possibly joining the queue)

$L$  = length of the road

$U(t, L)$  = number of drivers who have reached destination before time  $t$

$$U(t, L) = \min_{\tau} \left\{ Q(\tau) + L F^* \left( \frac{t - \tau}{L} \right) \right\}$$

# Characterization of a Nash equilibrium



$\beta \in [0, \kappa]$  = Lagrangian variable labeling one particular driver

$\tau^q(\beta)$  = time when driver  $\beta$  joins the queue

$\tau^a(\beta)$  = time when driver  $\beta$  arrives at destination

$$\text{Nash equilibrium} \quad \implies \quad \varphi(\tau^q(\beta)) + \psi(\tau^a(\beta)) = c$$

# Existence and Uniqueness of Nash equilibrium

## Theorem (A.B. - K. Han).

Let the flux  $f$  and cost functions  $\varphi, \psi$  satisfy the assumptions (A1)-(A2). Then, for every  $\kappa > 0$ , the Hamilton-Jacobi equation

$$U_x + F(U_t) = 0$$

admits a unique Nash equilibrium solution with

$$\text{total \# of drivers} = Q(+\infty) = \kappa$$

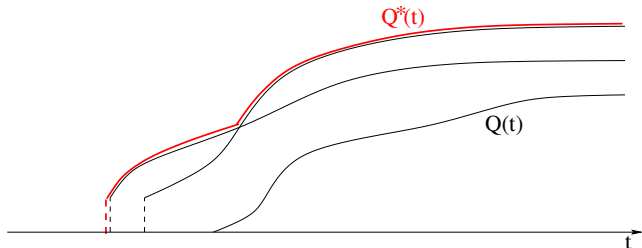


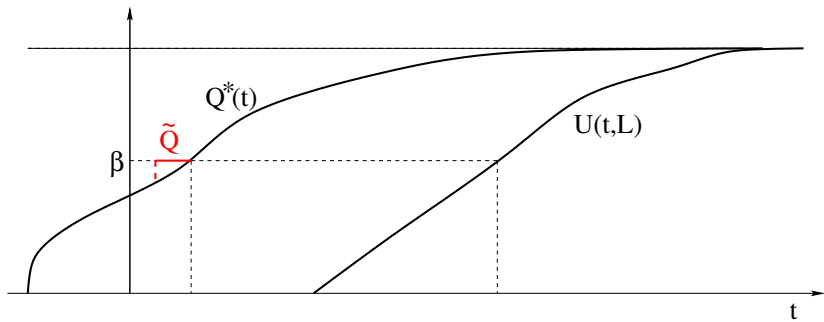
# Sketch of the proof

1. For a given cost  $c$ , let  $\mathcal{Q}_c^-$  be the set of all departure distributions  $Q(\cdot)$  for which every driver has a cost  $\leq c$ :

$$\varphi(\tau^q(\beta)) + \psi(\tau^a(\beta)) \leq c \quad \text{for a.e. } \beta \in [0, Q(+\infty)].$$

2. Claim:  $Q^*(t) \doteq \sup \{ Q(t); Q \in \mathcal{Q}_c^- \} \in \mathcal{Q}_c^-$   
is the initial data for a Nash equilibrium with common cost  $c$ .



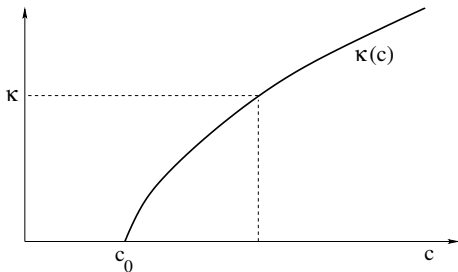


3. The Nash equilibrium with common cost  $c$  is unique.

4. Define:  $\kappa(c) \doteq \sup \left\{ Q(+\infty); Q \in \mathcal{Q}_c^- \right\}$   
= maximum number of cars in a solution where everyone pays  $\leq c$ .

There exists a minimum cost  $c_0$  such that  $\kappa(c) = 0$  for  $c \leq c_0$ .

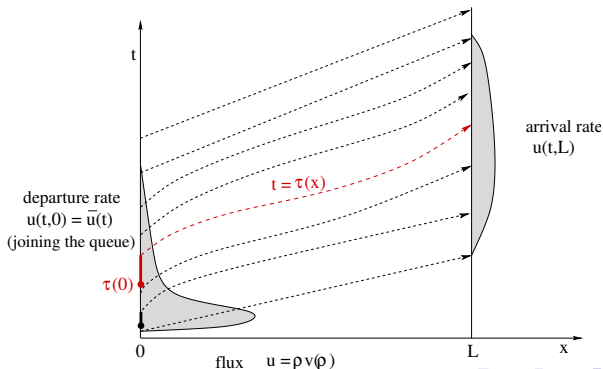
The map  $c \mapsto \kappa(c)$  is strictly increasing and continuous from  $[c_0, +\infty[$  to  $[0, +\infty[$ .



# Characterization of the Nash equilibrium solution

If  $\rho = \rho(t, x)$  is the density in a Nash equilibrium solution, then there exists a constant  $C$  such that, for any **particle line**  $t = \tau(x)$  (describing a car trajectory)

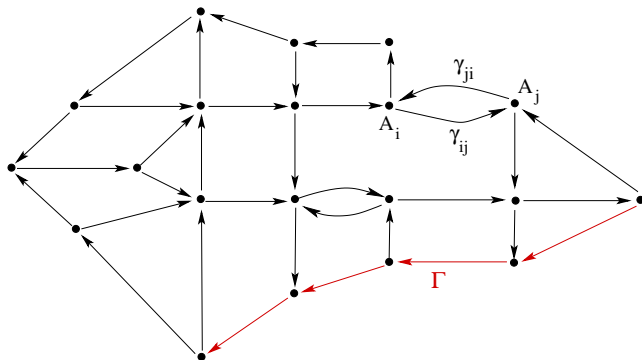
$$\begin{cases} \varphi(\tau(0)) + \psi(\tau(L)) = C & \text{if } \tau(0) \in \text{Supp}(\bar{u}) \\ \varphi(\tau(0)) + \psi(\tau(L)) \geq C & \text{if } \tau(0) \notin \text{Supp}(\bar{u}) \end{cases}$$



# Traffic Flow on a Network

Nodes:  $A_1, \dots, A_m$       arcs:  $\gamma_{ij}$

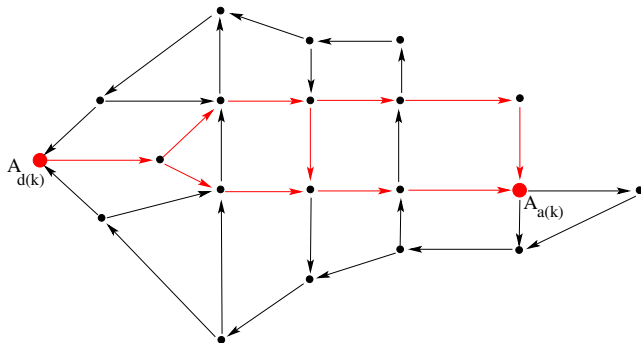
$n$  groups of drivers with different origins and destinations, and different costs



# Traffic Flow on a Network

Drivers in the  $k$ -th group depart from  $A_{d(k)}$  and arrive to  $A_{a(k)}$   
can use different paths  $\Gamma_1, \Gamma_2, \dots$  to reach destination

Departure cost:  $\varphi_k(t)$       arrival cost:  $\psi_k(t)$



# Admissible departure rates

$G_k$  = total number of drivers in the  $k$ -th group,  $k = 1, \dots, n$

$\Gamma_p$  = viable path (concatenation of viable arcs  $\gamma_{ij}$ ),  $p = 1, \dots, N$

$t \mapsto \bar{u}_{k,p}(t)$  = departure rate of  $k$ -drivers traveling along the path  $\Gamma_p$

The set of departure rates  $\{\bar{u}_{k,p}\}$  is **admissible** if

$$\bar{u}_{k,p}(t) \geq 0, \quad \sum_{\Gamma_p \in \mathcal{R}_k} \int_{-\infty}^{\infty} \bar{u}_{k,p}(t) dt = G_k \quad k = 1, \dots, n$$

$\mathcal{R}_k$  = set of admissible paths, from node  $A_{d(k)}$  to node  $A_{a(k)}$

$\tau_p(t)$  = arrival time for a driver starting at time  $t$ , traveling along  $\Gamma_p$

(depends on the overall traffic conditions)

# Optima and Equilibria

An admissible family  $\{\bar{u}_{k,p}\}$  of departure rates is **globally optimal** if it minimizes the sum of the total costs of all drivers

$$J(\bar{u}) \doteq \sum_{k,p} \int \left( \varphi_k(t) + \psi_k(\tau_p(t)) \right) \bar{u}_{k,p}(t) dt$$

An admissible family  $\{\bar{u}_{k,p}\}$  of departure rates is a **Nash equilibrium** if no driver of any group can lower his own total cost by changing departure time or switching to a different path to reach destination:

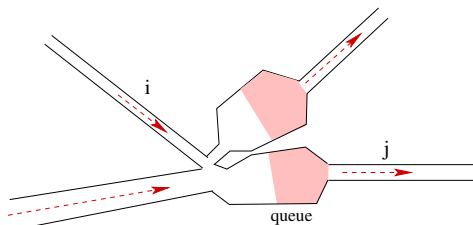
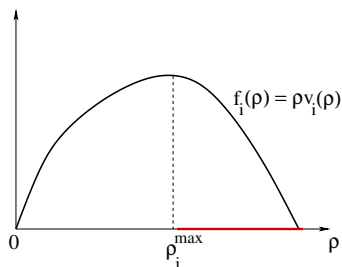
$$\varphi_k(t) + \psi_k(\tau_p(t)) = C_k \quad \text{for all } t \in \text{Supp}(\bar{u}_{k,p})$$

$$\varphi_k(t) + \psi_k(\tau_p(t)) \geq C_k \quad \text{for all } t \in \mathbb{R}$$



# Assumptions on the dynamics

**(A1)** On each arc, the flux function  $\rho \mapsto f_i(\rho) = \rho v_i(\rho)$  is concave down.

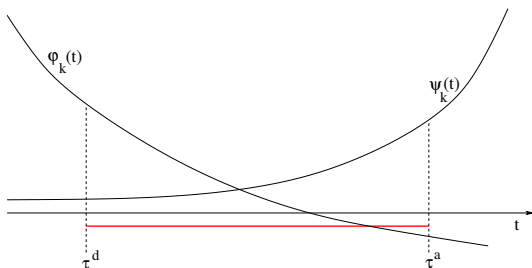


- All roads remain in a free state:  $\rho_i \in [0, \rho_i^{\max}]$
- At any given junction, if too many drivers wish to enter an outgoing road  $j \in \mathcal{O}$ , a queue forms at its entrance (in a buffer with infinite capacity). Incoming roads  $i \in \mathcal{I}$  are not affected

# Assumptions on the cost functions

(A2) For  $k$ -drivers, the departure cost  $\varphi_k$  and the arrival cost  $\psi_k$  satisfy

$$\varphi'_k < 0 \quad \psi'_k > 0, \quad \lim_{|t| \rightarrow \infty} (\varphi_k(t) + \psi_k(t)) = \infty,$$



**Theorem** (A.B. - Ke Han, *Netw. & Heter. Media*, 2013).

Under the assumptions (A1)-(A2), on a general network of roads, there exists at least one globally optimal solution, and at least one Nash equilibrium solution.

**Note:** For a single group of drivers on a single road, solutions are unique. Uniqueness is not expected to hold, on a general network.

**Theorem** (A.B. - K.Nguyen, 2014).

The existence of a globally optimal solution and a Nash equilibrium solution remain valid, if the flow at each intersection is modeled by a buffer of finite size.

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The existence of a globally optimal solution and a Nash equilibrium solution remain valid, if the flow at each intersection is modeled by a buffer of finite size.

# Existence of Nash equilibria

Proof: by finite dimensional approximations + topological methods

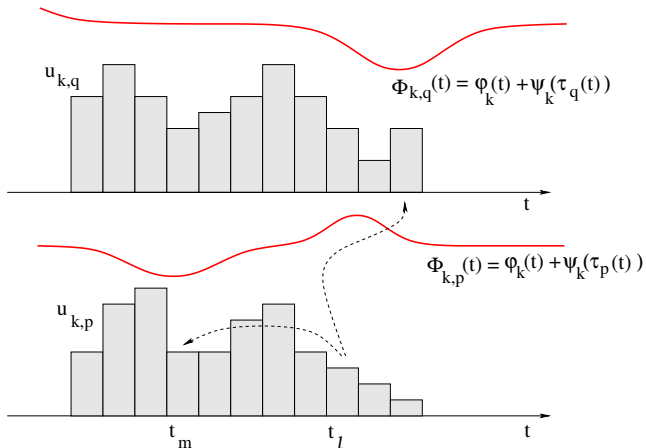
Fix a small time step  $\delta > 0$ , choose  $R$  sufficiently large.

$$\mathcal{U} = \left\{ \mathbf{u} = \{u_{k,p}(\cdot)\}; \quad \mathbf{u} \text{ is an admissible departure rate} \right. \\ \left. \begin{array}{l} \text{Supp}(u_{k,p}) \subseteq [-R, R], \\ u_{k,p} \text{ constant on each interval } [(n-1)\delta, n\delta] \end{array} \right\}$$

$\mathcal{U}$  is a compact, convex set in a finite dimensional space

# An evolution equation on $\mathcal{U}$

$$\frac{d}{d\theta} \mathbf{u} = \Lambda(\mathbf{u})$$



# An evolution equation on $\mathcal{U}$

$$\frac{d}{d\theta} \mathbf{u} = \Lambda(\mathbf{u}) \quad \mathbf{u} \in \mathcal{U}$$

$$\Phi_{k,p}^{\ell} = \int_{(\ell-1)\delta}^{\ell\delta} \left( \varphi_k(t) + \psi_k(\tau_p(t)) \right) dt$$

= average cost to a  $k$ -driver choosing path  $p$

$$u_{k,p}^{\ell} = u_{k,p}(t) \quad t \in ](\ell-1)\delta, \ell\delta]$$

$$\begin{aligned} \frac{d}{d\theta} u_{k,p}^\ell &= \sum_q \sum_m \left[ \Phi_{k,q}^m - \Phi_{k,p}^\ell \right]_+ \cdot u_{k,q}^m \\ &\quad - \sum_q \sum_m \cdot \left[ \Phi_{k,p}^\ell - \Phi_{k,q}^m \right]_+ \cdot u_{k,p}^\ell \end{aligned}$$

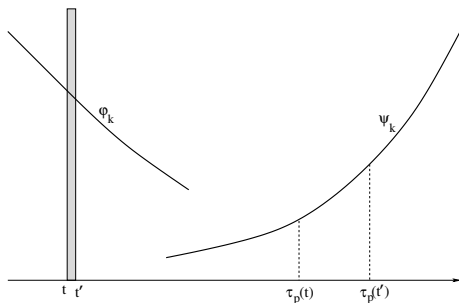
This continuous evolution equation on the compact convex set  $\mathcal{U}$  has at least one steady state  $\{\bar{u}_{k,p}\}$

(a Galerkin approximation to a Nash equilibrium)

Letting the time step  $\delta \rightarrow 0$ , the departure rates  $\{\bar{u}_{k,p}\}$  remain uniformly bounded in  $\mathbf{L}^\infty$ , with uniformly bounded support.



# A priori bounds



$$\text{Nash equilibrium} \implies \varphi_k(t) + \psi_k(\tau_p(t)) = \varphi_k(t') + \psi_k(\tau_p(t'))$$

Bound on the maximum flux on each arc

$$\implies \tau_p(t') - \tau_p(t) \geq \frac{1}{F_{\max}} \cdot \int_t^{t'} u_{k,p}(s) ds$$

# Existence of a Nash equilibrium solution

Let the time step  $\delta \rightarrow 0$  approach zero.

Taking subsequences, one has

$$\bar{u}_{k,p}^\nu(\cdot) \rightharpoonup \bar{u}_{k,p}(\cdot) \quad \text{weakly}$$

$$\implies \tau_p^\nu(\cdot) \rightarrow \tau_p(\cdot) \quad \text{uniformly on compact sets}$$

The departure rates  $\bar{u}_{k,p}(\cdot)$  provide a Nash equilibrium

**Note:** For any model of traffic flow at intersections that yields

- unique solutions, for general  $L^\infty$  data
- continuity of the travel time, w.r.t. weak convergence of the departure rates

the same argument provides the existence of a Nash equilibrium.

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# Traffic flow on networks - IV

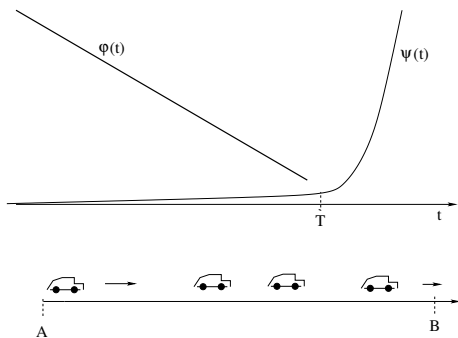
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# A Decision Problem

- Car drivers starting from a location  $A$  (a residential neighborhood) need to reach a destination  $B$  (a working place) at a given time  $T$ .
- There is a cost  $\varphi(\tau_d)$  for departing early and a cost  $\psi(\tau_a)$  for arriving late.

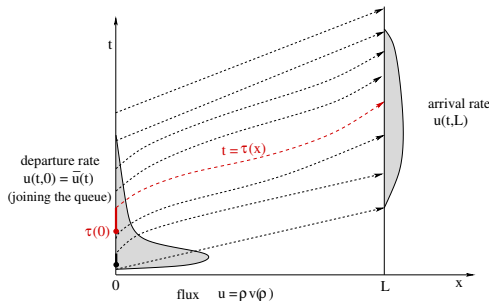


# The Nash equilibrium solution

A solution  $u = u(t, x)$  is a **Nash equilibrium** if no driver can reduce his/her own cost by choosing a different departure time.

This implies that **all drivers pay the same cost**.

$$\begin{cases} \varphi(\tau(0)) + \psi(\tau(L)) = C & \text{if } \tau(0) \in \text{Supp}(\bar{u}) \\ \varphi(\tau(0)) + \psi(\tau(L)) \geq C & \text{if } \tau(0) \notin \text{Supp}(\bar{u}) \end{cases}$$



# Continuous dependence of the Nash equilibrium solution

$\varphi_1(t), \varphi_2(t)$  costs for departing at time  $t$

$\psi_1(t), \psi_2(t)$  costs for arriving at time  $t$

$v_1(\rho), v_2(\rho)$  speeds of cars, when the density is  
 $\rho \in [0, \rho^*], \quad \rho_1^{\max}, \rho_2^{\max} \leq \rho^*$

$Q_1(t), Q_2(t)$  = number of cars that have departed up to time  $t$ , in the corresponding Nash equilibrium solutions

Theorem (A.B., C.Liu, W.Shen, F.Yu, *Quarterly Appl. Math.*, 2012)

Assume all cars depart and arrive within the interval  $[a, b]$ . Then

$$\begin{aligned} & \|Q_1 - Q_2\|_{L^1([a,b])} \\ & \leq C \cdot \left( \|\varphi_1 - \varphi_2\|_{L^\infty([a,b])} + \|\psi_1 - \psi_2\|_{L^\infty([a,b])} + \|v_1 - v_2\|_{L^\infty([0,\rho^*])}^{1/2} \right) \end{aligned}$$

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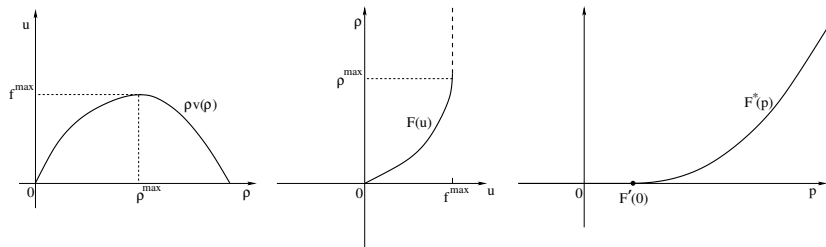
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# Back to the Lax formula



$$u = \rho v(\rho), \quad \rho = F(u)$$

Legendre transform: 
$$F^*(\rho) \doteq \max_u \{ \rho u - F(u) \}$$

$L$  = length of the road

$$\mu \doteq \frac{L}{f'(0)} = \frac{L}{v(0)} = \text{minimum travel time (if road is empty)}$$

$Q(t)$  = # of drivers departing before time  $t$  (possibly joining a queue)

$U(t, L)$  = # of drivers that have reached the end of the road before time  $t$

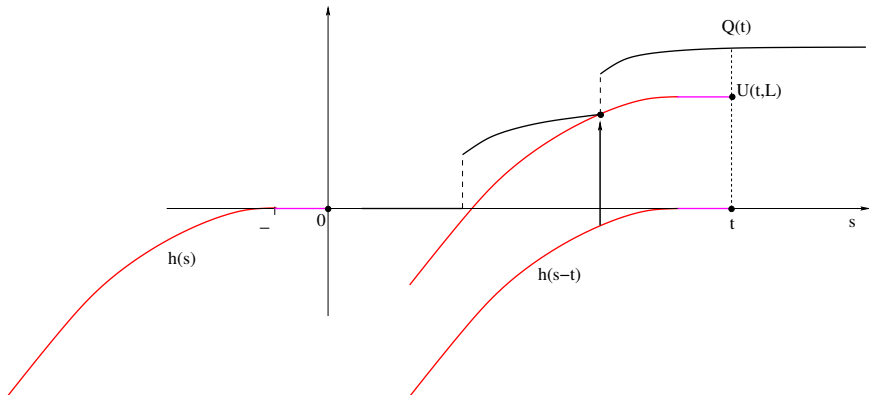
$$U(t, L) = \min_{\tau} \left\{ Q(\tau) + L F^* \left( \frac{t - \tau}{L} \right) \right\} = \min_{\tau} \left\{ Q(\tau) - h(\tau - t) \right\}$$

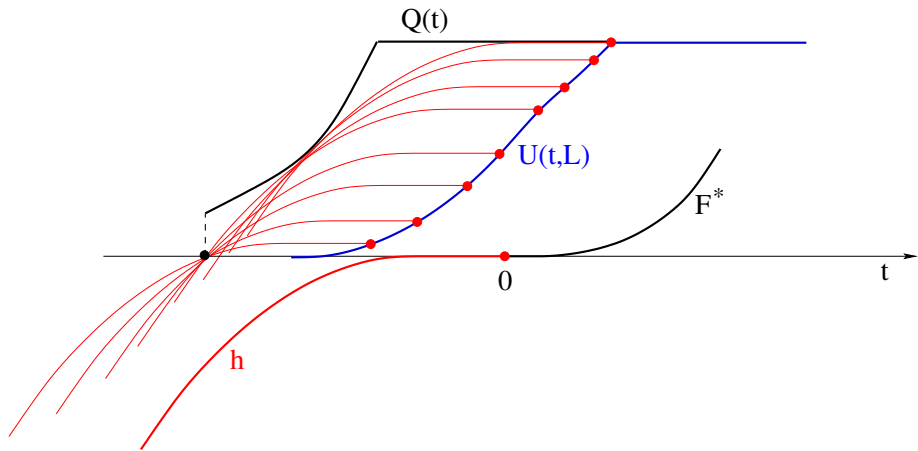
$$h(s) \doteq -L \cdot F^* \left( \frac{-s}{L} \right)$$

# Geometric interpretation of the Lax formula

$$U(t, L) = \min_{\tau} \left\{ Q(\tau) - h(\tau - t) \right\}$$

is the amount by which we can shift upward the graph of  $h(\cdot - t)$ , before hitting the graph of  $Q(\cdot)$



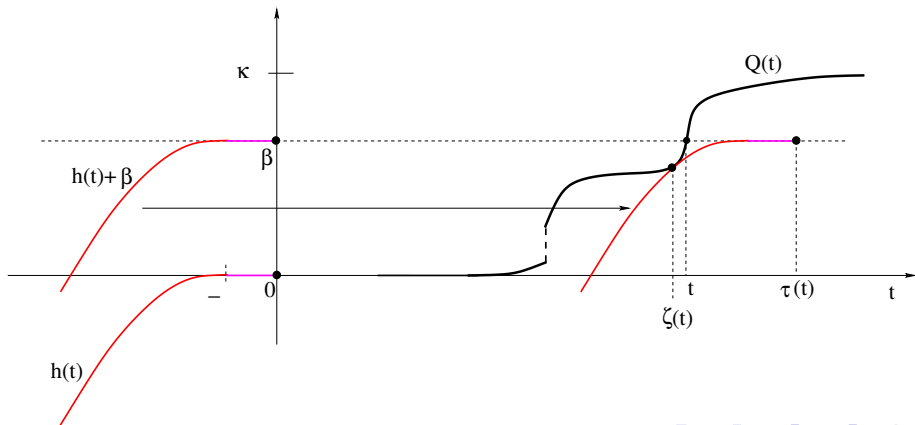


# Computing the arrival time

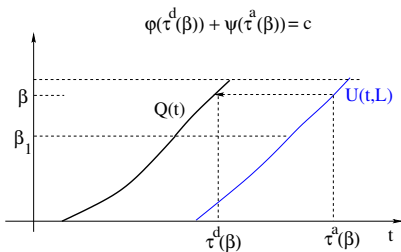
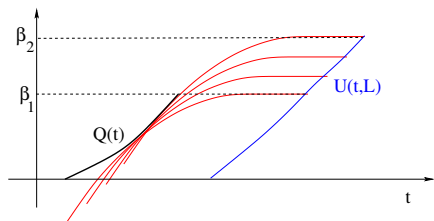
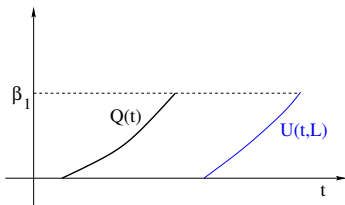
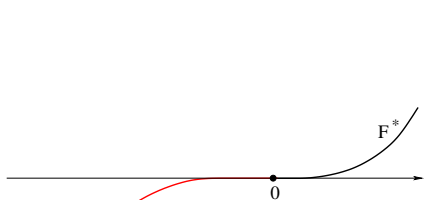
$t$  = departure time of driver  $\beta$

$\tau(t)$  = arrival time of driver  $\beta$

= horizontal shift, moving the graph of  $h + \beta$  below the graph of  $Q$   
depends only on the tangency point  $\zeta(t)$



# Constructing the Nash equilibrium with common cost = $c$



# A minimax property

A.B., C.Liu, W.Shen, F.Yu, *Quarterly Appl. Math.*, 2012

Set of all departure distributions for  $\kappa$  drivers:

$$\mathcal{Q}^\kappa = \{Q(\cdot); \quad Q \text{ nondecreasing, } Q(-\infty) = 0, \quad Q(+\infty) = \kappa\}$$

Maximum cost among all drivers

$$\Psi(Q) \doteq \text{ess - sup}_{\beta \in [0, \kappa]} \left\{ \varphi(\tau^q(\beta)) + \psi(\tau^a(\beta)) \right\}$$

The departure distribution  $Q^*$  for the Nash equilibrium yields the global minimum of  $\Psi$ , namely

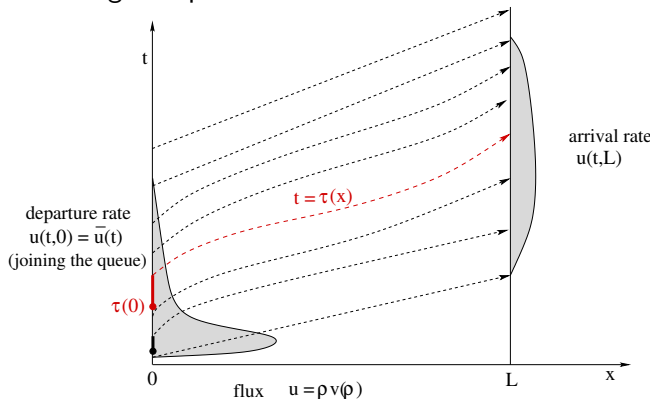
$$\Psi(Q^*) = \min \{ \Psi(Q); \quad Q \in \mathcal{Q}^\kappa \}$$

(converse is false, if a queue is present)

# Improving the Nash equilibrium

$Q^*(\cdot) =$  Nash equilibrium departure distribution

$Q^\sharp(\cdot) =$  new departure distribution, where drivers depart at the same time when they were clearing the queue





# Nash equilibrium as a variational inequality (T.Friesz)

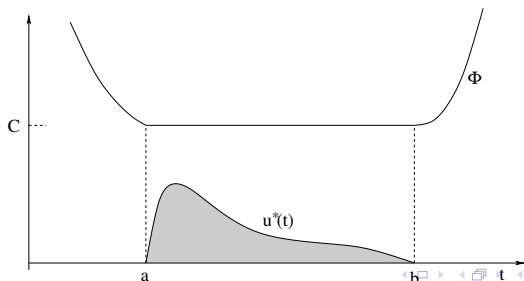
$u(t)$  = rate of departures at time  $t$

$$\mathcal{U} \doteq \left\{ u \geq 0; \int u(t) dt = \kappa \right\}$$

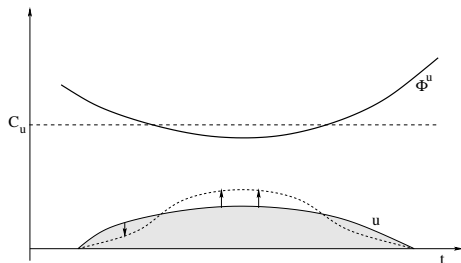
$u^*(\cdot)$  yields a Nash equilibrium if

$$\int \left[ u(t) - u^*(t) \right] \Phi^*(t) dt \geq 0 \quad \text{for all } u \in \mathcal{U}$$

$\Phi^*(t) = \varphi(t) + \psi(\tau(t))$  = departure cost + arrival cost to a driver departing at time  $t$ .



$$C_u = \frac{1}{\kappa} \int \Phi^u(t) u(t) dt = \text{average cost}$$



- increase  $u$  at points where  $\Phi^u(t) < C_u$
- decrease  $u$  at points where  $\Phi^u(t) > C_u$

# A numerical scheme

$$\frac{d}{d\theta} u(t) = \{u(t) - \Phi^u(t) - c\}_+ - u(t)$$

choose the constant  $c$  so that the integral of the right hand side is zero

$$u_\theta = \begin{cases} -u & \text{if } u + c - \Phi^u < 0, \\ c - \Phi & \text{if } u + c - \Phi^u \geq 0. \end{cases}$$

convergence ?

## Dynamic stability of Nash equilibrium ?

Assume: drivers can change their departure time on a day-to-day basis, in order to decrease their own cost (one group of drivers, one single road)

Introduce an additional variable  $\theta$  counting the number of days on the calendar.

$\bar{u}(t, \theta) \doteq$  [rate of departures at time  $t$ , on day  $\theta$ ]

$\Phi(t, \theta) \doteq$  [cost to a driver starting at time  $t$ , on day  $\theta$ ]

# A conservation law with non-local flux

**Model 1:** drivers gradually change their departure time, drifting toward times where the cost is smaller.

$\tau^d(\beta, \theta) =$  departure time of driver  $\beta \in [0, \kappa]$ , on day  $\theta$

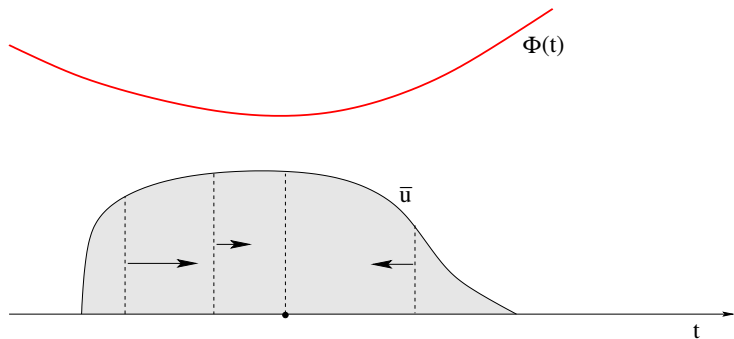
$\Phi(t, \theta) =$  total cost to a driver departing at time  $t$ , on day  $\theta$

$$\frac{\partial}{\partial \theta} \tau^d(\beta, \theta) = - \frac{\partial}{\partial t} \Phi(\tau^d(\beta, \theta), \theta)$$

If the rate of change is proportional to the gradient of the cost, this leads to the conservation law with nonlocal flux

$$\bar{u}_\theta - [\Phi_t \bar{u}]_t = 0$$

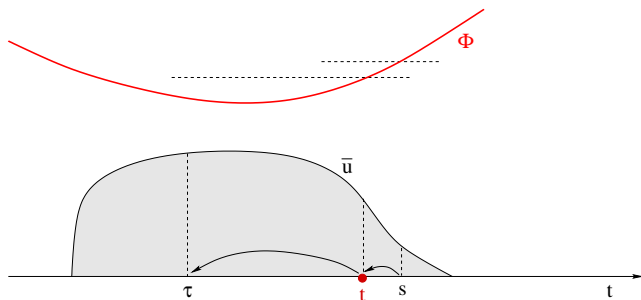
$$\bar{u}_\theta - [\Phi_t \bar{u}]_t = 0$$



# An integral evolution equation

**Model 2:** drivers jump to different departure times having a lower cost.  
If the rate of change is proportional to the difference between the costs, this leads to

$$\frac{d}{d\theta} \bar{u}(t) = \int \bar{u}(s) [\Phi(s) - \Phi(t)]_+ ds - \int \bar{u}(t) [\Phi(t) - \Phi(s)]_+ ds$$

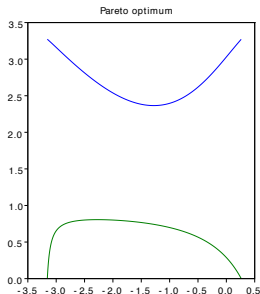
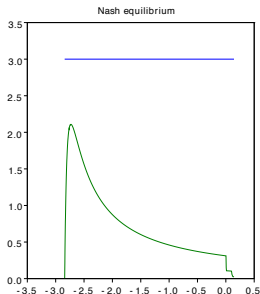


**Question:** as  $\theta \rightarrow \infty$ , does the departure rate  $\bar{u}(t, \theta)$  approach the unique Nash equilibrium?

# Some numerical experiments (Wen Shen, 2011)

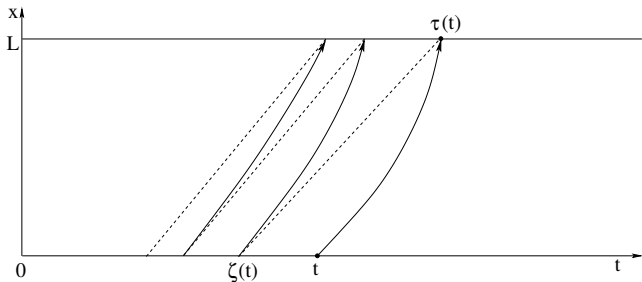
- departure and arrival costs:  $\varphi(x) = -x$ ,  $\psi(x) = e^x$
- velocity of cars:  $v(\rho) = 2 - \rho$       length of road = 2
- total number of cars = 2.2005
- common total cost in the Nash equilibrium = 3

$$\rho_t + (2\rho - \rho^2)_x = 0$$





- is the Nash equilibrium unstable ?
- what causes instability ?
- is there a chaotic attractor ?



main difficulty: non-local dependence

$$\Phi(t) = \varphi(t) + \psi(\tau(t))$$

The arrival time  $\tau(t)$  depends only on the values of

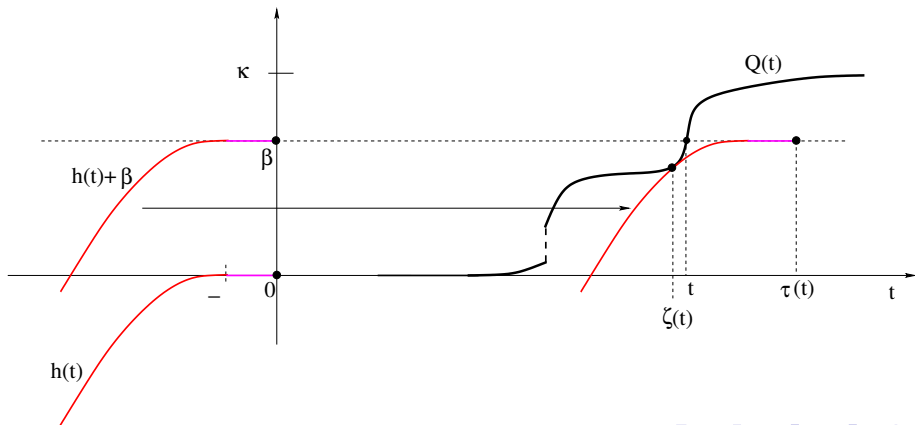
$$Q(t) = \int_{-\infty}^t u(s) ds \quad \text{near } \zeta(t)$$

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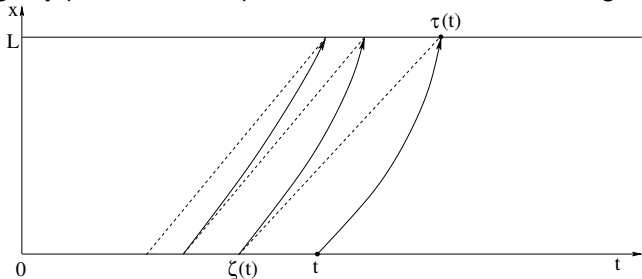
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= horizontal shift, moving the graph of  $h + \beta$  below the graph of  $Q$   
depends only on the tangency point  $\zeta(t)$



$\zeta(t)$  = tangency point = initial point on the characteristic ending at  $(t, L)$



$$u \mapsto Q \mapsto \Phi$$

$$Q(t) = \int_{-\infty}^t u(s) ds, \quad \Phi(t) = \varphi(t) + \psi(\tau(t))$$

Main difficulty in the analysis: pointwise, non-local dependence of the arrival time map  $t \mapsto \tau(t)$

linearized stability ?

