



HAL
open science

Traffic flow on networks I, II, III

Alberto Bressan

► **To cite this version:**

| Alberto Bressan. Traffic flow on networks I, II, III. NETCO 2014, 2014, Tours, France. hal-01023949

HAL Id: hal-01023949

<https://inria.hal.science/hal-01023949>

Submitted on 17 Jul 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Traffic flow on networks - I

Alberto Bressan

Department of Mathematics, Penn State University

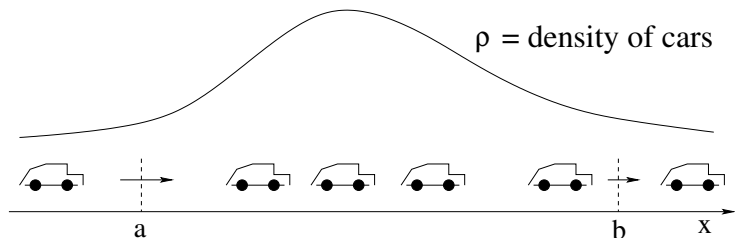
bressan@math.psu.edu

- Review of scalar conservation laws, the Lax formula
- Flow on a network of roads: boundary conditions at intersections
- Well posedness of the Cauchy problem for L^∞ data

- Optimal scheduling, with departure and arrival costs:
optima and equilibria on a single road
- Global optima and Nash equilibria on a network of roads

A conservation law describing traffic flow

(Lighthill-Whitham-Richards, 1955-56)



$$\rho_t + [\rho v(\rho)]_x = 0$$

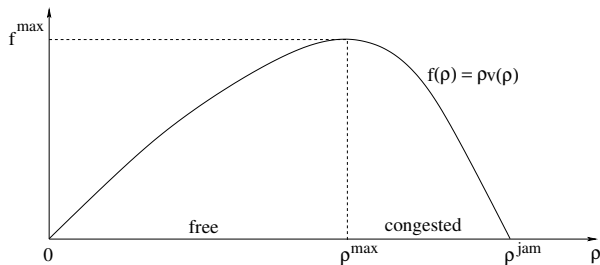
$v(\rho) = \text{velocity of cars (depending only on the density)}$

$f(\rho) = \rho v(\rho) = \text{flux}$

The flux function

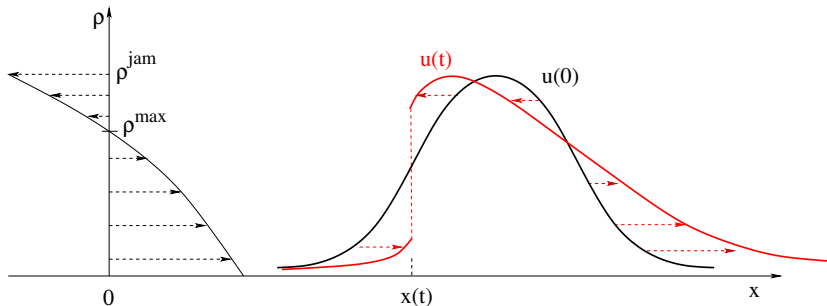
$$\rho_t + f(\rho)_x = 0, \quad f(\rho) = \rho v(\rho)$$

$$f \in \mathcal{C}^2, \quad f''(\rho) < 0, \quad f(0) = f(\rho^{\text{jam}}) = 0$$



The Cauchy problem

Characteristic speed: $f'(\rho)$ decreases with ρ

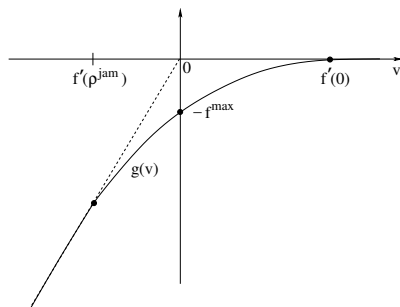
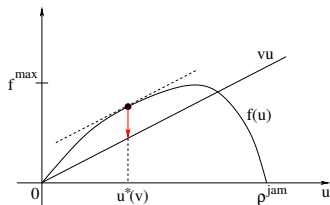


Shocks travel with Rankine-Hugoniot speed $\dot{x} = \frac{f(\rho^+) - f(\rho^-)}{\rho^+ - \rho^-}$

They are admissible if the Lax condition $\rho^- < \rho^+$ holds

The Lax formula

Legendre transform: $g(v) \doteq \inf_{u \in [0, \rho^{jam}]} \{vu - f(u)\}$



$$g(v) = u^*(v)v - f(u^*(v))$$

$$\text{where } f'(u^*(v)) = v$$

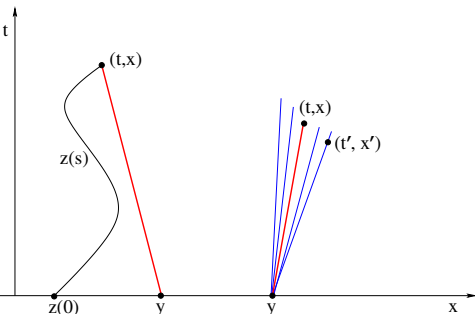
$$\begin{cases} v \leq f'(\rho^{jam}) \\ v \geq f'(0) \end{cases} \begin{matrix} \implies \\ \implies \end{matrix} \begin{cases} g(v) = \rho^{jam} v \\ g(v) = 0 \end{cases}$$

An optimization problem

$$u_t + f(u)_x = 0 \quad u(0, x) = \bar{u}(x) \in [0, \rho^{jam}]$$

Given the initial data $\bar{V}(x) \doteq \int_{-\infty}^x \bar{u}(y) dy$ and a terminal point (t, x) ,

maximize: $\bar{V}(z(0)) + \int_0^t g(\dot{z}(s)) ds$, subject to $z(t) = x$



- optimal trajectory is affine, with $\dot{x} \in [f'(\rho^{jam}), f'(0)]$

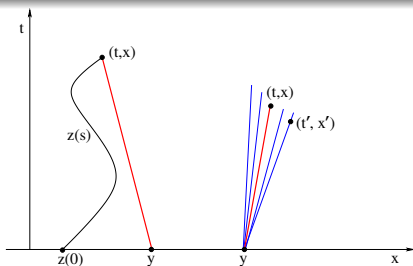
- Value function is

$$V(t, x) = \max_y \bar{V}(y) + t g\left(\frac{x - y}{t}\right)$$

Properties of the Legendre transform

$$f'(\rho) = v \iff g'(v) = \rho$$

In this case, $g(v) = \rho v - f(\rho)$



$$V(t', x') \geq V^-(t', x') = \bar{V}(y) + t' g\left(\frac{x' - y}{t'}\right)$$

Differentiating w.r.t. (t', x') , choosing ρ such that $f'(\rho) = \frac{x-y}{t}$ one obtains

$$\begin{aligned}(\partial_t V^-, \partial_x V^-)(t, x) &= \left(g\left(\frac{x-y}{t}\right) - \frac{x-y}{t} g'\left(\frac{x-y}{t}\right), g'\left(\frac{x-y}{t}\right) \right) \\ &= \left(\rho \cdot \frac{x-y}{t} - f(\rho) - \frac{x-y}{t} \cdot \rho, \rho \right) = (-f(\rho), \rho)\end{aligned}$$

A Hamilton-Jacobi equation

If $V(t, x) = \bar{V}(y) + t' g\left(\frac{x-y}{t}\right)$, choose ρ so that $f'(\rho) = \frac{x-y}{t}$.

Then $(\partial_t V, \partial_x V) = (-f(\rho), \rho)$ lies in the sub-differential of V at (t, x) .

Since V is Lipschitz continuous,

$$V_t + f(V_x) = 0 \quad \text{for a.e. } (t, x)$$

Moreover $\rho = V_x$ yields a solution of conservation law

$$\rho_t + f(\rho)_x = 0$$

satisfying the Oleinik admissibility conditions: only upward jumps

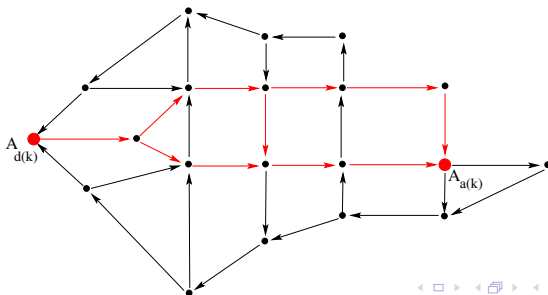
Traffic Flow on a Network of Roads

Vehicle density ρ is described by a scalar conservation law on each road

$$\rho_t + f_k(\rho)_x = 0$$

+ boundary conditions at road intersections

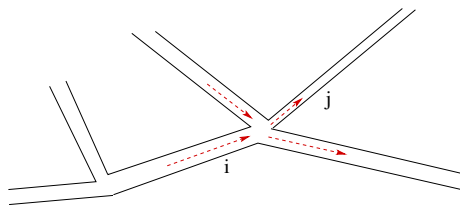
- Describe the evolution of traffic density
- Study optimization problems (for a central planner)
- Study Nash equilibrium problems



Boundary conditions at junctions

incoming roads: $i \in \mathcal{I}$

outgoing roads: $j \in \mathcal{O}$



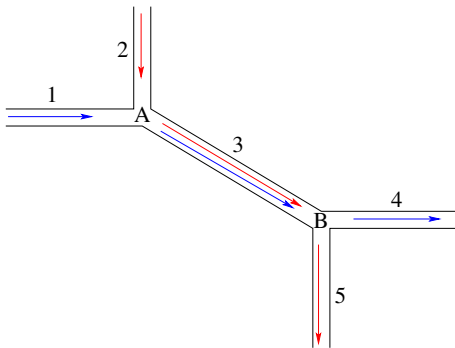
Boundary conditions account for:

- θ_{ij} = fraction of drivers from road i that turn into road j .
- c_i = relative priority of drivers from road i
(i.e., percentage of time drivers from road i get green light)

$$\sum_j \theta_{ij} = 1$$

$$\sum_i c_i = 1$$

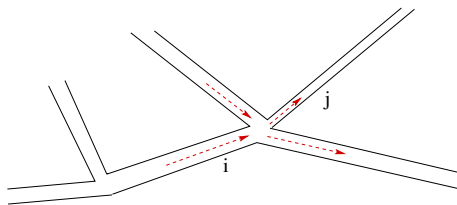
Drivers' turning preferences θ_{ij} must be determined as part of the solution



of vehicles on road i that wish to turn into road j is conserved:

$$(\rho\theta_{ij})_t + (\rho v_i(\rho)\theta_{ij})_x = 0$$

Traffic flow on a network of roads



On the i -th incoming road, car flow is described by

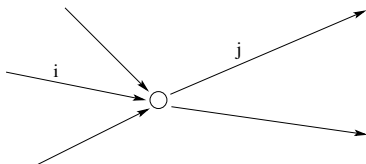
$$\begin{cases} \rho_t + f_i(\rho)_x = 0 & \text{conservation law} \\ \theta_{ij,t} + v_i(\rho)\theta_{ij,x} = 0 & \text{linear transport equation} \end{cases}$$

θ_{ij} are *passive scalars*, relevant only at intersection

Boundary conditions at junctions

incoming roads: $i \in \mathcal{I}$

outgoing roads: $j \in \mathcal{O}$



Boundary conditions should relate the pointwise limits

$$\begin{cases} \rho_i(t, 0-) \\ \rho_j(t, 0+) \end{cases} \quad \text{depending on } \theta_{ij}(t, 0-)$$

Conservation equations:

$$\sum_i f_i(\rho_i(t, 0-))\theta_{ij}(t, 0-) = f_j(\rho_j(t, 0+)) \quad j \in \mathcal{O}$$

- H.Holden, N.H.Risebro, A mathematical model of traffic flow on a network of unidirectional roads, *SIAM J. Math. Anal.* **26**, 1995.
- G.M.Coclite, M.Garavello, B.Piccoli, Traffic flow on a road network, *SIAM J. Math. Anal.* **36**, 2005.
- M.Herty, S.Moutari, M.Rasche, Optimization criteria for modeling intersections of vehicular traffic flow, *Netw. Heterog. Media* **1**, 2006.
- M.Garavello, B.Piccoli, Conservation laws on complex networks, *Ann.I.H.Poincaré* **26** 2009.
- M.Garavello, B.Piccoli, *Traffic Flow on Networks*, AIMS, 2006.

- Assign boundary conditions by describing how Riemann problems should be solved
- By taking limits of front tracking approximations, this uniquely determines how Cauchy problems should be solved

This approach works well if θ_{ij} are constants

What happens if $\theta_{ij} = \theta_{ij}(t, x)$??

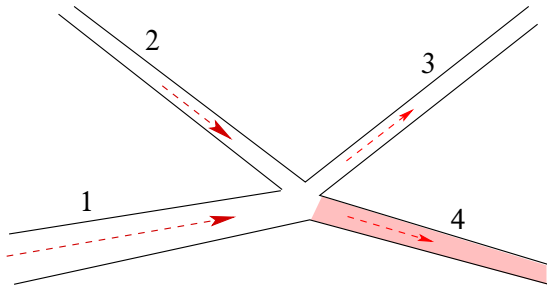
- Assign boundary conditions by describing how Riemann problems should be solved
- By taking limits of front tracking approximations, this uniquely determines how Cauchy problems should be solved

This approach works well if θ_{ij} are constants

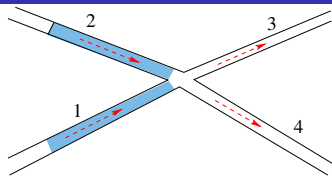
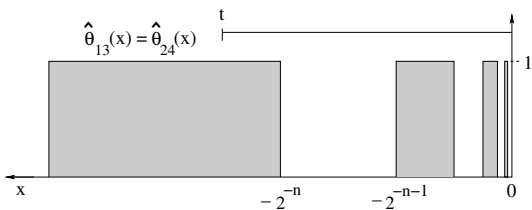
What happens if $\theta_{ij} = \theta_{ij}(t, x)$??

Modeling assumptions

- If all cars arriving at the intersection can immediately move to outgoing roads, no queue is formed.
- If outgoing roads are congested, the inflow of cars from road 1 is twice as large as the inflow from road 2.



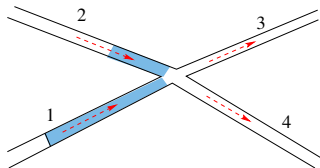
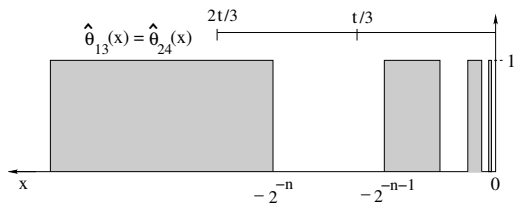
Example 1: θ_{ij} with unbounded variation, two solutions



$$f_k(\rho) = 2\rho - \rho^2 \quad \text{maximum flux: } f_k^{max} = 1$$

Initial data: $\hat{\rho}_k(x) = 1, \quad k = 1, 2, 3, 4$

$$\hat{\theta}_{13}(x) = \hat{\theta}_{24}(x) = \begin{cases} 1 & \text{if } -2^{-n} < x < -2^{-n-1}, \quad n \text{ even} \\ 0 & \text{if } -2^{-n} < x < -2^{-n-1}, \quad n \text{ odd} \end{cases}$$



Solution 1. Incoming fluxes: $f_1(t, 0) = 1$, $f_2(t, 0) = 1$

Solution 2. Incoming fluxes: $f_1(t, 0) = \frac{2}{3}$, $f_2(t, 0) = \frac{1}{3}$

Multiple solutions for a hyperbolic system

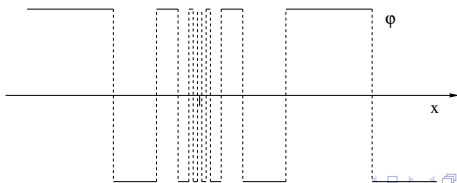
(A.B. & Wen Shen, *Nonlinear Analysis, T.M.A., 1998*)

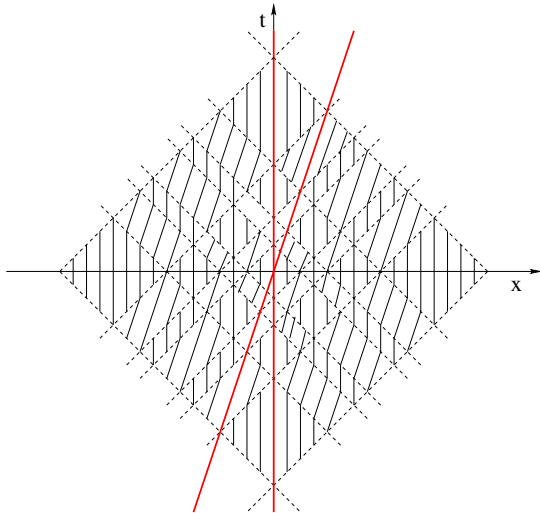
$$\left\{ \begin{array}{l} u_t + u_x = 0, \\ v_t - v_x = 0, \\ w_t + \left[\frac{1-uv}{6} \cdot w \right]_x = 0, \end{array} \right. \quad \left\{ \begin{array}{l} u(0, x) = \bar{u}(x), \\ v(0, x) = \bar{v}(x), \\ w(0, x) = \bar{w}(x). \end{array} \right.$$

strictly hyperbolic, linearly degenerate, with characteristic speeds

$-1 < \frac{1-uv}{6} < 1$. Initial data $\bar{u}, \bar{v}, \bar{w} \in BV \implies$ solution is unique.

$$\bar{u}(x) = \bar{v}(x) = \varphi(x) \doteq \begin{cases} 1 & \text{if } 2^{-2n-1} < |x| < 2^{-2n} \\ -1 & \text{if } 2^{-2n} < |x| < 2^{-2n+1} \end{cases}$$

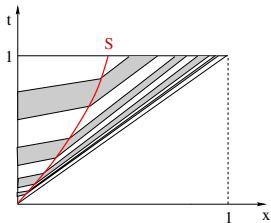
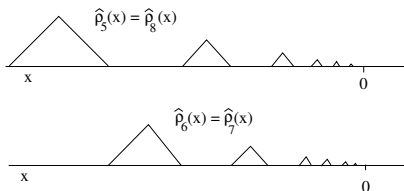
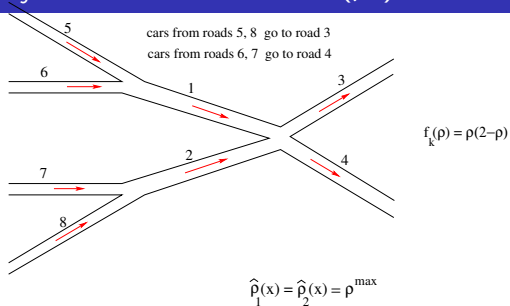




$\dot{x} = \frac{1-u(t,x)v(t,x)}{6}$ has infinitely many solutions through the origin

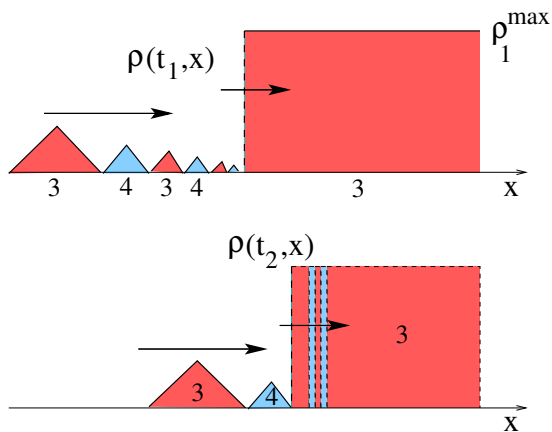
\implies Cauchy problem has infinitely many solutions

Example 2: θ_{ij} constant, $Tot.Var.(\rho_i)$ small, two solutions

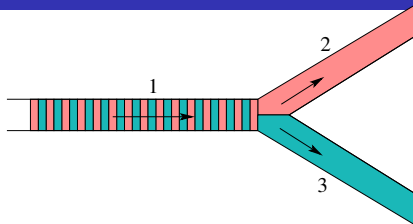


At some time $T > 0$, the same initial data as in Example 1 is created at the junction of roads 1 and 2

Instantaneous blow up of $Tot.Var.\{\theta_{13}\}$ on road 1



Example 3: lack of continuity w.r.t. weak convergence



$$f_1^{max} = 2, \quad f_2^{max} = f_3^{max} = 1.$$

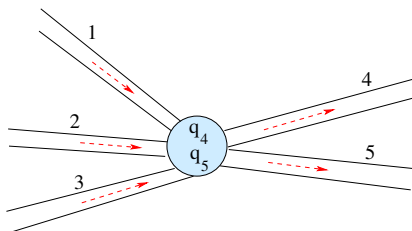
$$(\theta_{12}^n, \theta_{13}^n)(x) = \begin{cases} (1, 0) & \text{if } x \in](k-1)/n, k/n], \quad k \text{ even} \\ (0, 1) & \text{if } x \in](k-1)/n, k/n], \quad k \text{ odd.} \end{cases}$$

Flux exiting road 1: $f_1^{(n)}(t, 0) \equiv 1$

As $n \rightarrow \infty$, the weak limit is $\theta_{12} = \theta_{13} = \frac{1}{2}$

Flux out of road 1: $f_1(t, 0) = 2 \neq 1 = \lim_{n \rightarrow \infty} f_1^{(n)}(t, 0)$

An intersection model with buffers



- the intersection contains a buffer with finite size
- $t \mapsto q_j(t)$ = queues in front of outgoing roads $j \in \mathcal{O}$, within the buffer
- incoming drivers are admitted to the intersection at a rate depending on the size of these queues
- drivers already inside the intersection flow out to the road of their choice at the largest possible rate

- M. Herty, J. P. Lebacque, and S. Moutari, A novel model for intersections of vehicular traffic flow. *Netw. Heterog. Media* 2009.
- M. Garavello and P. Goatin, The Cauchy problem at a node with buffer. *Discrete Contin. Dyn. Syst.* 2012.
- M. Garavello and B. Piccoli, A multibuffer model for LWR road networks, in *Advances in Dynamic Network Modeling in Complex Transportation Systems*, 2013.

Toward the analysis of global optima and Nash equilibria, we need

- well posedness for L^∞ initial data $\hat{\rho}_k, \hat{\theta}_{ij}$
- continuity w.r.t. weak convergence

$$\begin{cases} \rho_{k,t} + f_k(\rho_k)_x = 0 & \text{conservation laws} \\ \theta_{ij,t} + v_i(\rho_i)\theta_{ij,x} = 0 & \text{linear transport equations} \end{cases}$$

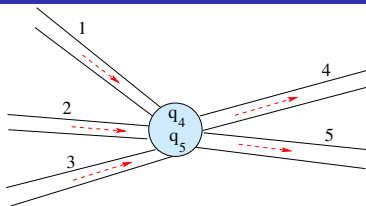
- M. Herty, J. P. Lebacque, and S. Moutari, A novel model for intersections of vehicular traffic flow. *Netw. Heterog. Media* 2009.
- M. Garavello and P. Goatin, The Cauchy problem at a node with buffer. *Discrete Contin. Dyn. Syst.* 2012.
- M. Garavello and B. Piccoli, A multibuffer model for LWR road networks, in *Advances in Dynamic Network Modeling in Complex Transportation Systems, 2013*.

Toward the analysis of global optima and Nash equilibria, we need

- well posedness for \mathbf{L}^∞ initial data $\hat{\rho}_k, \hat{\theta}_{ij}$
- continuity w.r.t. weak convergence

$$\left\{ \begin{array}{ll} \rho_{k,t} + f_k(\rho_k)_x = 0 & \text{conservation laws} \\ \theta_{ij,t} + v_i(\rho_i)\theta_{ij,x} = 0 & \text{linear transport equations} \end{array} \right.$$

The basic model



Incoming roads: $i \in \mathcal{I}$

Outgoing roads: $j \in \mathcal{O}$

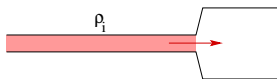
Boundary values at the junction:

$$\left\{ \begin{array}{ll} \bar{\theta}_{ij}(t) \doteq \lim_{x \rightarrow 0^-} \theta_{ij}(t, x), & i \in \mathcal{I}, j \in \mathcal{O} \\ \bar{\rho}_i(t) \doteq \lim_{x \rightarrow 0^-} \rho_i(t, x), & i \in \mathcal{I} \\ \bar{\rho}_j(t) \doteq \lim_{x \rightarrow 0^+} \rho_j(t, x), & j \in \mathcal{O} \\ \bar{f}_i(t) \doteq f_i(\bar{\rho}_i(t)) = \lim_{x \rightarrow 0^-} f_i(\rho_i(t, x)), & i \in \mathcal{I} \\ \bar{f}_j(t) \doteq f_j(\bar{\rho}_j(t)) = \lim_{x \rightarrow 0^+} f_j(\rho_j(t, x)), & j \in \mathcal{O} \end{array} \right.$$

$q_j(t) = \#$ of cars within the intersection, waiting to access road $j \in \mathcal{O}$.

Conservation of the total number of cars implies

$$\dot{q}_j = \frac{d}{dt}q_j(t) = \sum_{i \in \mathcal{I}} \bar{f}_i \bar{\theta}_{ij} - \bar{f}_j \quad j \in \mathcal{O}$$



$$\omega_i = \omega_i(\bar{\rho}_i) \doteq \begin{cases} f_i(\bar{\rho}_i) & \text{if } \bar{\rho}_i \text{ is a free state} \\ f_i^{\max} & \text{if } \bar{\rho}_i \text{ is a congested state} \end{cases} \quad i \in \mathcal{I},$$

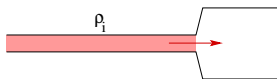
= maximum possible flux at the end of an incoming road

ω_i is the largest flux $f_i(\rho)$ among all states ρ that can be connected to $\bar{\rho}_i$ with a wave of negative speed

$q_j(t) = \#$ of cars within the intersection, waiting to access road $j \in \mathcal{O}$.

Conservation of the total number of cars implies

$$\dot{q}_j = \frac{d}{dt}q_j(t) = \sum_{i \in \mathcal{I}} \bar{f}_i \bar{\theta}_{ij} - \bar{f}_j \quad j \in \mathcal{O}$$



$$\omega_i = \omega_i(\bar{\rho}_i) \doteq \begin{cases} f_i(\bar{\rho}_i) & \text{if } \bar{\rho}_i \text{ is a free state} \\ f_i^{max} & \text{if } \bar{\rho}_i \text{ is a congested state} \end{cases} \quad i \in \mathcal{I},$$

= maximum possible flux at the end of an incoming road

ω_i is the largest flux $f_i(\rho)$ among all states ρ that can be connected to $\bar{\rho}_i$ with a wave of negative speed

(SBJ) - Single Buffer Junction

$M > 0$ = maximum number of cars that can occupy the intersection at any given time

$c_i > 0$, $i \in \mathcal{I}$, priorities given to different incoming roads

(SBJ) Incoming fluxes \bar{f}_i satisfy

$$\bar{f}_i = \min \left\{ \omega_i, c_i \left(M - \sum_{j \in \mathcal{O}} q_j \right) \right\}, \quad i \in \mathcal{I}$$

Outgoing fluxes \bar{f}_j satisfy

$$\begin{cases} \text{if } q_j > 0, \text{ then } \bar{f}_j = \omega_j \\ \text{if } q_j = 0, \text{ then } \bar{f}_j = \min \left\{ \omega_j, \sum_{i \in \mathcal{I}} \bar{f}_i \bar{\theta}_{ij} \right\} \end{cases} \quad j \in \mathcal{O}$$

(MBJ) - Multiple Buffer Junction

$M_j > 0$ = maximum number of cars that can occupy the intersection, waiting to enter road j

$c_i > 0$, $i \in \mathcal{I}$, priorities given to different incoming roads

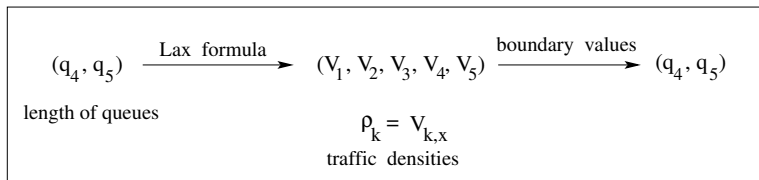
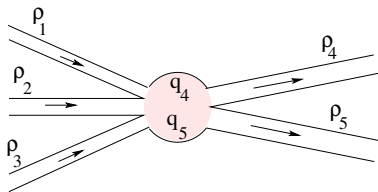
(MBJ) Incoming fluxes \bar{f}_i satisfy

$$\bar{f}_i = \min \left\{ \omega_i, \frac{c_i(M_j - q_j)}{\theta_{ij}}, j \in \mathcal{O} \right\}, \quad i \in \mathcal{I}$$

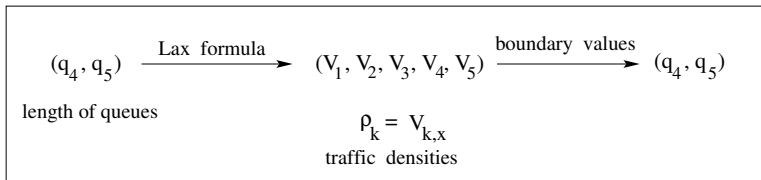
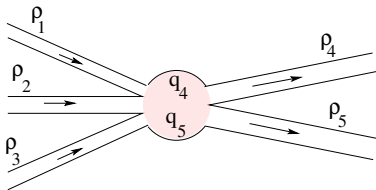
Outgoing fluxes \bar{f}_j satisfy

$$\begin{cases} \text{if } q_j > 0, \text{ then } \bar{f}_j = \omega_j, \\ \text{if } q_j = 0, \text{ then } \bar{f}_j = \min \left\{ \omega_j, \sum_{i \in \mathcal{I}} \bar{f}_i \bar{\theta}_{ij} \right\} \end{cases} \quad j \in \mathcal{O}$$

Variational formulation (A.B. - Khai Nguyen, 2014)



- If the queue sizes $q_j(t)$ within the buffer are known, then the initial-boundary value problems can be independently solved along each incoming road. These solutions can be computed by solving suitable variational problems. From the value functions V_k , the traffic density $\rho_k = V_{k,x}$ along each incoming or outgoing road is recovered by a Lax type formula.



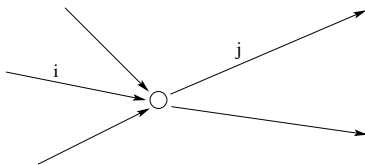
- Conversely, if these value functions V_k are known, then the queue sizes q_j can be determined by balancing the boundary fluxes of all incoming and outgoing roads
- The solution of the Cauchy problem is obtained as the unique fixed point of a contractive transformation
- The present model accounts for backward propagation of queues along roads leading to a crowded intersection, it achieves well-posedness for general L^∞ data, and continuity w.r.t. weak convergence

- P. Le Floch, Explicit formula for scalar non-linear conservation laws with boundary condition. *Math. Methods Appl. Sciences* (1988)

$$\rho_t + f(\rho)_x = 0 \quad \rho(0, x) = a(x), \quad \rho(t, 0) = b(t)$$

A variational inequality determines if the boundary value can be pointwise attained or not.

- C. Imbert, R. Monneau, and H. Zidani, A Hamilton-Jacobi approach to junction problems and application to traffic flows. *ESAIM Control Optim. Calc. Var.* (2013).



A single optimization problem determines the solution simultaneously on all roads

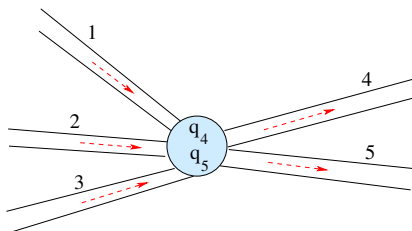
Traffic flow on networks - II

Alberto Bressan

Department of Mathematics, Penn State University

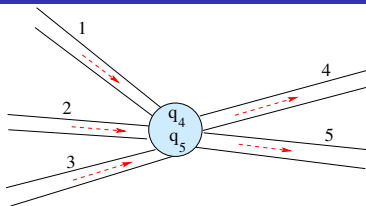
bressan@math.psu.edu

An intersection model with buffers



- the intersection contains a buffer with finite size
- $t \mapsto q_j(t)$ = queues in front of outgoing roads $j \in \mathcal{O}$, within the buffer
- incoming drivers are admitted to the intersection at a rate depending on the size of these queues
- drivers already inside the intersection flow out to the road of their choice at the largest possible rate

The basic model



Incoming roads: $i \in \mathcal{I}$

Outgoing roads: $j \in \mathcal{O}$

Boundary values at the junction:

$$\left\{ \begin{array}{ll} \bar{\theta}_{ij}(t) \doteq \lim_{x \rightarrow 0^-} \theta_{ij}(t, x), & i \in \mathcal{I}, j \in \mathcal{O} \\ \bar{\rho}_i(t) \doteq \lim_{x \rightarrow 0^-} \rho_i(t, x), & i \in \mathcal{I} \\ \bar{\rho}_j(t) \doteq \lim_{x \rightarrow 0^+} \rho_j(t, x), & j \in \mathcal{O} \\ \bar{f}_i(t) \doteq f_i(\bar{\rho}_i(t)) = \lim_{x \rightarrow 0^-} f_i(\rho_i(t, x)), & i \in \mathcal{I} \\ \bar{f}_j(t) \doteq f_j(\bar{\rho}_j(t)) = \lim_{x \rightarrow 0^+} f_j(\rho_j(t, x)), & j \in \mathcal{O} \end{array} \right.$$

Size of queues at intersection

$q_j(t) =$ # of cars within the intersection, waiting to access road $j \in \mathcal{O}$.

Conservation of the total number of cars implies

$$\dot{q}_j = \frac{d}{dt}q_j(t) = \sum_{i \in \mathcal{I}} \bar{f}_i \bar{\theta}_{ij} - \bar{f}_j \quad j \in \mathcal{O}$$

Upper bounds on the exiting or entering fluxes

$$\omega_i(\bar{\rho}_i) \doteq \begin{cases} f_i(\bar{\rho}_i) & \text{if } \bar{\rho}_i \text{ is a free state,} \\ f_i^{max} & \text{if } \bar{\rho}_i \text{ is a congested state,} \end{cases} \quad i \in \mathcal{I},$$

= maximum possible flux at the exit of an incoming road.

ω_i is the largest flux $f_i(\rho)$ among all states ρ that can be connected to $\bar{\rho}_i$ with a wave of negative speed



$$\omega_j(\bar{\rho}_j) \doteq \begin{cases} f_j^{max} & \text{if } \bar{\rho}_j \text{ is a free state,} \\ f_j(\bar{\rho}_j) & \text{if } \bar{\rho}_j \text{ is a congested state,} \end{cases} \quad j \in \mathcal{O},$$

= maximum possible flux at the entrance of an outgoing road.

(SBJ) - Single Buffer Junction

$M > 0$ = maximum number of cars that can occupy the intersection

$c_i > 0$, $i \in \mathcal{I}$, priorities given to different incoming roads

(SBJ) Incoming fluxes \bar{f}_i satisfy

$$\bar{f}_i = \min \left\{ \omega_i(\bar{\rho}_i), c_i \left(M - \sum_{j \in \mathcal{O}} q_j \right) \right\}, \quad i \in \mathcal{I}$$

Outgoing fluxes \bar{f}_j satisfy

$$\begin{cases} \text{if } q_j > 0, \text{ then } \bar{f}_j = \omega_j(\bar{\rho}_j), \\ \text{if } q_j = 0, \text{ then } \bar{f}_j = \min \left\{ \omega_j(\bar{\rho}_j), \sum_{i \in \mathcal{I}} \bar{f}_i \bar{\theta}_{ij} \right\} \end{cases} \quad j \in \mathcal{O}$$

(MBJ) - Multiple Buffer Junction

$M_j > 0$ = maximum number of cars that can occupy the intersection, waiting to enter road j

$c_i > 0$, $i \in \mathcal{I}$, priorities given to different incoming roads

(MBJ) Incoming fluxes \bar{f}_i satisfy

$$\bar{f}_i = \min \left\{ \omega_i(\bar{\rho}_i), \frac{c_i(M_j - q_j)}{\theta_{ij}}, j \in \mathcal{O} \right\}, \quad i \in \mathcal{I}$$

Outgoing fluxes \bar{f}_j satisfy

$$\begin{cases} \text{if } q_j > 0, \text{ then } \bar{f}_j = \omega_j(\bar{\rho}_j), \\ \text{if } q_j = 0, \text{ then } \bar{f}_j = \min \left\{ \omega_j(\bar{\rho}_j), \sum_{i \in \mathcal{I}} \bar{f}_i \bar{\theta}_{ij} \right\} \end{cases} \quad j \in \mathcal{O}$$

The Cauchy problem

$$\left\{ \begin{array}{l} \rho_{k,t} + f_k(\rho_k)_x = 0 \\ \theta_{ij,t} + v_i(\rho_i)\theta_{ij,x} = 0 \\ \dot{q}_j = \sum_{i \in \mathcal{I}} \bar{f}_i \bar{\theta}_{ij} - \bar{f}_j \end{array} \right. \quad \left\{ \begin{array}{l} \rho_k(0, x) = \hat{\rho}_k(x) \\ \theta_{ij}(0, x) = \hat{\theta}_{ij}(x) \\ q_j(0) = \hat{q}_j \end{array} \right. \quad \begin{array}{l} k \in \mathcal{I} \cup \mathcal{O} \\ i \in \mathcal{I}, j \in \mathcal{O} \\ j \in \mathcal{O} \end{array}$$

+ boundary conditions at the intersection

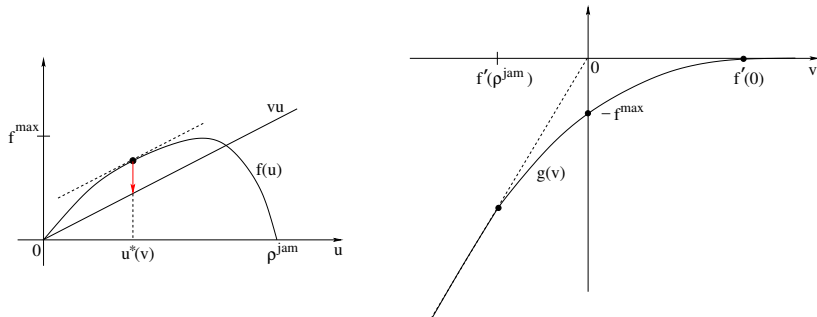
Construction of solutions

Assume: size of queues $q_j(t)$ is known

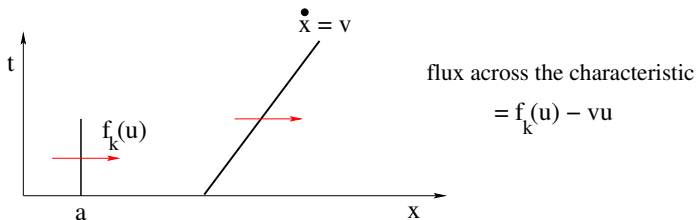
- Introduce an optimization problem whose value function V_i yields the density of cars on each incoming road: $\rho_i = V_{i,x}$, for model **(SBJ)**
- Same for model **(MBJ)**
- Introduce a variational problem whose value function V_j yields the density of cars on each outgoing road: $\rho_j = V_{j,x}$.
- Compute an updated value for the queues: $\tilde{q}_j = \Lambda_j(q)$
- Construct the solution by finding a fixed point of Λ

The Legendre transform of f_k

Legendre transform:
$$g_k(v) \doteq \inf_{u \in [0, \rho_k^{\text{jam}}]} \{uv - f_k(u)\}, \quad k \in \mathcal{I} \cup \mathcal{O}$$



A variational problem on each road



$$g_k(v) = - [\text{flux of cars from left to right, across the characteristic}]$$

For boundary conditions **(SBJ)**, define the **maximum permissible flux at the end of road i**

$$h_i(\mathbf{q}) \doteq \min \left\{ f_i^{\max}, c_i \cdot \left(M - \sum_{j \in \mathcal{O}} q_j \right) \right\} \quad i \in \mathcal{I}$$

Optimization Problem 1 - incoming roads with boundary condition (SBJ)

For any $i \in \mathcal{I}$, given a terminal point (\bar{t}, \bar{x}) , and boundary data

$$\bar{V}_i(x) \doteq \int_{-\infty}^x \hat{\rho}_i(y) dy \quad q_j(t), \quad j \in \mathcal{O}$$

consider the optimization problem:

$$\text{maximize:} \quad \bar{V}_i(x(0)) + \int_0^{\bar{t}} L_i(x(t), \dot{x}(t)) dt$$

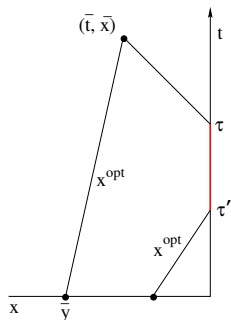
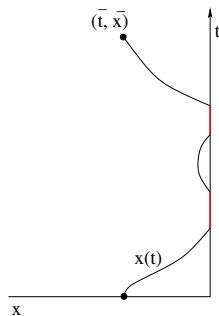
$$\text{payoff:} \quad L_i(x(t), \dot{x}(t)) = \begin{cases} g_i(\dot{x}(t)) & \text{if } x(t) < 0 \\ -h_i(\mathbf{q}(t)) & \text{if } x(t) = 0 \end{cases}$$

Optimization Problem 1

$$\text{maximize: } \bar{V}_i(x(0)) + \int_0^{\bar{t}} L_i(x(t), \dot{x}(t)) dt$$

among all absolutely continuous functions $x : [0, \bar{t}] \mapsto \mathbb{R}$ such that

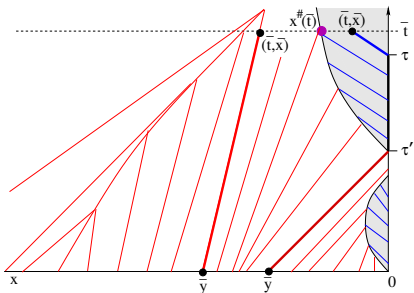
$$x(\bar{t}) = \bar{x}, \quad x(t) \leq 0 \quad \text{for all } t \in [0, \bar{t}]$$



The Value Function V_i

- an optimal solution x^{opt} exists, and is the concatenation of at most three affine functions
- $\dot{x}^{opt} \in [f'_i(0), f'_i(\rho_i^{jam})]$
- the traffic density $\rho_i(t, x) = V_{i,x}(t, x)$ is an entropy solution of the conservation law, satisfying initial + boundary conditions
- Lax formula: for a.e. (\bar{t}, \bar{x}) , the optimal solution is unique and

$$f'_i(\rho_k(\bar{t}, \bar{x})) = \dot{x}^{opt}(\bar{t}) = \begin{cases} \frac{\bar{x} - \bar{y}}{\bar{t}} \\ \bar{x} \\ \frac{\bar{x}}{\bar{t} - \tau} \end{cases} \quad \text{or}$$



$$V_i(t, x) \doteq \max \left\{ \max_{y \leq 0} \left[\bar{V}_i(y) + t g_i \left(\frac{x-y}{t} \right) \right], \right. \\ \left. \max_{0 \leq \tau' \leq \tau \leq t, y \leq 0} \left[\bar{V}_i(y) + \tau' g_i \left(\frac{-y}{\tau'} \right) - \int_{\tau'}^{\tau} h_i(\mathbf{q}(s)) ds + (t - \tau) g_i \left(\frac{x}{t - \tau} \right) \right] \right\}. \quad \text{(LF1)}$$

$V_i(t, x)$ = total amount of cars which at time t are still inside the half line $] - \infty, x]$

$\bar{V}_i(0) - V_i(t, 0)$ = total amount of cars which have exited from road i during the time interval $[0, t]$

Properties of the Legendre transform

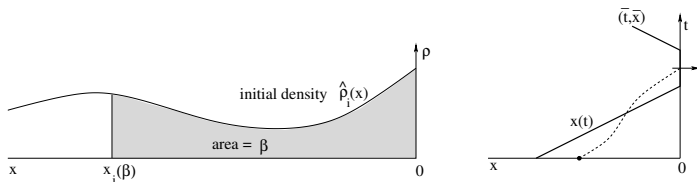
$$f'(\rho) = v \iff g'(v) = \rho$$

In this case, $g(v) = \rho v - f(\rho)$

$$\begin{aligned} 0 &= \frac{d}{d\tau} \left[- \int_{\tau'}^{\tau} h_i(\mathbf{q}(s)) ds + (t - \tau) g_i \left(\frac{x}{t - \tau} \right) \right] \\ &= - h_i(\mathbf{q}(\tau)) - g_i \left(\frac{x}{t - \tau} \right) + \frac{x}{t - \tau} g_i' \left(\frac{x}{t - \tau} \right) \\ &= - h_i(\mathbf{q}(\tau)) - \left[\rho \frac{x}{t - \tau} - f_i(\rho) \right] + \frac{x}{t - \tau} \rho \\ &= - h_i(\mathbf{q}(\tau)) + f_i(\rho) \end{aligned}$$

Optimization Problem 2 - incoming roads with boundary condition (MBJ)

$$h_i(\mathbf{q}, \theta) \doteq \min \left\{ f_i^{max}, c_i \cdot \frac{M_j - q_j}{\theta_{ij}}; j \in \mathcal{O} \right\} \quad (\text{boundary flux})$$



Define $\beta \mapsto x_i(\beta)$ implicitly by $\int_{x(\beta)}^0 \hat{\rho}_i(x) dx = \beta$, then set $\theta_{ij}(\beta) \doteq \hat{\theta}_{ij}(x_i(\beta))$

Let $t \mapsto \beta(t)$ be the solution to the ODE

$$\dot{\beta}(t) = \begin{cases} g_i(\dot{x}(t)) & \text{if } x(t) < 0 \\ h_i(\mathbf{q}(t), \theta(\beta(t))) & \text{if } x(t) = 0 \end{cases} \quad \text{with } \beta(0) = \bar{V}_i(x(0))$$

Optimization Problem 2

For any $i \in \mathcal{I}$, given a terminal point (\bar{t}, \bar{x}) , and initial-boundary data

$$\bar{V}_i(x) \doteq \int_{-\infty}^x \hat{\rho}_i(y) dy \quad q_j(t), \quad j \in \mathcal{O}$$

consider the optimization problem

$$\text{maximize:} \quad \bar{V}_i(x(0)) + \int_0^{\bar{t}} L_i(x(t), \dot{x}(t)) dt$$

$$\text{payoff:} \quad L_i(x(t), \dot{x}(t)) \doteq \begin{cases} g_i(\dot{x}(t)) & \text{if } x(t) < 0 \\ -h_i(\mathbf{q}(t), \theta(\beta(t))) & \text{if } x(t) = 0 \end{cases}$$

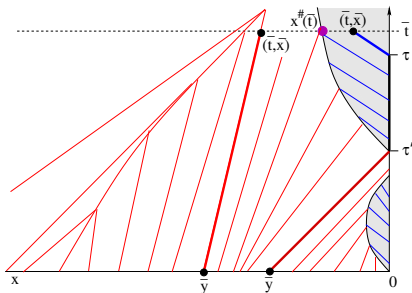
among all absolutely continuous functions $x : [0, \bar{t}] \mapsto \mathbb{R}$ such that

$$x(\bar{t}) = \bar{x}, \quad x(t) \leq 0 \quad \text{for all } t \in [0, \bar{t}]$$

The Value Function V_i

- an optimal solution x^{opt} exists, and is the concatenation of at most three affine functions
- $\dot{x}^{opt} \in [f'_i(0), f'_i(\rho_i^{jam})]$
- the traffic density $\rho_i(t, x) = V_{i,x}(t, x)$ is an entropy solution of the conservation law, satisfying initial + boundary conditions
- Lax formula: for a.e. (\bar{t}, \bar{x}) , the optimal solution is unique and

$$f'_i(\rho_i(\bar{t}, \bar{x})) = \dot{x}^{opt}(\bar{t}) = \begin{cases} \frac{\bar{x} - \bar{y}}{\bar{t}} \\ \bar{x} \\ \frac{\bar{x}}{\bar{t} - \tau} \end{cases} \quad \text{or}$$

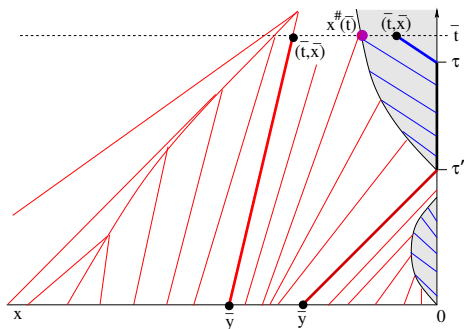


$$V_i(t, x) \doteq \max \left\{ \max_{y \leq 0} \left[\widehat{V}_i(y) + t g_i \left(\frac{x-y}{t} \right) \right], \right. \\ \left. \max_{0 \leq \tau' \leq \tau \leq t, y \leq 0} \left[G_i(\tau, \tau'; y) + (t - \tau) g_i \left(\frac{x}{t - \tau} \right) \right] \right\}, \quad (\text{LF2})$$

$G_i(\tau, \tau'; y) \doteq -\beta(\tau)$, where

$$\frac{d}{ds} \beta(s) = h_i(\mathbf{q}(s), \theta_{ij}(\beta(s))), \quad s \in [\tau', \tau], \quad \beta(\tau') = \widehat{V}_i(y) + \tau' g_i \left(\frac{-y}{\tau'} \right)$$

Motivation



Choosing ρ such that $f'_i(\rho) = \frac{x}{t - \tau}$, the optimality condition

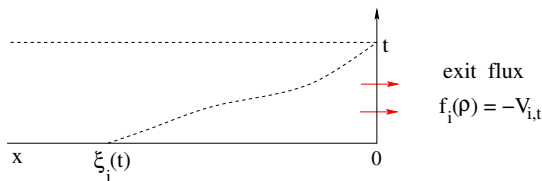
$$\frac{d}{d\tau} \left[G_i(\tau, \tau', y) + (t - \tau) g_i \left(\frac{x}{t - \tau} \right) \right] = 0$$

implies $f_i(\rho) = h_i(\mathbf{q}(\tau), \theta_{ij}(\beta(\tau)))$

Optimization Problem 3 - outgoing roads

The initial position of the car on road i that reaches the intersection at time t is

$$\xi_i(t) \doteq \max \left\{ z \in]-\infty, 0]; \int_z^0 \hat{\rho}_i(y) dy = \bar{V}_i(0) - V_i(t, 0) \right\}$$



The total number of drivers that have reached the intersection before time t and wanted to turn into road j is

$$F_j(t) = \hat{q}_j + \sum_{i \in \mathcal{I}} \int_{\xi_i(t)}^0 \hat{\rho}_i(y) \hat{\theta}_{ij}(y) dy$$

Optimization Problem 3

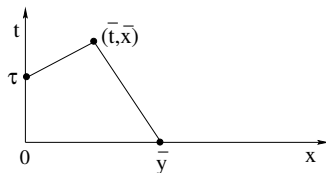
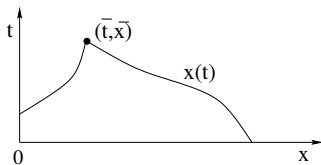
For any $j \in \mathcal{O}$, given a terminal point (\bar{t}, \bar{x}) , and initial-boundary data

$$\bar{V}_j(x) \doteq \int_0^x \hat{\rho}_j(y) dy, \quad F_j(t)$$

consider the problem of maximizing the functional

$$\max \left\{ \bar{V}_j(x(0)) + \int_0^{\bar{t}} g_j(\dot{x}(t)) dt, \quad \max_{\tau \geq 0, x(\tau)=0} \left(-F_j(\tau) + \int_{\tau}^{\bar{t}} g_j(\dot{x}(t)) dt \right) \right\}$$

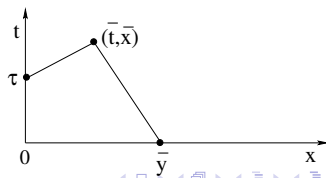
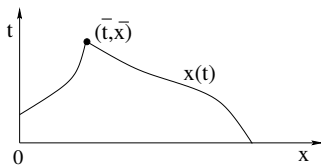
subject to $x(\bar{t}) = \bar{x}$, $x(t) \geq 0$ for all $t \in [0, \bar{t}]$

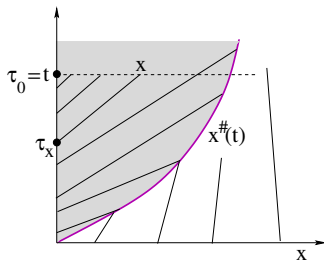
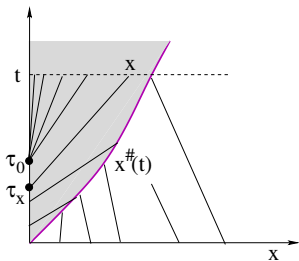
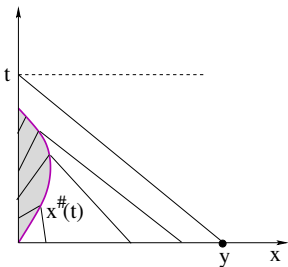


The Value Function V_j

- an optimal solution x^{opt} exists, and is affine
- $\dot{x}^{opt} \in [f'_j(0), f'_j(\rho_j^{jam})]$
- the traffic density $\rho_j(t, x) = V_{j,x}(t, x)$ is an entropy solution of the conservation law, satisfying initial + boundary conditions
- Lax formula: for a.e. (\bar{t}, \bar{x}) , the optimal solution is unique and

$$f'_j(\rho_j(\bar{t}, \bar{x})) = \dot{x}^{opt}(\bar{t}) = \begin{cases} \frac{\bar{x} - \bar{y}}{\bar{t}} \\ \frac{\bar{x}}{\bar{t} - \tau} \end{cases} \quad \text{or}$$





$$V_j(t, x) \doteq \max \left\{ \max_{y \geq 0} \left[\widehat{V}_j(y) + t g_j \left(\frac{x-y}{t} \right) \right], \max_{0 \leq \tau \leq t} \left[-F_j(\tau) + (t-\tau) g_j \left(\frac{x}{t-\tau} \right) \right] \right\} \quad (\text{LF3})$$

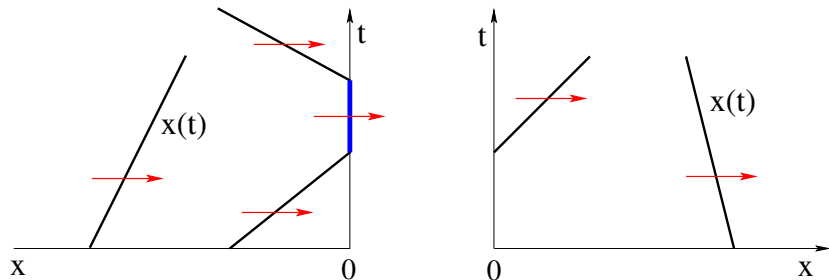
$\widehat{V}_j(x) - V_j(t, x) = \#$ of cars that have crossed the point x during the time interval $[0, t]$.

$-V_j(t, 0) = \#$ of cars that have entered road j (possibly after waiting in a queue) during the time interval $[0, t]$.

Remark on the payoff functionals

In all cases, for optimal trajectories the integrand functional is

$L(x, \dot{x}) = -$ [flux of cars from left to right, across the line $x(t)$]



$$f'_k(\rho) = \dot{x} \quad \Longrightarrow \quad g_k(\dot{x}) = - \left(f_k(\rho) - \dot{x} \rho \right)$$

Length of the queues

\hat{q}_j = initial size of the queue in front of road j

$F_j(t)$ = total number of cars that have reached the intersection before time t and wish to turn into road j

$-V_j(t, 0)$ = total number of cars that have entered road j during the interval $[0, t]$

Size of the queue at the entrance of road j

$$q_j(t) = F_j(t) + V_j(t, 0) \quad (\mathbf{Q})$$

The Cauchy problem near an intersection

$$\left\{ \begin{array}{l} \rho_{k,t} + f_k(\rho_k)_x = 0 \\ \theta_{ij,t} + v_i(\rho_i)\theta_{ij,x} = 0 \\ \dot{q}_j = \sum_{i \in \mathcal{I}} \bar{f}_i \bar{\theta}_{ij} - \bar{f}_j \end{array} \right. \quad \left\{ \begin{array}{l} \rho_k(0, x) = \hat{\rho}_k(x) \\ \theta_{ij}(0, x) = \hat{\theta}_{ij}(x) \\ q_j(0) = \hat{q}_j \end{array} \right. \quad \begin{array}{l} k \in \mathcal{I} \cup \mathcal{O} \\ i \in \mathcal{I}, j \in \mathcal{O} \\ j \in \mathcal{O} \end{array} \quad \text{(CP)}$$

+ boundary conditions at the intersection

Admissible solutions to the Cauchy problem near an intersection

Definition. We say that the functions $\rho_k = \rho_k(t, x)$ and $q_j = q_j(t)$ provide an **admissible solution** to the Cauchy problem **(CP)** with junction conditions **(SBJ)** if there exist Lipschitz continuous functions $V_k = V_k(t, x)$ such that

- (i) $\rho_k = V_{k,x}$
- (ii) For $i \in \mathcal{I}$, the functions V_i satisfy **(LF1)**
- (iii) For $j \in \mathcal{O}$, the functions V_j satisfy **(LF3)**
- (iv) For $j \in \mathcal{O}$, the functions q_j satisfy **(Q)**

In case of the junction conditions **(MBJ)**, instead of **(LF1)** the functions V_i are required to satisfy **(LF2)**

Well posedness of the Cauchy problem with L^∞ data

Theorem (A.B. & Khai Nguyen, 2014).

Consider any measurable initial data $\hat{\rho}_k(x) \in [0, \rho_k^{jam}]$, $\hat{\theta}_{ij}(x) \in [0, 1]$, and \hat{q}_j , with

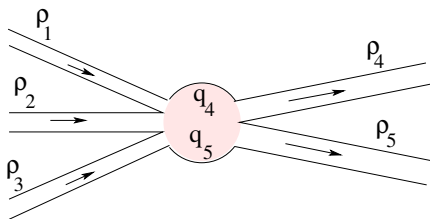
$$\sum_{j \in \mathcal{O}} \hat{q}_j < M \quad \text{or} \quad \hat{q}_j < M_j \text{ for all } j \in \mathcal{O}$$

respectively in the cases **(SBJ)** or **(MBJ)**.

Then the Cauchy problem **(CP)** has a unique admissible solution, globally defined for all $t \geq 0$.

One then checks that the functions $\rho_k = V_{k,x}$ are weak solutions to the conservation laws, satisfy the usual (Lax, Oleinik, ...) entropy admissibility conditions, together with the required boundary conditions **(SBJ)** or **(MBJ)**.

Sketch of the proof



$$(q_4, q_5) \xrightarrow{\text{Lax formula}} (V_1, V_2, V_3, V_4, V_5) \xrightarrow{\text{boundary values}} (q_4, q_5)$$

length of queues

$$\rho_k = V_{k,x}$$

traffic densities

find a fixed point !

- $q_j(t)$ = size of queue in front of j -th road
- $V_i(t, x)$ = value function on i -th incoming road
- $F_j(t)$ = # of cars that have entered the intersection up to time t and wish to go into road j
- $V_j(t, x)$ = value function on j -th outgoing road
- $\Lambda_j(\mathbf{q})(t) = F_j(t) + V_j(t, 0)$ = updated size of the queue

The sequence of maps

$$\mathbf{q} = (q_j)_{j \in \mathcal{O}} \mapsto (V_i)_{i \in \mathcal{I}} \mapsto (F_j)_{j \in \mathcal{O}} \mapsto (V_j)_{j \in \mathcal{O}} \mapsto (\Lambda_j(\mathbf{q}))_{j \in \mathcal{O}}$$

defines a strict contraction, on a suitably small time interval $[0, T]$.
Hence it has a unique fixed point, which provides the required solution.

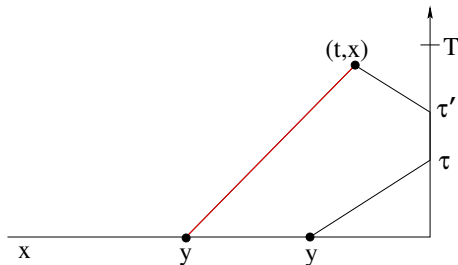
The argument can then be iterated on $[0, T_1]$, $[T_1, T_2]$, \dots ,
with $T_\nu \rightarrow \infty$.

Contraction property

Assume: $\sup_{t \in [0, T], j \in \mathcal{O}} |q_j(t) - \tilde{q}_j(t)| = \delta$

$$V_i(t, x) \doteq \max \left\{ \max_{y \leq 0} \left[\bar{V}_i(y) + t g_i \left(\frac{x-y}{t} \right) \right], \right. \\ \left. \max_{0 \leq \tau' \leq \tau \leq t, y \leq 0} \left[\bar{V}_i(y) + \tau' g_i \left(\frac{-y}{\tau'} \right) - \int_{\tau'}^{\tau} h_i(\mathbf{q}(s)) ds + (t - \tau) g_i \left(\frac{x}{t - \tau} \right) \right] \right\}. \quad (\text{LF1})$$

$$|V_i(t, x) - \tilde{V}_i(t, x)| \leq C T \delta.$$



Continuity w.r.t. weak convergence

Theorem (A.B. & Khai Nguyen, 2014).

Consider a sequence of initial data $(\hat{\rho}_k^\nu, \hat{\theta}_{ij}^\nu, \hat{q}_j^\nu)_{\nu \geq 1}$ such that, as $\nu \rightarrow \infty$,

$$\hat{q}_j^\nu \rightarrow \hat{q}_j, \quad j \in \mathcal{O},$$

together with the weak convergence

$$\begin{cases} \hat{\rho}_i^\nu \hat{\theta}_{ij}^\nu \rightharpoonup \hat{\rho}_i \hat{\theta}_{ij}, & i \in \mathcal{I}, j \in \mathcal{O}, \\ \hat{\rho}_j^\nu \rightharpoonup \hat{\rho}_j, & j \in \mathcal{O}. \end{cases}$$

Calling $\rho_k^\nu = V_{k,x}^\nu$ and q_j^ν the corresponding solutions, for every $t > 0$ one has

- uniform convergence: $q_j^\nu(t) \rightarrow q_j(t)$ for t on bounded sets,
- strong convergence in \mathbf{L}_{loc}^1 : $\rho_k^\nu(t, \cdot) \rightarrow \rho_k(t, \cdot) \quad k \in \mathcal{I} \cup \mathcal{O}$

The result holds both in the case **(SBJ)** of a single buffer and in the case **(MBJ)** of multiple buffers.

Traffic flow on networks - III

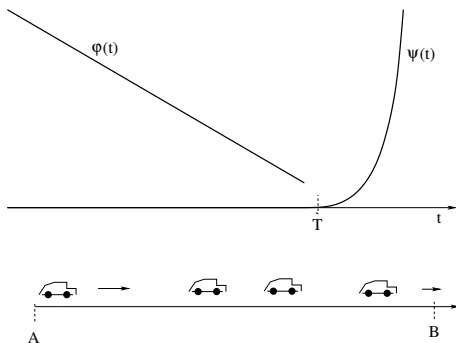
Alberto Bressan

Department of Mathematics, Penn State University

bressan@math.psu.edu

An Optimal Scheduling Problem

- Car drivers starting from a location A (a residential neighborhood) need to reach a destination B (a working place) at a given time T .
- There is a cost $\varphi(\tau_d)$ for departing early and a cost $\psi(\tau_a)$ for arriving late.



Elementary solution

Assume: the travel time Δt is constant, hence $\tau_a = \tau_d + \Delta t$

Then the optimal departure time is

$$\tau_d^{\text{opt}} = \operatorname{argmin}_t \left\{ \varphi(t) + \psi(t + \Delta t) \right\}$$

If everyone departs exactly at the same optimal time,
a traffic jam is created and this strategy is not optimal anymore

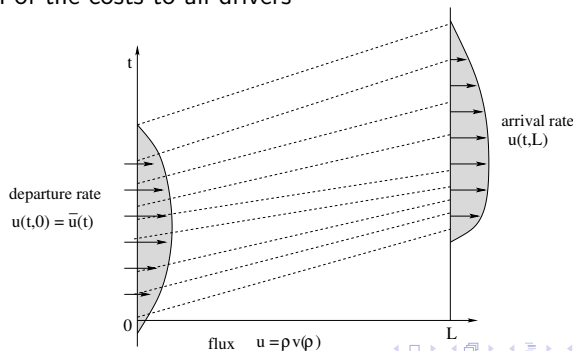
Speed of cars depends on traffic density !

An optimization problem for traffic flow

Given **departure cost** $\varphi(t)$ and **arrival cost** $\psi(t)$,
choose a departure rate $\bar{u}(t)$ so that the solution of the conservation law

$$\begin{cases} \rho_t + [\rho v(\rho)]_x = 0 & x \in [0, L] \\ \rho(t, 0)v(\rho(t, 0)) = \bar{u}(t) & t \in \mathbb{R} \end{cases}$$

minimizes the sum of the costs to all drivers



$$u(t, x) \doteq \rho(t, x) v(\rho(t, x)) = \text{flux of cars}$$

Minimize total cost:
$$J(\bar{u}) \doteq \int \varphi(t) u(t, 0) dt + \int \psi(t) u(t, L) dt$$

Choose the optimal departure rate $\bar{u}(t)$, subject to the constraints

$$\bar{u}(t) \geq 0, \quad \int \bar{u}(t) dt = \kappa = [\text{total number of drivers}]$$

Equivalent formulations

Boundary value problem for the density ρ :

$$\text{conservation law: } \rho_t + [\rho v(\rho)]_x = 0, \quad (t, x) \in \mathbb{R} \times [0, L]$$

$$\text{control (on the boundary data): } \rho(t, 0)v(\rho(t, 0)) = \bar{u}(t)$$

Cauchy problem for the flux u :

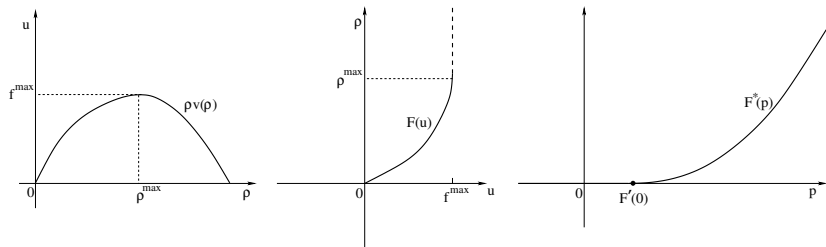
$$\text{conservation law: } u_x + F(u)_t = 0, \quad u = \rho v(\rho), \quad F(u) = \rho$$

$$\text{control (on the initial data): } u(t, 0) = \bar{u}(t)$$

$$\text{Cost: } J(u) = \int_{-\infty}^{+\infty} \varphi(t)u(t, 0) dt + \int_{-\infty}^{+\infty} \psi(t)u(t, L) dt$$

$$\text{Constraint: } \int_{-\infty}^{+\infty} \bar{u}(t) dt = \kappa$$

The flux function and its Legendre transform



$$u = \rho v(\rho), \quad \rho = F(u)$$

Legendre transform:
$$F^*(p) \doteq \max_u \{ pu - F(u) \}$$

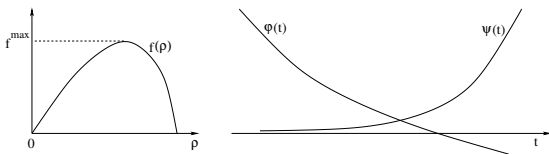
Existence of a globally optimal solution

(A1) The flux function $\rho \mapsto f(\rho) = \rho v(\rho)$ is strictly concave down.

$$f(0) = f(\rho^{jam}) = 0, \quad f'' < 0.$$

(A2) The cost functions φ, ψ satisfy

$$\varphi' < 0, \quad \psi, \psi' \geq 0, \quad \lim_{|t| \rightarrow \infty} (\varphi(t) + \psi(t)) = +\infty$$



Theorem (A.B. & Ke Han, *SIAM J. Math. Anal.*, 2012).

Let (A1)-(A2) hold. Then, for any $\kappa > 0$, there exists a unique admissible initial data \bar{u} minimizing the total cost $J(\cdot)$.

Sketch of the proof

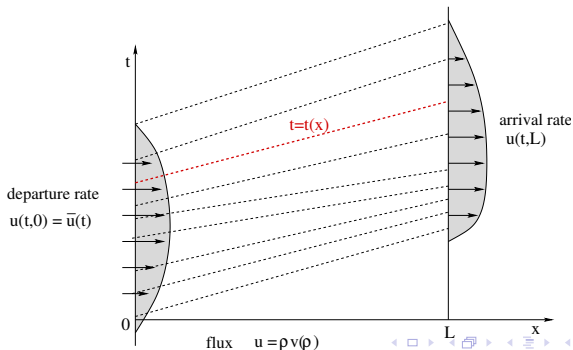
- Consider a minimizing sequence of departure rates \bar{u}_n
- It is not restrictive to assume $\bar{u}_n(t) \in [0, f^{max}]$
- The supports of \bar{u}_n are uniformly bounded
- Take a weakly convergent subsequence $\bar{u}_n \rightharpoonup \bar{u}$
- The functional is continuous w.r.t. weak convergence, hence \bar{u} is optimal

Characterization of the globally optimal solution

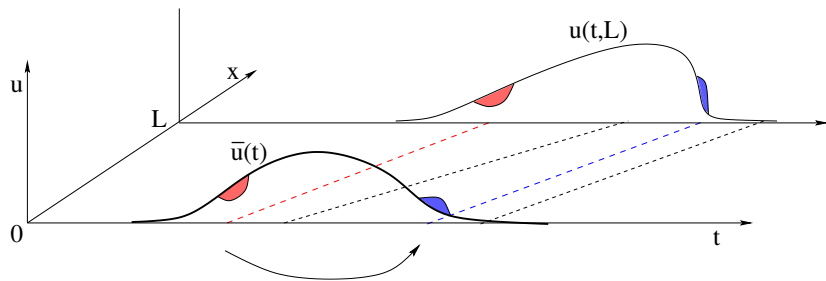
If $\rho = \rho(t, x)$ is the density in a globally optimal solution, then

- ρ is continuous (no shocks appear)
- there exists a constant C such that, for any **characteristic** line $t = t(x)$

$$\begin{cases} \varphi(t(0)) + \psi(t(L)) = C & \text{if } t(0) \in \text{Supp}(\bar{u}) \\ \varphi(t(0)) + \psi(t(L)) \geq C & \text{if } t(0) \notin \text{Supp}(\bar{u}) \end{cases}$$

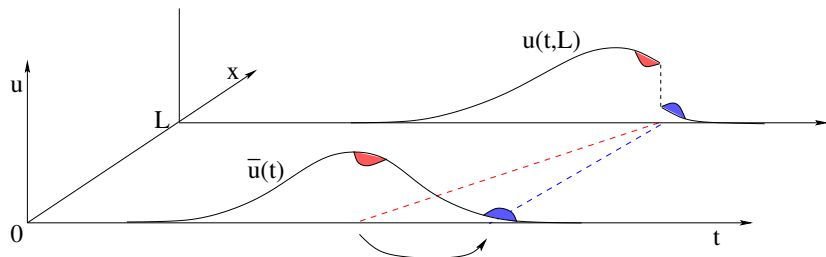


Derivation of necessary conditions



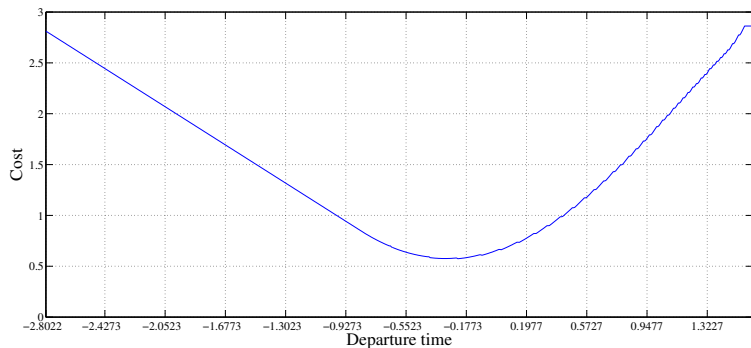
shift mass to a characteristic with lower total cost

Ruling out shocks



shift mass to a characteristic with lower total cost

Does everyone pay the same cost?



Cost vs. departure time in a globally optimal solution

The Nash equilibrium solution

A solution $u = u(t, x)$ is a **Nash equilibrium** if no driver can reduce his/her own cost by choosing a different departure time.

This implies that **all drivers pay the same cost**.

To find a Nash equilibrium, introduce the integrated variable

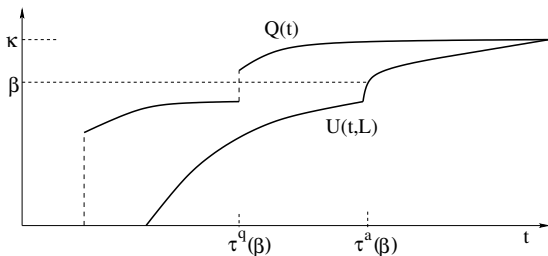
$$U(t, x) \doteq \int_{-\infty}^t \rho(s, x) v(\rho(s, x)) ds = [\text{number of drivers}$$

that have crossed the point x on the road within time t]

This solves a Hamilton-Jacobi equation

$$U_x + F(U_t) = 0$$

Note: a **queue** can form at the entrance of the highway



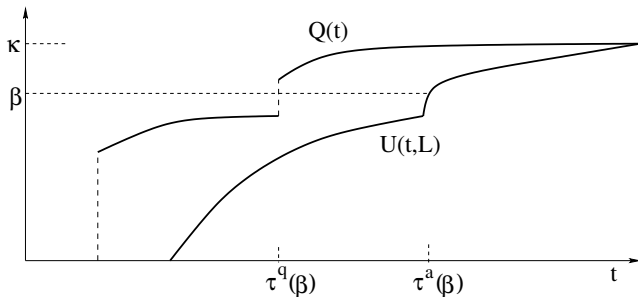
$Q(t)$ = number of drivers who have started their journey before time t
(possibly joining the queue)

L = length of the road

$U(t, L)$ = number of drivers who have reached destination before time t

$$U(t, L) = \min_{\tau} \left\{ Q(\tau) + L F^* \left(\frac{t - \tau}{L} \right) \right\}$$

Characterization of a Nash equilibrium



$\beta \in [0, \kappa]$ = Lagrangian variable labeling one particular driver

$\tau^q(\beta)$ = time when driver β joins the queue

$\tau^a(\beta)$ = time when driver β arrives at destination

$$\text{Nash equilibrium} \quad \implies \quad \varphi(\tau^q(\beta)) + \psi(\tau^a(\beta)) = c$$

Existence and Uniqueness of Nash equilibrium

Theorem (A.B. - K. Han).

Let the flux f and cost functions φ, ψ satisfy the assumptions (A1)-(A2). Then, for every $\kappa > 0$, the Hamilton-Jacobi equation

$$U_x + F(U_t) = 0$$

admits a unique Nash equilibrium solution with

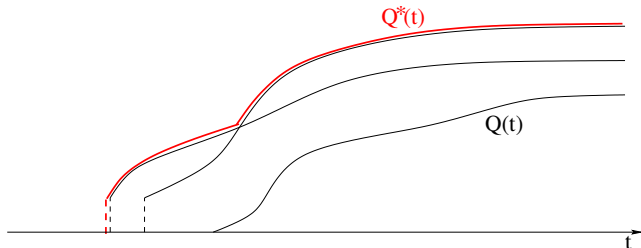
$$\text{total \# of drivers} = Q(+\infty) = \kappa$$

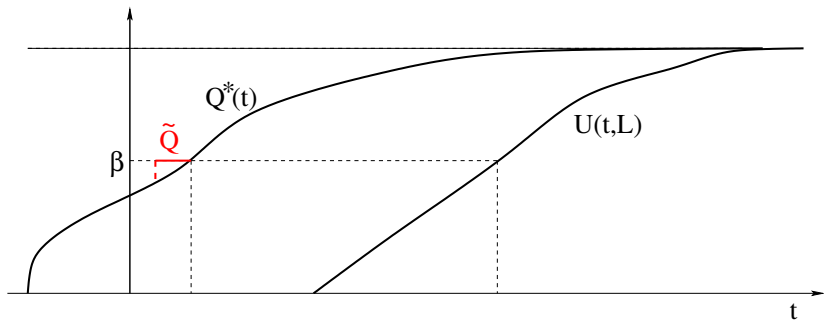
Sketch of the proof

1. For a given cost c , let \mathcal{Q}_c^- be the set of all departure distributions $Q(\cdot)$ for which every driver has a cost $\leq c$:

$$\varphi(\tau^q(\beta)) + \psi(\tau^a(\beta)) \leq c \quad \text{for a.e. } \beta \in [0, Q(+\infty)].$$

2. Claim: $Q^*(t) \doteq \sup \{ Q(t); Q \in \mathcal{Q}_c^- \} \in \mathcal{Q}_c^-$
is the initial data for a Nash equilibrium with common cost c .



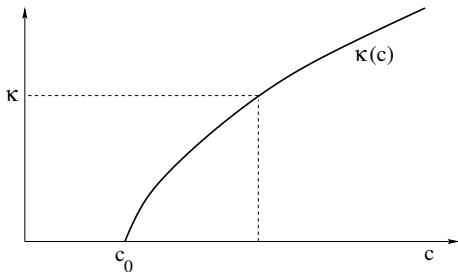


3. The Nash equilibrium with common cost c is unique.

4. Define: $\kappa(c) \doteq \sup \left\{ Q(+\infty); Q \in \mathcal{Q}_c^- \right\}$
= maximum number of cars in a solution where everyone pays $\leq c$.

There exists a minimum cost c_0 such that $\kappa(c) = 0$ for $c \leq c_0$.

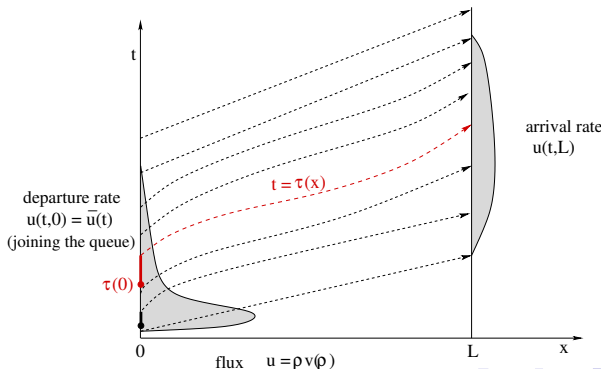
The map $c \mapsto \kappa(c)$ is strictly increasing and continuous from $[c_0, +\infty[$ to $[0, +\infty[$.



Characterization of the Nash equilibrium solution

If $\rho = \rho(t, x)$ is the density in a Nash equilibrium solution, then there exists a constant C such that, for any **particle line** $t = \tau(x)$ (describing a car trajectory)

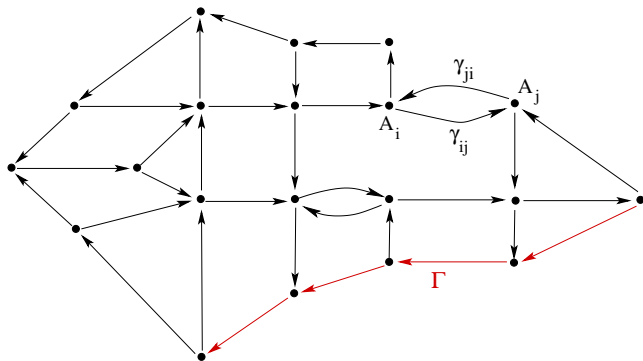
$$\begin{cases} \varphi(\tau(0)) + \psi(\tau(L)) = C & \text{if } \tau(0) \in \text{Supp}(\bar{u}) \\ \varphi(\tau(0)) + \psi(\tau(L)) \geq C & \text{if } \tau(0) \notin \text{Supp}(\bar{u}) \end{cases}$$



Traffic Flow on a Network

Nodes: A_1, \dots, A_m arcs: γ_{ij}

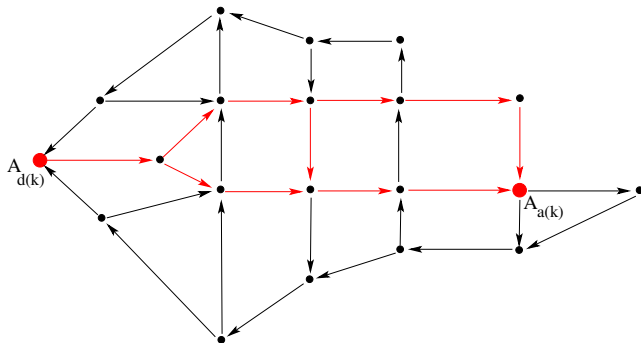
n groups of drivers with different origins and destinations, and different costs



Traffic Flow on a Network

Drivers in the k -th group depart from $A_{d(k)}$ and arrive to $A_{a(k)}$
can use different paths $\Gamma_1, \Gamma_2, \dots$ to reach destination

Departure cost: $\varphi_k(t)$ arrival cost: $\psi_k(t)$



Admissible departure rates

G_k = total number of drivers in the k -th group, $k = 1, \dots, n$

Γ_p = viable path (concatenation of viable arcs γ_{ij}), $p = 1, \dots, N$

$t \mapsto \bar{u}_{k,p}(t)$ = departure rate of k -drivers traveling along the path Γ_p

The set of departure rates $\{\bar{u}_{k,p}\}$ is **admissible** if

$$\bar{u}_{k,p}(t) \geq 0, \quad \sum_{\Gamma_p \in \mathcal{R}_k} \int_{-\infty}^{\infty} \bar{u}_{k,p}(t) dt = G_k \quad k = 1, \dots, n$$

\mathcal{R}_k = set of admissible paths, from node $A_{d(k)}$ to node $A_{a(k)}$

$\tau_p(t)$ = arrival time for a driver starting at time t , traveling along Γ_p

(depends on the overall traffic conditions)

Optima and Equilibria

An admissible family $\{\bar{u}_{k,p}\}$ of departure rates is **globally optimal** if it minimizes the sum of the total costs of all drivers

$$J(\bar{u}) \doteq \sum_{k,p} \int \left(\varphi_k(t) + \psi_k(\tau_p(t)) \right) \bar{u}_{k,p}(t) dt$$

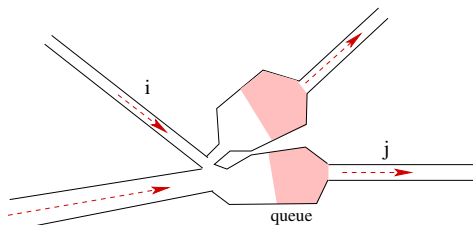
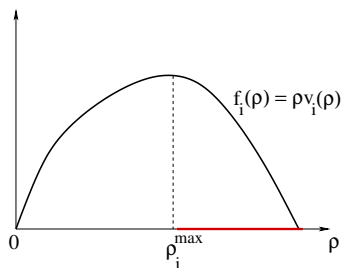
An admissible family $\{\bar{u}_{k,p}\}$ of departure rates is a **Nash equilibrium** if no driver of any group can lower his own total cost by changing departure time or switching to a different path to reach destination:

$$\varphi_k(t) + \psi_k(\tau_p(t)) = C_k \quad \text{for all } t \in \text{Supp}(\bar{u}_{k,p})$$

$$\varphi_k(t) + \psi_k(\tau_p(t)) \geq C_k \quad \text{for all } t \in \mathbb{R}$$

Assumptions on the dynamics

(A1) On each arc, the flux function $\rho \mapsto f_i(\rho) = \rho v_i(\rho)$ is concave down.

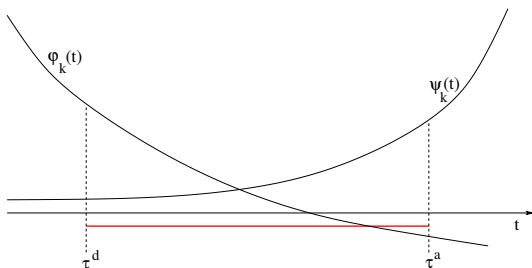


- All roads remain in a free state: $\rho_i \in [0, \rho_i^{\max}]$
- At any given junction, if too many drivers wish to enter an outgoing road $j \in \mathcal{O}$, a queue forms at its entrance (in a buffer with infinite capacity). Incoming roads $i \in \mathcal{I}$ are not affected

Assumptions on the cost functions

(A2) For k -drivers, the departure cost φ_k and the arrival cost ψ_k satisfy

$$\varphi'_k < 0 \quad \psi'_k > 0, \quad \lim_{|t| \rightarrow \infty} (\varphi_k(t) + \psi_k(t)) = \infty,$$



Theorem (A.B. - Ke Han, *Netw. & Heter. Media*, 2013).

Under the assumptions (A1)-(A2), on a general network of roads, there exists at least one globally optimal solution, and at least one Nash equilibrium solution.

Note: For a single group of drivers on a single road, solutions are unique. Uniqueness is not expected to hold, on a general network.

Theorem (A.B. - K.Nguyen, 2014).

The existence of a globally optimal solution and a Nash equilibrium solution remain valid, if the flow at each intersection is modeled by a buffer of finite size.

Theorem (A.B. - Ke Han, *Netw. & Heter. Media*, 2013).

Under the assumptions (A1)-(A2), on a general network of roads, there exists at least one globally optimal solution, and at least one Nash equilibrium solution.

Note: For a single group of drivers on a single road, solutions are unique. Uniqueness is not expected to hold, on a general network.

Theorem (A.B. - K.Nguyen, 2014).

The existence of a globally optimal solution and a Nash equilibrium solution remain valid, if the flow at each intersection is modeled by a buffer of finite size.

Existence of Nash equilibria

Proof: by finite dimensional approximations + topological methods

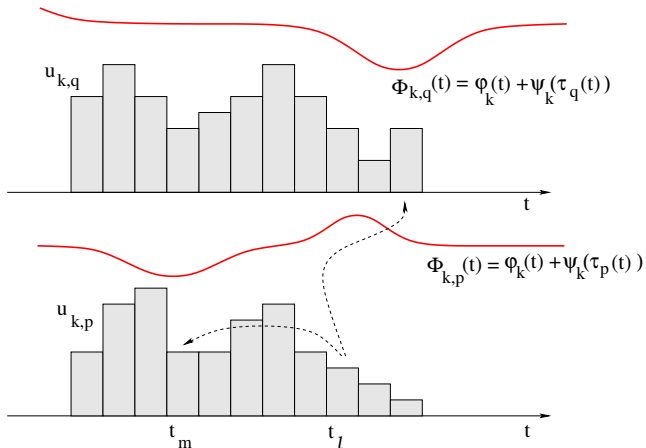
Fix a small time step $\delta > 0$, choose R sufficiently large.

$$\mathcal{U} = \left\{ \mathbf{u} = \{u_{k,p}(\cdot)\}; \quad \mathbf{u} \text{ is an admissible departure rate} \right. \\ \left. \begin{array}{l} \text{Supp}(u_{k,p}) \subseteq [-R, R], \\ u_{k,p} \text{ constant on each interval } [(n-1)\delta, n\delta] \end{array} \right\}$$

\mathcal{U} is a compact, convex set in a finite dimensional space

An evolution equation on \mathcal{U}

$$\frac{d}{d\theta} \mathbf{u} = \Lambda(\mathbf{u})$$



An evolution equation on \mathcal{U}

$$\frac{d}{d\theta} \mathbf{u} = \Lambda(\mathbf{u}) \quad \mathbf{u} \in \mathcal{U}$$

$$\Phi_{k,p}^{\ell} = \int_{(\ell-1)\delta}^{\ell\delta} \left(\varphi_k(t) + \psi_k(\tau_p(t)) \right) dt$$

= average cost to a k -driver choosing path p

$$u_{k,p}^{\ell} = u_{k,p}(t) \quad t \in](\ell-1)\delta, \ell\delta]$$

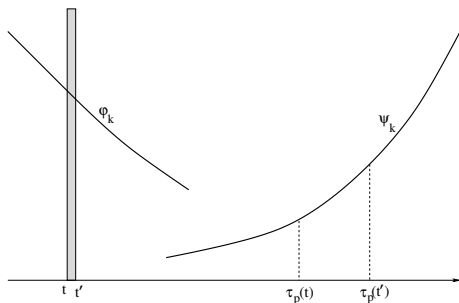
$$\begin{aligned} \frac{d}{d\theta} u_{k,p}^\ell &= \sum_q \sum_m \left[\Phi_{k,q}^m - \Phi_{k,p}^\ell \right]_+ \cdot u_{k,q}^m \\ &\quad - \sum_q \sum_m \cdot \left[\Phi_{k,p}^\ell - \Phi_{k,q}^m \right]_+ \cdot u_{k,p}^\ell \end{aligned}$$

This continuous evolution equation on the compact convex set \mathcal{U} has at least one steady state $\{\bar{u}_{k,p}\}$

(a Galerkin approximation to a Nash equilibrium)

Letting the time step $\delta \rightarrow 0$, the departure rates $\{\bar{u}_{k,p}\}$ remain uniformly bounded in \mathbf{L}^∞ , with uniformly bounded support.

A priori bounds



$$\text{Nash equilibrium} \implies \varphi_k(t) + \psi_k(\tau_p(t)) = \varphi_k(t') + \psi_k(\tau_p(t'))$$

Bound on the maximum flux on each arc

$$\implies \tau_p(t') - \tau_p(t) \geq \frac{1}{F_{\max}} \cdot \int_t^{t'} u_{k,p}(s) ds$$

Existence of a Nash equilibrium solution

Let the time step $\delta \rightarrow 0$ approach zero.

Taking subsequences, one has

$$\bar{u}_{k,p}^\nu(\cdot) \rightharpoonup \bar{u}_{k,p}(\cdot) \quad \text{weakly}$$

$$\implies \tau_p^\nu(\cdot) \rightarrow \tau_p(\cdot) \quad \text{uniformly on compact sets}$$

The departure rates $\bar{u}_{k,p}(\cdot)$ provide a Nash equilibrium

Note: For any model of traffic flow at intersections that yields

- unique solutions, for general L^∞ data
- continuity of the travel time, w.r.t. weak convergence of the departure rates

the same argument provides the existence of a Nash equilibrium.

Existence of a Nash equilibrium solution

Let the time step $\delta \rightarrow 0$ approach zero.

Taking subsequences, one has

$$\bar{u}_{k,p}^\nu(\cdot) \rightharpoonup \bar{u}_{k,p}(\cdot) \quad \text{weakly}$$

$$\implies \tau_p^\nu(\cdot) \rightarrow \tau_p(\cdot) \quad \text{uniformly on compact sets}$$

The departure rates $\bar{u}_{k,p}(\cdot)$ provide a Nash equilibrium

Note: For any model of traffic flow at intersections that yields

- unique solutions, for general \mathbf{L}^∞ data
- continuity of the travel time, w.r.t. weak convergence of the departure rates

the same argument provides the existence of a Nash equilibrium.

Traffic flow on networks - IV

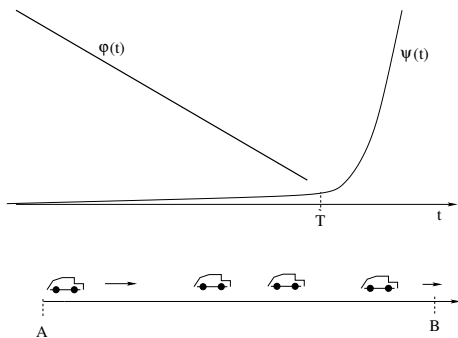
Alberto Bressan

Department of Mathematics, Penn State University

bressan@math.psu.edu

A Decision Problem

- Car drivers starting from a location A (a residential neighborhood) need to reach a destination B (a working place) at a given time T .
- There is a cost $\varphi(\tau_d)$ for departing early and a cost $\psi(\tau_a)$ for arriving late.

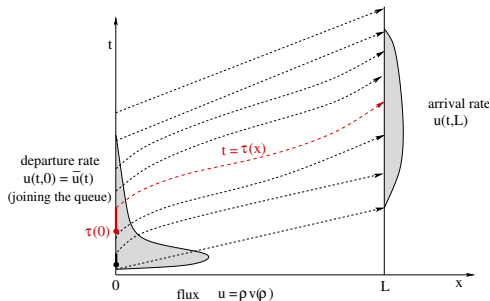


The Nash equilibrium solution

A solution $u = u(t, x)$ is a **Nash equilibrium** if no driver can reduce his/her own cost by choosing a different departure time.

This implies that **all drivers pay the same cost**.

$$\begin{cases} \varphi(\tau(0)) + \psi(\tau(L)) = C & \text{if } \tau(0) \in \text{Supp}(\bar{u}) \\ \varphi(\tau(0)) + \psi(\tau(L)) \geq C & \text{if } \tau(0) \notin \text{Supp}(\bar{u}) \end{cases}$$



Continuous dependence of the Nash equilibrium solution

$\varphi_1(t), \varphi_2(t)$ costs for departing at time t

$\psi_1(t), \psi_2(t)$ costs for arriving at time t

$v_1(\rho), v_2(\rho)$ speeds of cars, when the density is
 $\rho \in [0, \rho^*], \quad \rho_1^{\max}, \rho_2^{\max} \leq \rho^*$

$Q_1(t), Q_2(t)$ = number of cars that have departed up to time t , in the corresponding Nash equilibrium solutions

Theorem (A.B., C.Liu, W.Shen, F.Yu, *Quarterly Appl. Math.*, 2012)

Assume all cars depart and arrive within the interval $[a, b]$. Then

$$\begin{aligned} & \|Q_1 - Q_2\|_{L^1([a,b])} \\ & \leq C \cdot \left(\|\varphi_1 - \varphi_2\|_{L^\infty([a,b])} + \|\psi_1 - \psi_2\|_{L^\infty([a,b])} + \|v_1 - v_2\|_{L^\infty([0,\rho^*])}^{1/2} \right) \end{aligned}$$

Continuous dependence of the Nash equilibrium solution

$\varphi_1(t), \varphi_2(t)$ costs for departing at time t

$\psi_1(t), \psi_2(t)$ costs for arriving at time t

$v_1(\rho), v_2(\rho)$ speeds of cars, when the density is
 $\rho \in [0, \rho^*], \quad \rho_1^{\max}, \rho_2^{\max} \leq \rho^*$

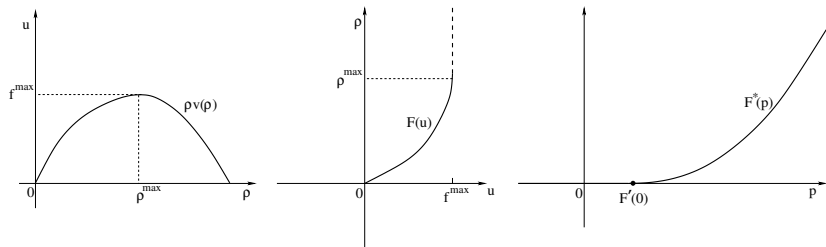
$Q_1(t), Q_2(t)$ = number of cars that have departed up to time t , in the corresponding Nash equilibrium solutions

Theorem (A.B., C.Liu, W.Shen, F.Yu, *Quarterly Appl. Math.*, 2012)

Assume all cars depart and arrive within the interval $[a, b]$. Then

$$\begin{aligned} & \|Q_1 - Q_2\|_{L^1([a,b])} \\ & \leq C \cdot \left(\|\varphi_1 - \varphi_2\|_{L^\infty([a,b])} + \|\psi_1 - \psi_2\|_{L^\infty([a,b])} + \|v_1 - v_2\|_{L^\infty([0,\rho^*])}^{1/2} \right) \end{aligned}$$

Back to the Lax formula



$$u = \rho v(\rho), \quad \rho = F(u)$$

Legendre transform:
$$F^*(p) \doteq \max_u \{ pu - F(u) \}$$

L = length of the road

$$\mu \doteq \frac{L}{f'(0)} = \frac{L}{v(0)} = \text{minimum travel time (if road is empty)}$$

$Q(t)$ = # of drivers departing before time t (possibly joining a queue)

$U(t, L)$ = # of drivers that have reached the end of the road before time t

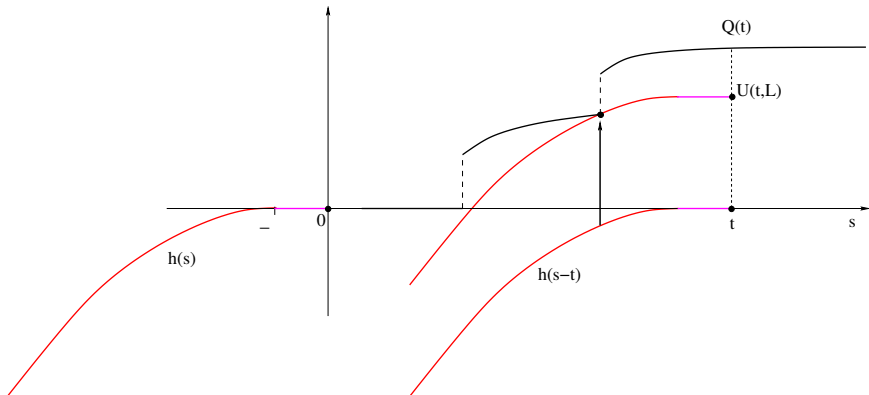
$$U(t, L) = \min_{\tau} \left\{ Q(\tau) + L F^* \left(\frac{t - \tau}{L} \right) \right\} = \min_{\tau} \left\{ Q(\tau) - h(\tau - t) \right\}$$

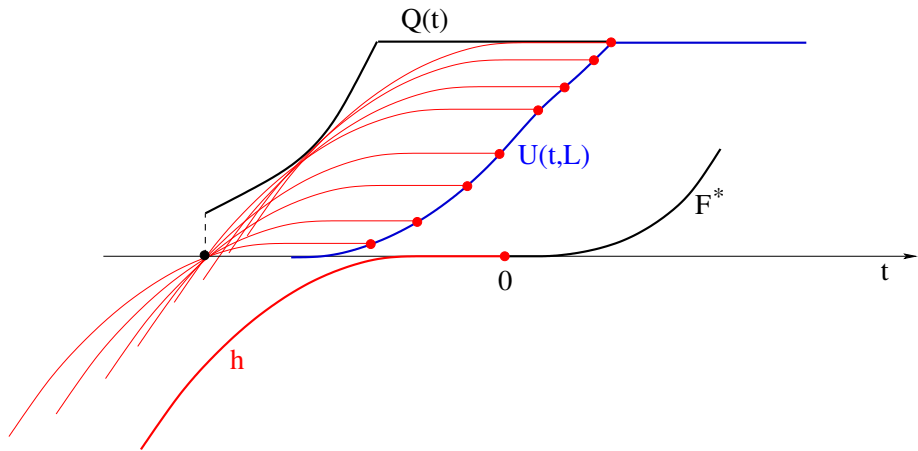
$$h(s) \doteq -L \cdot F^* \left(\frac{-s}{L} \right)$$

Geometric interpretation of the Lax formula

$$U(t, L) = \min_{\tau} \left\{ Q(\tau) - h(\tau - t) \right\}$$

is the amount by which we can shift upward the graph of $h(\cdot - t)$, before hitting the graph of $Q(\cdot)$



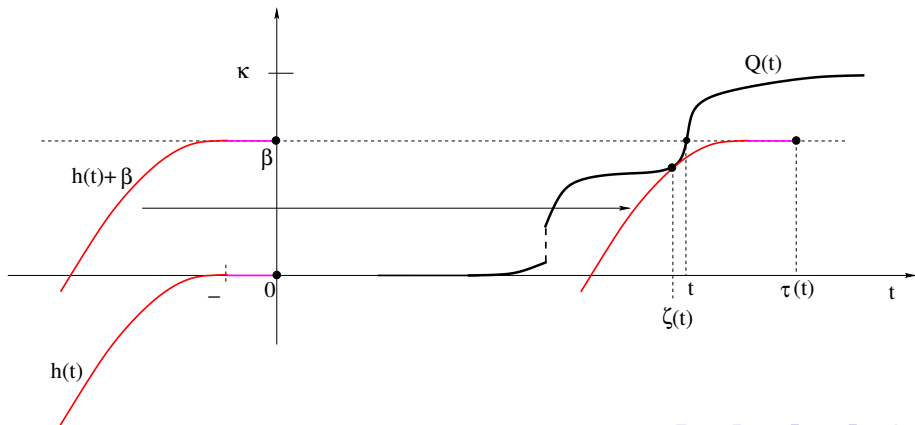


Computing the arrival time

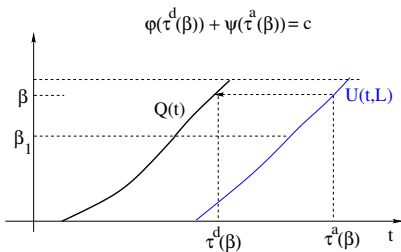
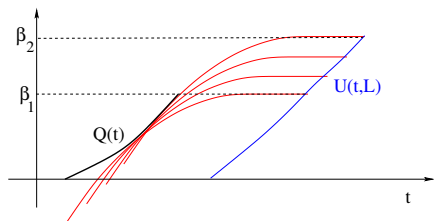
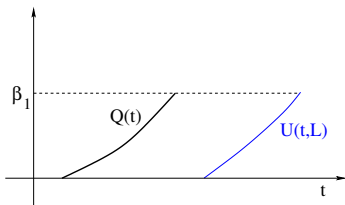
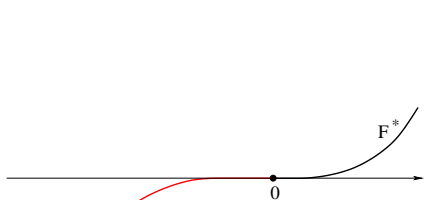
t = departure time of driver β

$\tau(t)$ = arrival time of driver β

= horizontal shift, moving the graph of $h + \beta$ below the graph of Q
depends only on the tangency point $\zeta(t)$



Constructing the Nash equilibrium with common cost = c



A minimax property

A.B., C.Liu, W.Shen, F.Yu, *Quarterly Appl. Math.*, 2012

Set of all departure distributions for κ drivers:

$$\mathcal{Q}^\kappa = \{Q(\cdot); \quad Q \text{ nondecreasing, } Q(-\infty) = 0, \quad Q(+\infty) = \kappa\}$$

Maximum cost among all drivers

$$\Psi(Q) \doteq \text{ess - sup}_{\beta \in [0, \kappa]} \left\{ \varphi(\tau^q(\beta)) + \psi(\tau^a(\beta)) \right\}$$

The departure distribution Q^* for the Nash equilibrium yields the global minimum of Ψ , namely

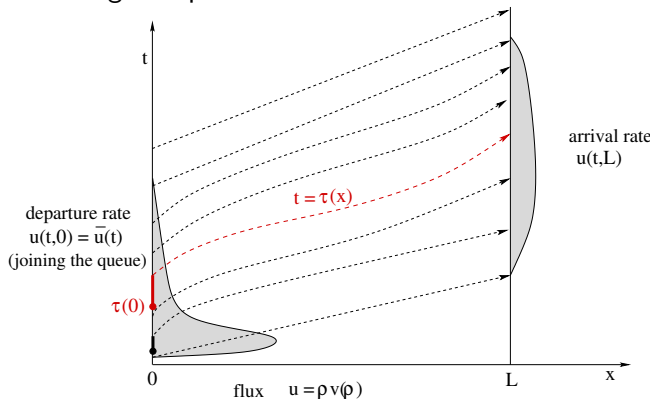
$$\Psi(Q^*) = \min \{ \Psi(Q); \quad Q \in \mathcal{Q}^\kappa \}$$

(converse is false, if a queue is present)

Improving the Nash equilibrium

$Q^*(\cdot)$ = Nash equilibrium departure distribution

$Q^\sharp(\cdot)$ = new departure distribution, where drivers depart at the same time when they were clearing the queue



Nash equilibrium as a variational inequality (T.Friesz)

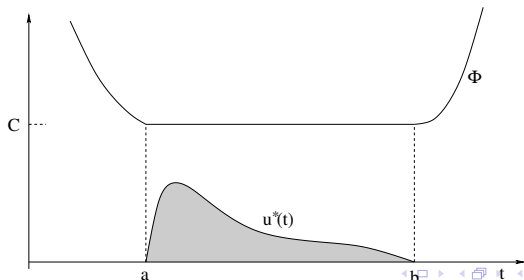
$u(t)$ = rate of departures at time t

$$\mathcal{U} \doteq \left\{ u \geq 0; \int u(t) dt = \kappa \right\}$$

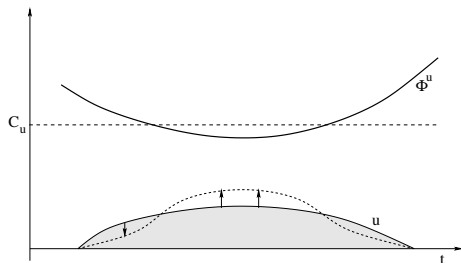
$u^*(\cdot)$ yields a Nash equilibrium if

$$\int \left[u(t) - u^*(t) \right] \Phi^*(t) dt \geq 0 \quad \text{for all } u \in \mathcal{U}$$

$\Phi^*(t) = \varphi(t) + \psi(\tau(t))$ = departure cost + arrival cost to a driver departing at time t .



$$C_u = \frac{1}{\kappa} \int \Phi^u(t) u(t) dt = \text{average cost}$$



- increase u at points where $\Phi^u(t) < C_u$
- decrease u at points where $\Phi^u(t) > C_u$

A numerical scheme

$$\frac{d}{d\theta} u(t) = \{u(t) - \Phi^u(t) - c\}_+ - u(t)$$

choose the constant c so that the integral of the right hand side is zero

$$u_\theta = \begin{cases} -u & \text{if } u + c - \Phi^u < 0, \\ c - \Phi & \text{if } u + c - \Phi^u \geq 0. \end{cases}$$

convergence ?

Dynamic stability of Nash equilibrium ?

Assume: drivers can change their departure time on a day-to-day basis, in order to decrease their own cost (one group of drivers, one single road)

Introduce an additional variable θ counting the number of days on the calendar.

$\bar{u}(t, \theta) \doteq$ [rate of departures at time t , on day θ]

$\Phi(t, \theta) \doteq$ [cost to a driver starting at time t , on day θ]

A conservation law with non-local flux

Model 1: drivers gradually change their departure time, drifting toward times where the cost is smaller.

$\tau^d(\beta, \theta) =$ departure time of driver $\beta \in [0, \kappa]$, on day θ

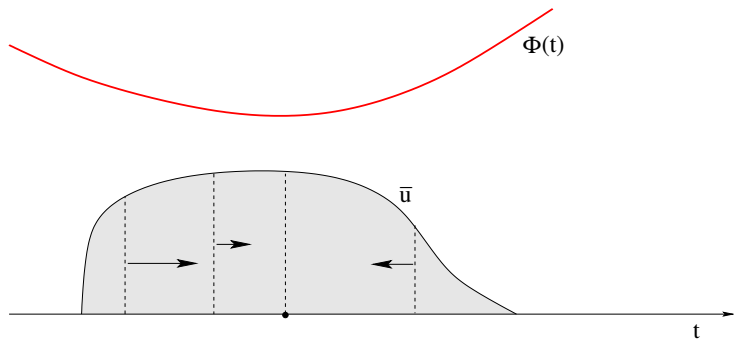
$\Phi(t, \theta) =$ total cost to a driver departing at time t , on day θ

$$\frac{\partial}{\partial \theta} \tau^d(\beta, \theta) = - \frac{\partial}{\partial t} \Phi(\tau^d(\beta, \theta), \theta)$$

If the rate of change is proportional to the gradient of the cost, this leads to the conservation law with nonlocal flux

$$\bar{u}_\theta - [\Phi_t \bar{u}]_t = 0$$

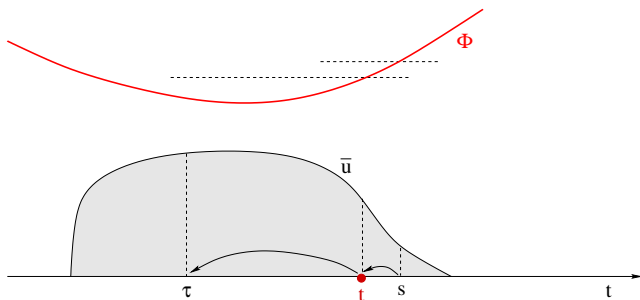
$$\bar{u}_\theta - [\Phi_t \bar{u}]_t = 0$$



An integral evolution equation

Model 2: drivers jump to different departure times having a lower cost.
If the rate of change is proportional to the difference between the costs, this leads to

$$\frac{d}{d\theta} \bar{u}(t) = \int \bar{u}(s) [\Phi(s) - \Phi(t)]_+ ds - \int \bar{u}(t) [\Phi(t) - \Phi(s)]_+ ds$$

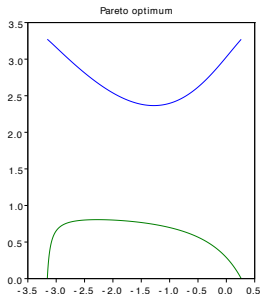
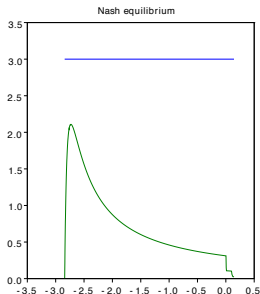


Question: as $\theta \rightarrow \infty$, does the departure rate $\bar{u}(t, \theta)$ approach the unique Nash equilibrium?

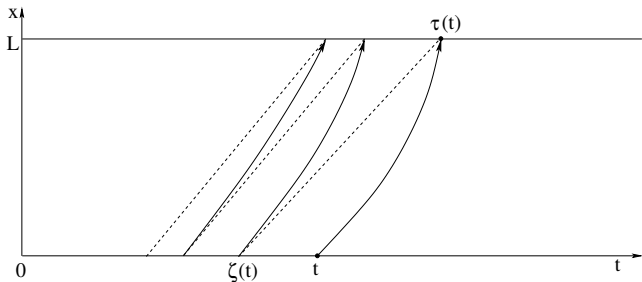
Some numerical experiments (Wen Shen, 2011)

- departure and arrival costs: $\varphi(x) = -x$, $\psi(x) = e^x$
- velocity of cars: $v(\rho) = 2 - \rho$ length of road = 2
- total number of cars = 2.2005
- common total cost in the Nash equilibrium = 3

$$\rho_t + (2\rho - \rho^2)_x = 0$$



- is the Nash equilibrium unstable ?
- what causes instability ?
- is there a chaotic attractor ?



main difficulty: non-local dependence

$$\Phi(t) = \varphi(t) + \psi(\tau(t))$$

The arrival time $\tau(t)$ depends only on the values of

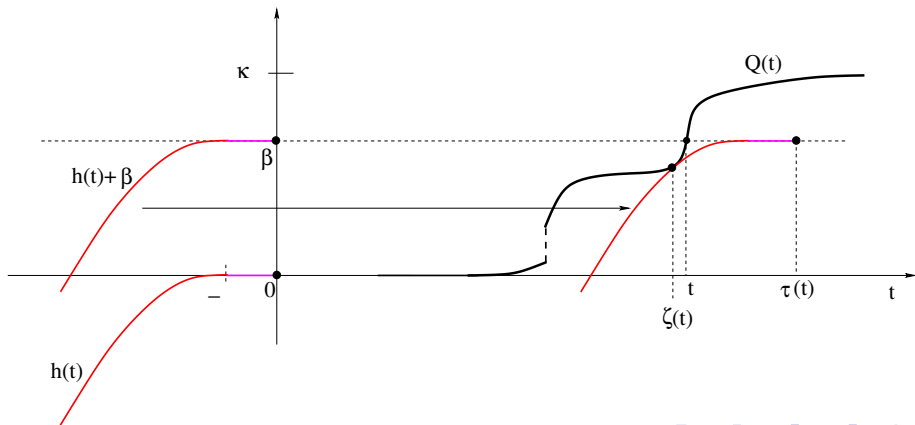
$$Q(t) = \int_{-\infty}^t u(s) ds \quad \text{near } \zeta(t)$$

Computing the arrival time

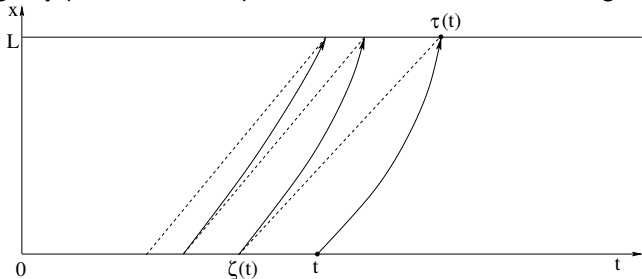
t = departure time of driver β

$\tau(t)$ = arrival time of driver β

= horizontal shift, moving the graph of $h + \beta$ below the graph of Q
depends only on the tangency point $\zeta(t)$



$\zeta(t)$ = tangency point = initial point on the characteristic ending at (t, L)



$$u \mapsto Q \mapsto \Phi$$

$$Q(t) = \int_{-\infty}^t u(s) ds, \quad \Phi(t) = \varphi(t) + \psi(\tau(t))$$

Main difficulty in the analysis: pointwise, non-local dependence of the arrival time map $t \mapsto \tau(t)$

linearized stability ?

