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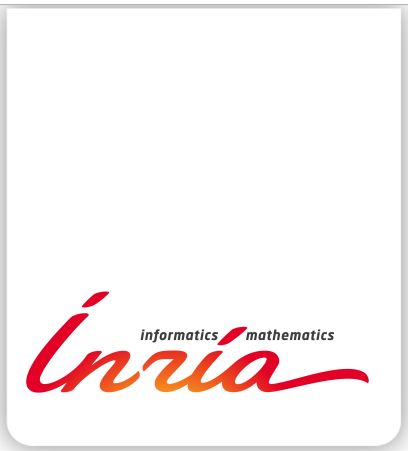

Mapped Fourier Methods for stiff problems in toroidal geometry

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Abstract: Fourier spectral or pseudo-spectral methods are usually extremely efficient for periodic problems. However this efficiency is lost if the solutions have zones of rapid variations or internal layers. For these cases, a large number of Fourier modes are required and this makes the Fourier method unpractical in many cases. This work investigates the use of mapped Fourier method as a way to circumvent this problem. Mapped Fourier method uses instead of the usual Fourier interpolant the composition of the Fourier interpolant with a mapping in such a way that in the computational space, the functions to represent are not stiff. This work gives some examples of the usefulness of this method and apply it to a simple model of pellet injection in tokamaks as an example of its potential interest for complex multi-dimensional problem.

Key-words: Spectral methods, Fourier approximation, mapped Fourier method, pellet injection, Tokamaks

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Méthodes Spectrales Fourier transformées pour les problèmes raides en géométrie toroidale

Résumé : Les méthodes spectrales ou pseudo-spectrales de Fourier sont habituellement extrèmement efficaces pour les problèmes périodiques. Cette efficacité est cependant perdue lorsque les solutions ont des zones localisées de variations rapides ou des couches internes. Dans ces cas, un grand nombre de modes de Fourier doit être utilisé pour représenter la solution et les méthodes de Fourier deviennent inutilisables. Ce travail étudie l'utilisation des méthodes de Fourier transformées comme moyen de régler ce problème. Les méthodes de Fourier transformées utilisent à la place de l'interpolant de Fourier, la composition de cet interpolant avec un changement de variables construit pour que dans l'espace de calcul, les fonctions à représenter n'aient pas de zones à variations rapides. Ce travail donne quelques exemples de l'utilité de cette méthode et l'applique à un modèle simple d'injection de glaçons dans les tokamaks comme exemple de son intéret potentiel pour des problèmes complexes multidimensionnels.

Mots-clés : Méthodes Spectrales, Approximation de Fourier, méthode de Fourier transformée, injection de glaçon, Tokamaks

1 Introduction

Many problems in science and engineering involve one or several periodicity directions. In these directions, the numerical approximation of the mathematical models can be performed in an efficient way using Fourier decomposition of the solutions in term of trigonometric polynomials. For infinitely smooth solution, the convergence of Fourier methods is more rapid than any finite power of $1/N$ where N is the number of Fourier modes : this property is called spectral accuracy and is one of the main advantage of Fourier methods versus finite difference (FD) or finite element (FE) local approximation methods since it allows to reach a very good accuracy using a small number of modes. Moreover, the existence of fast algorithms (Fast Fourier Transform) commonly available on vector and parallel computers allows an efficient implementation of these method in term of CPU cost.
Therefore in different branches of science and engineering, a large number of simulation codes rely on the use of Fourier techniques. This is for instance the case in astrophysics or in plasma physics and tokamak research where mature simulation codes designed, validated and tuned against experimental and observational results are used in a routine or production way in many different teams worldwide.
However, all these advantages of Fourier methods break down if the solution possesses localized internal layers and region of rapid changes which are small in comparison of the computational domain. For these cases, even if the solution is infinitely smooth[1], spectral accuracy is only recovered for a large number of Fourier modes, making the use of these methods unpractical.
To simulate these cases, one possibility is to use local FD/FE methods. However, this can imply a large re-coding effort that is not always possible due to time and money constraints. Another possibility is to alter the physics of the problem by enlarging artificially the region of large variations or modifying the governing parameters of the problem to obtain a less stiff problem. For complex problem governed by non linear partial differential equation, the correctness of the results obtained in this way is not guaranteed.
In this work, we experiment another possibility whose advantage is to not require large modification of existing codes. Only the parts of the code performing the computations of the spatial derivatives in the periodicity directions need to be changed and moreover these modifications can be light (depending of the structure of the existing code however). The basic idea is to change the global trigonometric interpolant by functions much more adapted to the representation of functions exhibiting rapid internal variations. This is done by composing the Fourier modes with an appropriate change of variables. Another way to present the same idea is to write the initial problem in a computational domain where the solution is not stiff.
This idea to use a change of variable to map a physical domain to a periodic one for computational purpose is not new. The term mapped Fourier methods to designate these methods have been used in quantum computations studies [5] almost 20 years ago and it is for instance discussed at length in the well-known Boyd's text book [3]. In the cases where the localization of the internal zones of rapid variations is not know but is part of the problem, algorithms to construct an adapted mapping from the knowledge of the current solution or of the residual have been designed see [7] and [1]. For some theoretical development on the convergence of these type of methods in a non-periodic context, one can consult [10, 11].
The structure of this paper is as follows. In section 2, we give some examples of the behavior of Fourier methods when the solutions exhibit localized internal layers to emphasize the loss of

[1]The case where the solution possesses shocks or discontinuities is different : for shock, spectral accuracy is never recovered, even if N is very large. The present work does not address this problem and concentrates on problems with smooth solutions.

fast convergence of Fourier methods in these problems. Section 3 recall the principle of mapped Fourier methods and revisits the examples given in section 2 to show the potentialities of the method. Section 4 apply this technique for the simulation of a crude model of pellet injection, a technique used to refuel the core plasma in tokamaks. Finally we conclude by some remarks.

2 Spectral accuracy and spatial resolution

2.1 Accuracy of the numerical derivatives

We begin with the very simple problem of computing the successive derivatives of a function possessing a region of rapid variation. The test function considered is the Maxwellian :

$$u(x) = e^{-\frac{(x-\pi)^2}{\Delta^2}}$$

Computing the derivatives of this function by a Fourier method is done by sampling u on a set of discrete points $x_j = 2\pi/N, j = 0, \ldots, N-1$ and replacing u by its Fourier interpolant $I_N u$ that is a member of $\mathcal{S}_N$ the set of trigonometric polynomials of degree N. The Fourier interpolant is defined by

$$I_N u = \sum_{k=-N/2}^{N/2-1} \hat{u}_k e^{ikx}$$

where the coefficients $\hat{u}$ are computed by the Discrete Fourier Transform (DFT):

$$\hat{u}_k = \frac{1}{N} \sum_{j=0}^{N-1} u(x_j) e^{-ikx_j}$$

Then computing the pseudo-spectral derivative of u consists in computing the exact derivative of the Fourier interpolant $I_N u$. Fig. 1 compares the exact first derivative of u with the first derivative of $I_N u$ while Fig. 2 plots the results for the second derivative. It is seen that if the spatial resolution is not sufficient and not able to capture the zone of rapid variation of the function, then the results are extremely inaccurate and that the Fourier interpolant (the green line) is an extremely oscillatory function. This phenomena (the Gibbs phenomena) is well known and studied in details for instance in [4]. Fig. 3 that displays the discrete L2 norm of the error vs Δ is another way to consider the same results. It shows that either the resolution is sufficient and spectral methods are extremely accurate, either the spatial resolution is not sufficient and the accuracy of the methods methods is extremely bad. Spectral methods have thus a "binary" character : They are then either very good or very bad.

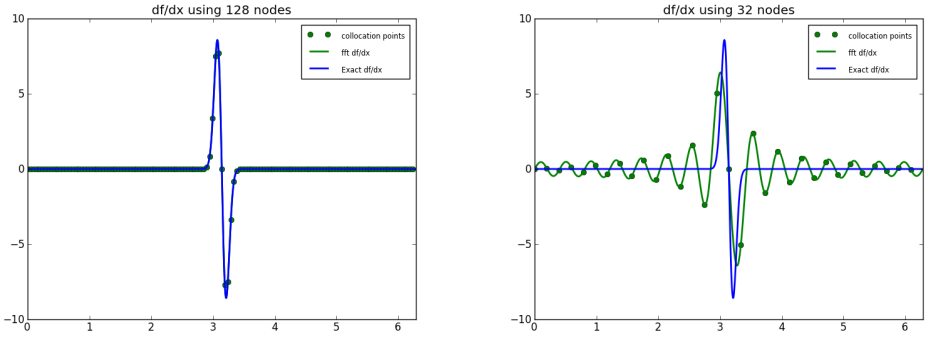

Figure 1: Pseudo Spectral representation of the first derivative of $e^{-\frac{(x-\pi)^2}{\Delta^2}}$ for $\Delta = 0.1$

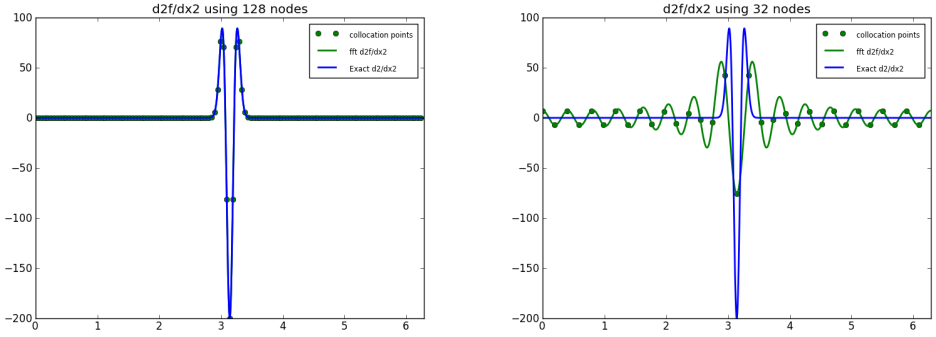

Figure 2: Pseudo Spectral representation of the second derivative of $e^{-\frac{(x-\pi)^2}{\Delta^2}}$ for $\Delta = 0.1$

2.2 Laplace equation

Here we consider in some sense the inverse problem of the one considered in section 2.1 and solve by a Fourier Spectral method the problem :

$$-u_{xx} = f \tag{1}$$

The right hand side f is chosen such that the exact solution of this problem is $u = e^{-\frac{(x-\pi)^2}{\Delta^2}}$. In the following numerical experiments, $\Delta = 0.1$ and the Laplace problem can be considered as moderately stiff. The figure 4 displays the trigonometric interpolating polynomial of the right hand side of equation (1) sampled respectively on 32, 64 and 128 collocation nodes. The collocation nodes are displayed by blue dots on this figure and it is seen that if the spatial resolution is not sufficient, the trigonometric interpolating polynomial passing through these points is highly oscillatory. For this quite large value of the width parameter Δ, 128 nodes at least are necessary to avoid large oscillations.

Solving 1 by a Fourier pseudo-spectral method on a grid of N nodes is equivalent to solve the problem : Find $u_N \in S_N$ such that

$$-u_N^{''} = I_N f \tag{2}$$

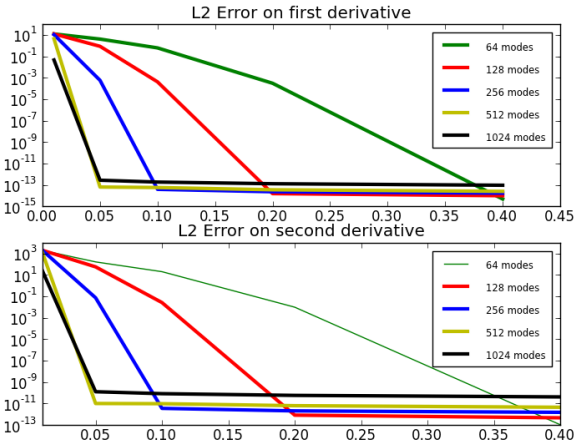

Figure 3: Evolution of the error vs Δ

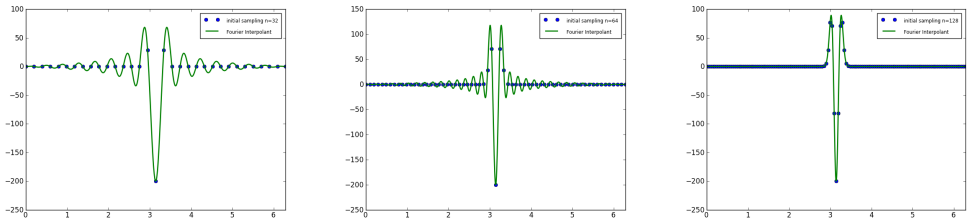

Figure 4: Fourier Interpolant of the RHS of equation 1

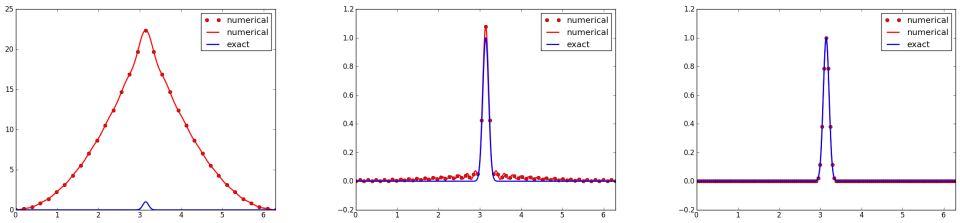

Figure 5: Numerical solution of equation 1 : left 32 nodes, middle 64, right 128 nodes

where I_N stands for the Fourier interpolant. Due to the highly oscillatory character of the trigonometric interpolating polynomial when the spatial resolution is not sufficient, one can expect the error to be extremely large in this case. This is what is shown in figure 5 that displays the solution of 2 for 32, 64 and 128 collocation nodes respectively. The red dots displays the solutions on the collocation points where the solution is effectively computed while the red solid lines display u_N the interpolating polynomial passing through these points. For $N = 32$, the error of the numerical solution is extremely large and the Fourier method is clearly useless in this case. Even for $N = 64$ where there are 5 collocation points in the width Δ of the Maxwellian,

small oscillations can be noticed in the solution and at least 128 are needed to solve this problem with a sufficient accuracy.

3 Mapped Fourier methods

3.1 Principle of the method

Consider a partial differential equation on $[0, 2\pi]$ that we write symbolically as

$$\mathcal{L}(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots) = f \tag{3}$$

The principle of Fourier methods is to look for the solution u of this PDE in the space $\mathcal{S}_N$ of the trigonometric polynomials of order N. Although there are many variations of spectral methods (Galerkin, Tau, collocation), a common characteristics of Fourier methods is to compute the approximate solution by sampling u (or the RHS f) on a set of discrete points $x_j = 2\pi/N, j = 0, \dots, N-1$ and to replace it by its Fourier interpolant

$$I_N u = \sum_{k=-N/2}^{N/2-1} \hat{u}_k T_k(x)$$

where the basis functions $T_k = e^{ikx}$ are the Fourier modes and where the coefficients $\hat{u}$ are computed by the Discrete Fourier Transform (DFT):

$$\hat{u}_k = \frac{1}{N} \sum_{j=0}^{N-1} u(x_j) e^{-ikx_j}$$

As shown in section 2 a drawback of this type of method is that the Fourier modes are unable to represent a function displaying small localized features when the spatial resolution is not sufficient : either the resolution is sufficient and Fourier methods are extremely efficient and accurate, either the spatial resolution is not sufficient and the accuracy of Fourier methods is extremely low.
Since the Fourier modes are not adapted to the representation of stiff functions with localized features, a simple remedy is simply to change the Fourier modes for functions that are better adapted to the characteristics of the functions that are to be computed. This can be simply done by introducing a smooth change of variables $h : \xi \rightarrow x$ and to consider instead of the Fourier modes $T_k(x)$, **mapped Fourier** modes defined by $T_k \; o \; h^{-1}(x)$. The function $u(x)$ is then developed on the basis of the mapped Fourier modes :

$$I_N(h)u = \sum_{k=-N/2}^{N/2-1} \hat{u}_k T_k \; o \; h^{-1}(x)$$

and standard (Galerkin or collocation) methods can be used as with usual methods.

Another way to describe this technique is to consider that to each function $u(x)$ defined on $\Omega_x = [0, 2\pi]$ is associated another function $\mathcal{U}(\xi)$ defined on a computational space Ω_ξ by :

$$\mathcal{U}(\xi) = u(x(\xi)) = u \; o \; h(\xi)$$

The mapping h is chosen to provide an adequate resolution in the computational space Ω_ξ or in other words, such that the function $\mathcal{U}(\xi)$ is not stiff. The solution $u(x)$ of (3) can then be obtained from the solution of the transformed pde

$$\mathcal{L}_\xi(u, \frac{\partial \mathcal{U}}{\partial \xi}, \frac{\partial^2 u}{\partial \xi^2}, \dots) = f_\xi \tag{4}$$

obtained from repeated application of the chain rule to the partial derivatives present in (3). Equation (4) is then discretized using a standard Fourier method. This procedure can look difficult to implement as the pde (4) can be considerably more complex than (3) especially if (3) contains high order derivatives.
However if a collocation method is used, to solve (3), only the expression of the derivatives at the collocation points x_j are needed. Then, let D be the matrix representing the Fourier derivative at the collocation points, i.e

$$(I_N u)'(x_i) = \sum D_{ij} u(x_j)$$

and denote $\boldsymbol{u}$ (resp $\boldsymbol{U}$) the N-dimensional vector of the values of u at the collocation points x_i i.e $\boldsymbol{u}_i = u(x_i)$ (resp. the N-dimensional vector of the values of $\mathcal{U}$ at the collocation points ξ_i i.e $\boldsymbol{U}_i = \mathcal{U}(\xi_i)$) then the application of the chain rule give :

$$u'(x_j) = \frac{\mathcal{U}'(\xi_j)}{h'(\xi_j)} = \frac{1}{h'(\xi_j)}(D\boldsymbol{U})_j = \frac{1}{h'(\xi_j)}(D\boldsymbol{u})_j \tag{5}$$

since $u(x_j) = \mathcal{U}(\xi_j)$ and the vectors $\boldsymbol{U}$ and $\boldsymbol{u}$ are identical. The mapped Fourier technique thus reduces to multiply by a diagonal matrix the result of the standard Fourier derivative. This last operation is usually done without assembling the matrix D and using instead Fast Fourier Transform.
Let us now consider the evaluation of higher order derivatives, one possibility is to evaluate them using the Faà di Bruno formula. For instance, we have

$$u''(x_j) = \frac{1}{(h'(\xi_j))^3}[h'(\xi_j)\mathcal{U}''(\xi_j) - h''(\xi_j)\mathcal{U}'(\xi_j)] \tag{6}$$

and $\mathcal{U}''(\xi_j)$ can be evaluated as $\mathcal{U}'(\xi_j)$ by FFT. However, an alternate way to evaluate high order derivatives is simply to apply in a repeated way the expression (5). This is the strategy that has been followed in this paper.
In the following two sections, we repeat the numerical experiments of sections 2.1 and 2.2 to demonstrate the benefits of using mapped Fourier methods. The mapping considered is

$$x = h(\xi) = atan(Ltan\xi) \tag{7}$$

Use of this mapping has been proposed and advocated in [2] that gives good arguments to prefer it against alternate choices in the periodic case.

3.2 Accuracy of the numerical derivatives

We repeat here the numerical experiments of the section 2.1 but use for the evaluation of the derivatives the mapped Fourier method. The value of the parameter L in (7) has been chosen as $L = \Delta$. Fig. 6 and 7 have to be compared with the figures 1 and 2 and illustrates the improvement that can be expected from the use of the mapped Fourier method. On these figures, note that not only the values of the derivatives on the collocation points are extremely accurate but that also the interpolating function (that is not a trigonometric interpolant but a member of the space $\mathcal{S}_N(h)$ of the function composing a Fourier interpolant with the mapping h) is not oscillatory and superpose exactly with the exact derivatives.

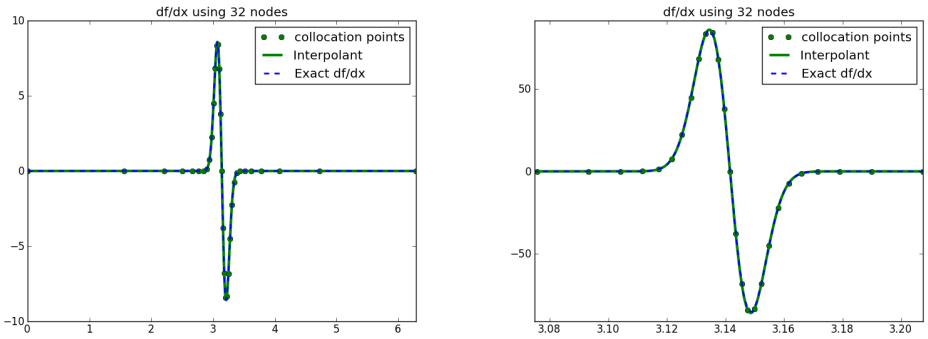

Figure 6: Mapped Fourier representation of the first derivative of $e^{-\frac{(x-\pi)^2}{\Delta^2}}$ for : left plot $\Delta = 0.1$, right plot $\Delta = 0.01$, note that the right plot is a zoom on the central region

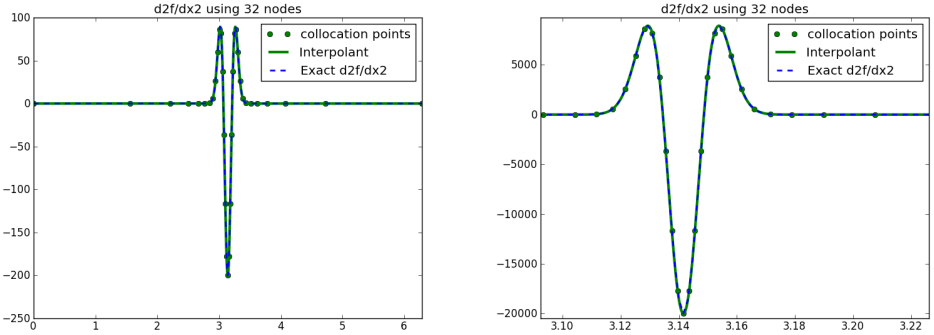

Figure 7: Mapped Fourier representation of the second derivative of $e^{-\frac{(x-\pi)^2}{\Delta^2}}$ for : left plot $\Delta = 0.1$, right plot $\Delta = 0.01$, note that the right plot is a zoom on the central region

3.3 Laplace equation

In this section, we again consider the solution of equation (1). The exact solution is the Maxwellian $u = e^{-\frac{(x-\pi)^2}{\Delta^2}}$ and as $\Delta \to 0$ this solution becomes harder and harder to catch with conventional Fourier methods. However, the use of transformed Fourier series as explained in section 3.1 can cure this problem in an efficient way. We again use the change of variables (7) with $L = \Delta$. Figure 8 shows the results obtained with 32 nodes for values of the width parameter Δ decreasing from 0.1 to 0.01. As in figure 5 the red dots displays the solutions on the collocation points where the solution is effectively computed while the red solid lines display u_N the interpolating function passing through these points. Note that in contrast with conventional Fourier method, the interpolating function is not a trigonometric polynomial in the physical space : it is a member of the space $S_N(h)$ and results from the composition of a trigonometric polynomial with a change of variables.

As shown in these figures, the accuracy obtained with the MF method is very good. Not only the solution computed on the collocation points coincides with the exact solution but the interpolating function matches the exact solution almost exactly. Figure 9 presents a zoom of

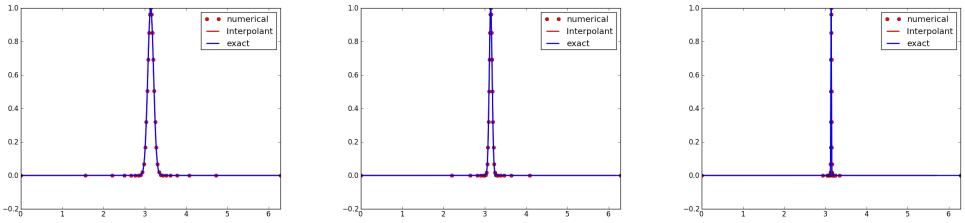

Figure 8: Numerical solution of equation (1) with 32 nodes : left $\Delta = 0.1$, middle $\Delta = 0.05$, right $\Delta = 0.01$

the solution obtained with only 32 nodes for $\Delta = 0.01$. Referring to figure 3, we note that standard Fourier methods are unable to compute this solution even with thousands of modes. For the graphical representation, the interpolating and exact solution are evaluated on 4096 points. It is seen on this figure that the interpolating function and the exact solution are identical : the blue line (the exact solution) superposes exactly the red one (the interpolating function).

From a computational point of view, the computational cost of solving (1) with a transformed method is identical to the cost of solving a problem of the type :

$$-\frac{d}{dx}[a(x)\frac{du}{dx}] = f \tag{8}$$

This requires essentially 4 FFT while the use of conventional Fourier methods require only 2 FFT for the Laplace problem. However, in practice this additional cost is largely balanced by the reduction in the number of necessary nodes. Moreover, in many practical application, the diffusion coefficients are non-constant and the real problems to solve are of the type (8) instead of (1). The cost of the transformed method is thus almost identical to the cost of standard methods.

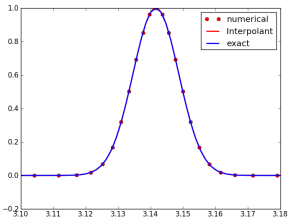

Figure 9: Zoom of the Numerical solution of equation (1) with 32 nodes : $\Delta = 0.01$

4 Pellet Injection

In magnetic confinement experiments for fusion research, the injection of cryogenic DT pellet in the plasma is considered as one of the most interesting way to fuel the core plasma. Pellets which are composed of neutral gas are rapidly heated by the plasma electrons and ionize to refuel the plasma. Moreover present tokamaks based on the so-called H-mode regime are known to be subject to some kind of MHD instabilities known as ELMs for Edge Localized Modes that can cause an excessive erosion of the plasma facing component. The triggering of ELMS by pellet injection has been demonstrated experimentally to be a possible technique to reduce the amplitude of naturally occurring large ELMs. However, a lot of uncertainties remains for the practical application of this technique in the future ITER tokamak and reliable numerical simulations can be extremely useful to reduce these uncertainties. The simulation of this problem is complex and involves the coupling of a non-linear MHD model with a pellet ablation model. This is usually done by introducing in the MHD model a density source term to model the ablation of the pellet. One of the difficulties of this type of simulation comes from the fact that the pellet size is much smaller than the tokamak size. A typical ratio of these dimensions is of order 10^{-3}. Consequently the number of numerical simulations for this type of problem is relatively small : Among the few works devoted to this question one can notice the work of Samtaney *et al* [9] that use AMR technique to deal with the size problem or Futatani *et al* [6] that artificially increase the pellet size.
In this preliminary work, we do not intend to use a complex MHD model to deal with this question but rather we want to study how the use of mapped Fourier methods can help to deal with this problem.

4.1 One dimensional problem

In this preliminary section, we consider a very simple diffusion problem with a source term localized in space. More specifically the mathematical model writes :

$$\frac{\partial u}{\partial t} - u_{xx} = S(x,t) \tag{9}$$

where the source term is

$$S(x,t) = 5ue^{-(\frac{t-0.3}{0.15\sqrt{2}})^2} \times e^{-(\frac{(x-\pi))}{\Delta})^2} \tag{10}$$

As $\Delta \to 0$, the solution of this equation becomes more and more difficult to compute accurately by Fourier methods. We have used here $\Delta = 0.03$ that corresponds to a difficult problem for Fourier methods. Equation (9) is discretized in space using the mapped Fourier method described in the previous section. Again the mapping is given by (7). The results of the mapped Fourier technique will be compared with those obtained by the standard Fourier pseudo-spectral method (that corresponds here to $L = 1$ in the formula (7)). A simple explicit first-order time advancing scheme have been used. More sophisticated and accurate schemes can be used but this was unnecessary for these preliminary experiments. We have found that a time step constraint of the form :

$$\Delta t \sim \frac{1}{max_j(h'(\xi_j))^2 n^2} \tag{11}$$

where n is the number of collocation points, is sufficient to ensure the stability of the scheme.
The results for standard Fourier methods are displayed in Fig. 10 for different times between $t = 0$ where the solution is uniform and equal to 2 and $t = 0.5$ for different numbers of Fourier

modes. Due to the specific form of the source term, (10) this one is maximum for $t = 0.3$ and almost zero for $t > 0.5$ and thus after $t > 0.5$ the solution simply relaxes to a constant. Fig. 10 shows that the computed solutions are characterized by the presence of huge oscillations whose amplitude exceed the maximum value of the true solution (Fig. 10 d.). As the number of Fourier modes increases, the amplitude of the oscillations are reduced. However this reduction is slow and 512 modes are necessary to obtain an oscillation free solution.

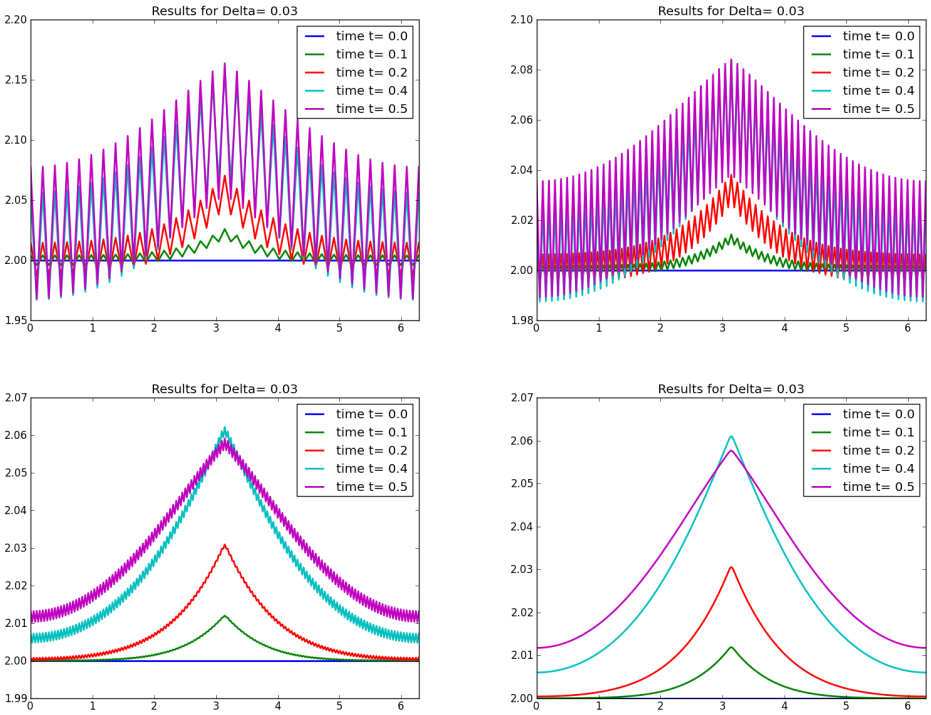

Figure 10: 1D diffusion problem. Standard Fourier : a) left top $n = 64$, b) right top $n = 128$, c) left bottom $n = 256$, d) right bottom $n = 512$

In contrast, mapped Fourier methods can capture the localized source term quite accurately with a very moderate number of Fourier modes. Fig. 11 compares the results obtained with 64 modes and $L = 0.1$ with the result obtained by the standard Fourier technique with 512 modes. It is seen that the results obtained with the mapped Fourier technique is not only free from oscillations but also as accurate as the results obtained with standard techniques and 512 modes.

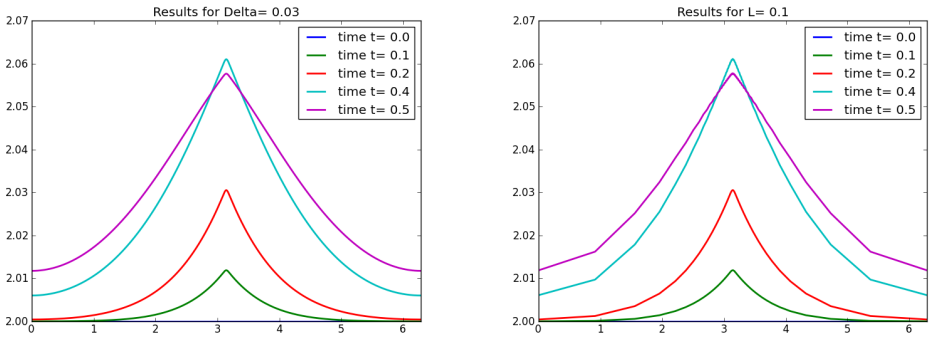

Figure 11: 1D diffusion problem Mapped Fourier (MF) compared to standard method : a) left Standard Fourier with 512 modes, b) right MF using 64 modes and $L = 0.1$

4.2 Two dimensional problem

In this section, a crude model of pellet injection in tokamaks is considered. This simulation do not intend to represent accurately the physical problem but tries to solve some of the numerical difficulties present in a complete model of pellet injection. For more complete modeling and numerical simulation, we refer to [9], [6]. The computational domain is a disk $(r, \phi) \in [1, 2] \times [0, 2\pi]$ with r the radial direction and ϕ the toroidal one that corresponds to the midplane of a tokamak. The temperature [2] evolution is governed by the heat equation and the ablation rate of the injected pellet during its passage through the background plasma is expressed in terms of a source term localized in time and space. More specifically the mathematical model writes :

$$\frac{\partial u}{\partial t} - \nabla.\boldsymbol{q} = S(x,t) \tag{12}$$

where $\boldsymbol{q}$ is the heat flux. Due to the anisotropic character of magnetized plasma, the heat flux in the parallel direction to the magnetic field far exceeds the one in the perpendicular direction. Considering in our problem that the magnetic field is purely parallel to the toroidal direction, we have used :

$$\boldsymbol{q} = \frac{\kappa_{//}}{r}\frac{\partial u}{\partial \phi}\boldsymbol{e}_\phi + \kappa_\perp \frac{\partial u}{\partial r}\boldsymbol{e}_r \tag{13}$$

with $\kappa_{//} = 10(u/u_0)^{5./2}$ and $\kappa_\perp = 0.01$. The initial temperature varies linearly between $u = 2$ at $r = 1$ and $u = 1$ at $r = 2$. Due to the highly non-linear behavior of the parallel heat diffusion coefficient, this one varies between 1.76 and 10 and is larger in the center of the disk. where the temperature is large. The ratio between parallel and perpendicular diffusion is between 176 and ~ 1000. The source term modeling pellet injection is given by :

$$S(x,t) = -250ue^{-(\frac{t-0.3}{0.15\sqrt{2}})^2} \times e^{-\frac{(\phi-\pi)^2+(r-1.5)^2)^2}{\Delta}} \tag{14}$$

More sophisticated models do exist and are available, a commonly used model for these studies being the one by Parks and Turnbull [8]. However the simple model (14) contains the essential numerical difficulties that we want to address and is sufficient for our purpose. Note that the

[2] alternatively this equation can be considered as a density evolution equation but in this case the sign in front of the source term must be changed

source term is negative as the injection of cryogenic pellets in a very hot plasma gives rise to a local drop of the plasma temperature. The parameter Δ is related to the pellet radius that is by several orders of magnitude smaller than the tokamak size. In this work, we have taken $\Delta = 0.03$. A mapped Fourier method has been used in the toroidal direction and a simple finite difference one in the radial direction. Fig 12 presents the distribution of collocations points in the computational domain. The right plot shows that the points are highly concentrated at the location of the source.

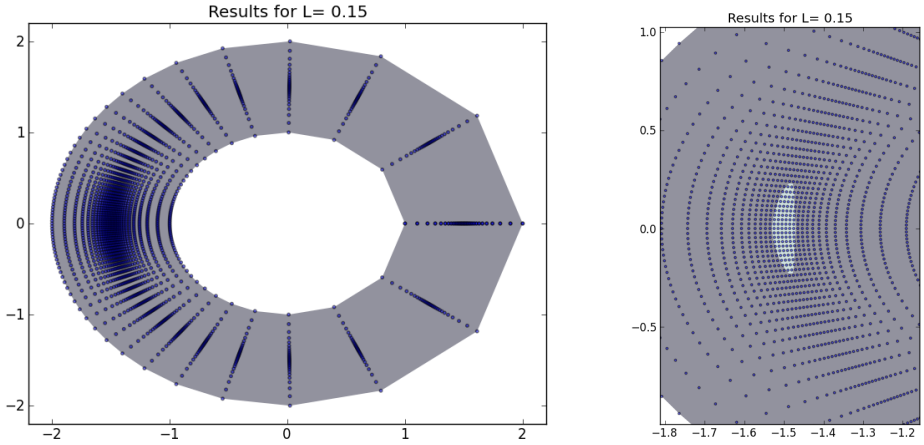

Figure 12: Left : Distribution of collocation points in the physical space, Right : Zoom on the source region

In Fig. 13 are compared results obtained from standard Fourier methods using from 64 to 256 modes with the results obtained with a mapped Fourier method with 32 points. In all these computations, the number of nodes in the radial direction is 32. The standard computations done with 64 and 128 Fourier are clearly unusable due to the presence of large oscillations on the opposite side of the disk. This is particularly true for tokamak researches since pellet injection is considered as a possible way to mitigate the occurrence of large scale ELMs (Edge Localized modes) that can potentially affect the integrity of the vacuum chamber of tokamaks by triggering small amplitude instabilities. The presence of large scale oscillations on the opposite side of the tokamak with respect to the location of the pellet source can then lead potentially to erroneous conclusions on the triggering of instabilities by pellet injections. For a number of mode equal to 256, these oscillations almost disappear (actually some small amplitude fluctuations are still present in these computations). However this large number of modes is far beyond what can be done for these computations considering the large number of nodes that must be used in the poloidal section. In contrast, mapped Fourier methods are particularly efficient in this context and only 32 nodes have been required to obtain oscillation free computations. Fig. 14 presents an enlargement of the source region showing that the results obtained with mapped Fourier methods compare very favorably to the results obtained by standard methods with a much larger number of points.

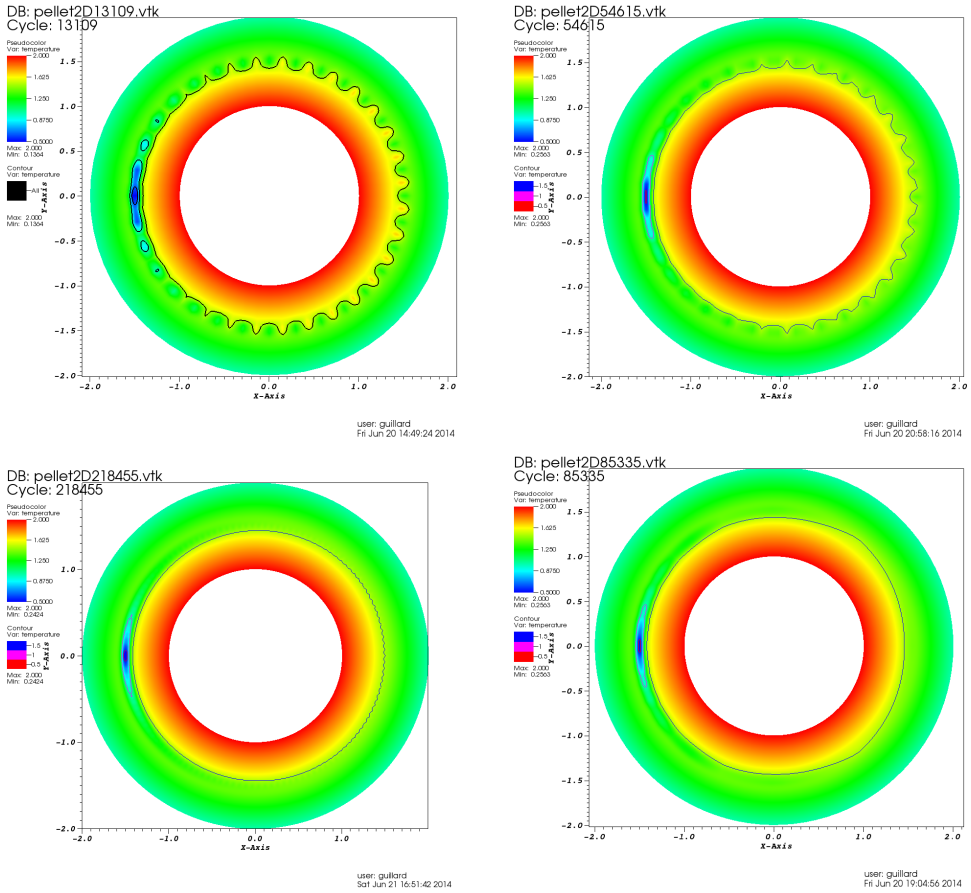

Figure 13: Top Left : Standard Fourier with 64 modes, Top Right Standard Fourier with 128 modes, Bottom Left : Standard Fourier with 256 modes, Bottom Right : Mapped Fourier with 32 modes

5 Conclusion and remarks

This preliminary work intends to demonstrate the possibilities of mapped Fourier methods for improving the computation in periodic domains of the solution of PDE when these solutions possesses internal layers of rapid variation. We have concentrated this study on the simple case where the localization of this zone is a priori known and do not evolve in time. This is a simplifying assumption but it already covers many practical situations. In particular, for Fusion Plasma, we believe that this technique can be applied to pellet injection simulations in tokamak with great benefit. Of course, this remains to be proved by using more sophisticated models than the one used here. This will be the purpose of a next series of investigations.

We also note that problem of adapting the mapping to the characteristics of the solution is also of great interest for the development of new numerical methods. The design of adaptive meshes linked with a priori or a posteriori error estimates is a mature field for local methods (FD/FE ones). This is not true for global (spectral) methods. In this area some works do exist,

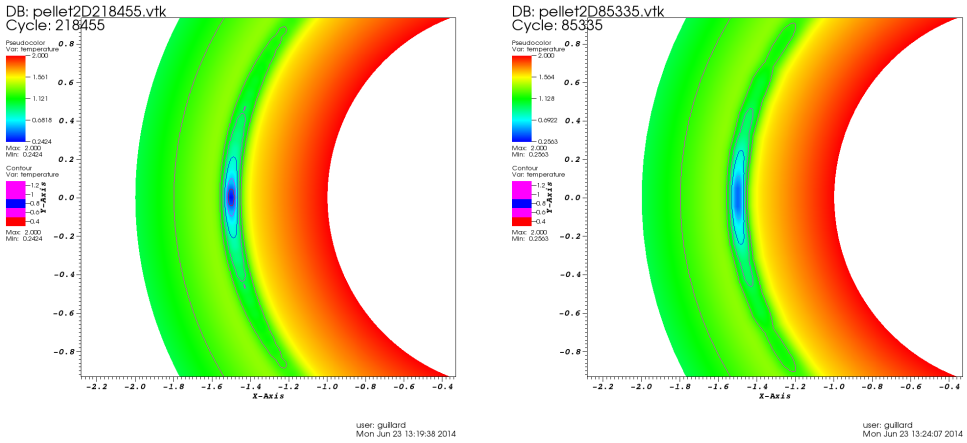

Figure 14: Left : Standard Fourier with 256 modes, Right : Mapped Fourier with 32 modes

see [7, 1] for instance. Boyd [3] table 16.2 gives a list of works performed in this direction. However, the majority of these works are in a simple 1D setting and real large scale applications using these techniques are still missing. We hope that this work can help to show that this area of investigation may have some interest.

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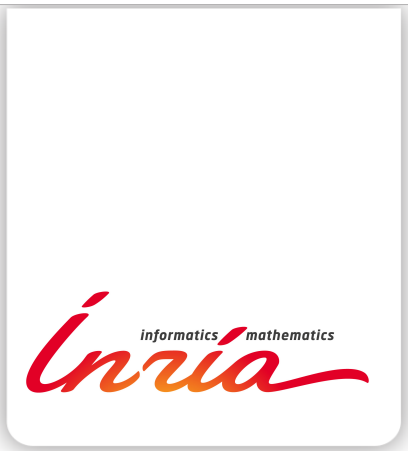

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