



**HAL**  
open science

# Investigation on High-Order Harmonic Generation of Guided Waves Using Local Computation Approaches: Theory and Comparison with Analytical Modelling

Rafal Radecki, Michael J. Leamy, Tadeusz Uhl, Wieslaw J. Staszewski,  
Zhongqing Su, Li Cheng, Pawel Packo

## ► To cite this version:

Rafal Radecki, Michael J. Leamy, Tadeusz Uhl, Wieslaw J. Staszewski, Zhongqing Su, et al.. Investigation on High-Order Harmonic Generation of Guided Waves Using Local Computation Approaches: Theory and Comparison with Analytical Modelling. EWSHM - 7th European Workshop on Structural Health Monitoring, IFFSTTAR, Inria, Université de Nantes, Jul 2014, Nantes, France. hal-01020401

**HAL Id: hal-01020401**

**<https://inria.hal.science/hal-01020401>**

Submitted on 8 Jul 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## INVESTIGATION ON HIGH-ORDER HARMONIC GENERATION OF GUIDED WAVES USING LOCAL COMPUTATION APPROACHES: THEORY AND COMPARISON WITH ANALYTICAL MODELLING

Rafal Radecki<sup>1</sup>, Michael J. Leamy<sup>2</sup>, Tadeusz Uhl<sup>3</sup>, Wieslaw J. Staszewski<sup>3</sup>, Zhongqing Su<sup>1,4,\*</sup>,  
Li Cheng<sup>1,4</sup>, Pawel Packo<sup>3</sup>

<sup>1</sup> Department of Mechanical Engineering, The Hong Kong Polytechnic University,  
Hung Hom, Kowloon, Hong Kong

<sup>2</sup> G. W. Woodruff School of Mechanical Engineering, Georgia Institute of Technology,  
Atlanta, GA, 30332, USA

<sup>3</sup> Department of Robotics and Mechatronics, AGH University of Science and Technology,  
Al. Mickiewicza 30, 30-059 Krakow, Poland

<sup>4</sup> The Hong Kong Polytechnic University Shenzhen Research Institute, Shenzhen 518057, China

\*mmsu@polyu.edu.hk

### ABSTRACT

The paper presents numerical modelling approaches for computing wave propagation and high-order harmonic generation in nonlinear media. This approach is based on two local computational methods, *i.e.* the Local Interaction Simulation Approach and the Cellular Automata for Elastodynamics. Discretized numerical dispersion curves are calculated and used for the entire analysis. A quantitative study of velocity matching between the fundamental and high-order harmonic waves is performed. The influence of propagation distance on the magnitude of high-order harmonics is evaluated for various excitation frequencies. Numerical analyses show divergence from analytical models for dispersion curves, especially for high-order harmonics. These preliminary investigations demonstrate that the numerical modelling approaches applied can be used effectively for nonlinear wavefield analysis; however, numerical instead of analytical spectral characteristics should be considered.

**KEYWORDS :** *guided elastic waves, nonlinear wavefield, modelling, numerical simulations, LISA, Cellular Automata*

### INTRODUCTION

Maintenance of engineering structures requires accurate and robust methods for damage detection and monitoring. Various approaches - based on elastic wave propagation - have been developed during the last few decades [1]. This includes widely-exploited methods based on guided ultrasonic waves used for damage detection in plate-like structures. Many research studies indicate that methods based on linear wave fields are often not sensitive to small damage severities. This important issue can be solved by increasing the frequency of propagating waves, so the relevant wavelengths are short enough to interact with possible small damage. However, simple wave propagation features - that relate to changes of amplitude and time-of-flight - are not easy to use in this case. This is mainly due to the multi-modal nature of guided ultrasonic waves that makes them difficult to analyse, particularly in the high-frequency range. Therefore more and more often nonlinear guided ultrasonic wave fields are considered for damage detection. High-order harmonic generation is one of the most commonly used approaches in this area of research, as reviewed in [2, 3]. It is well known that this nonlinear phenomenon results not only from structural discontinuities - due to damage-wave interactions [4,5] - but also from distortions of propagating waves due to material microstructures [6].

A major research effort is required to explain and separate both nonlinear phenomena. It is clear that a nonlinear framework of mechanics is needed. However, this is a challenging task since exact analytical solutions associated with the problem are difficult - and often not possible - to obtain in practice. It is clear that reliable and efficient numerical simulation tools are needed for the analysis of elastic wave propagation in nonlinear media. Recent years have shown very encouraging - in terms of accuracy and computational performance - research studies on local approaches for wave propagation modelling. Two methods are of particular interest in the work presented in this paper due their parallel computational algorithms. These are: the Local Interaction Simulation Approach (LISA) [7] and the Cellular Automata for Elastodynamics (CAFE) [8].

The paper investigates nonlinear wave propagation due to pre-defined nonlinear material properties. Numerical simulations - based on the LISA and CAFE - are used to analyse the second-order harmonic generation of guided ultrasonic waves. Amplitudes of the nonlinear second harmonic are analysed with respect to propagation distances and excitation frequencies. The effect of numerical discretization on guided wave propagation is also investigated. The results from numerical simulations are compared with analytical solutions to demonstrate the accuracy of the methods used. Analytical and numerical dispersion characteristics are also compared and discussed. The latter focuses on the selection of excitation frequency that is required for the highest possible energy transfer between the fundamental and second harmonic wave modes. The results from this selection analysis are in good agreement with the work presented in [9].

## 1. NUMERICAL MODELLING OF ELASTIC WAVES UNDER QUADRATIC CONSTITUTIVE RELATIONSHIP CONDITIONS

### 1.1 Theoretical background

It is well known that nonlinear propagation wave fields can involve various types of classical contact, geometrical and material nonlinearities. However, the work presented in this paper considers only the nonlinear constitutive behaviour. This is due to low excitation amplitudes investigated that preserve small response displacement amplitudes of the wave field and that no potential contacting interfaces are considered. With these assumptions, the constitutive relationship is considered nonlinear up to the quadratic term. The explicit relationship between stresses and strains involves the fourth order Hooke tensor  $\Lambda_4$ , and the sixth order tensor  $\Lambda_6$  related to this quadratic term. The material is assumed isotropic with the Lamé constants  $\lambda$  and  $\mu$  for the linear part, and three third-order Landau constants  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  for the quadratic term. Thus, the stress tensor can be defined as

$$\sigma = \Lambda_4 : \varepsilon + \frac{1}{2} \varepsilon : \Lambda_6 : \varepsilon \quad (1)$$

For numerical implementation, the general tensorial equation can be written in a matrix notation and for the two-dimensional case takes the following form

$$\begin{aligned} \sigma_{xx} &= (\lambda + 2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy} + (\mathcal{A} + 3\mathcal{B} + \mathcal{C})\varepsilon_{xx}^2 + (\mathcal{B} + \mathcal{C})\varepsilon_{yy}^2 + (\mathcal{A} + 2\mathcal{B})\varepsilon_{xy}^2 + (2\mathcal{B} + 2\mathcal{C})\varepsilon_{xx}\varepsilon_{yy} \\ \sigma_{yy} &= (\lambda + 2\mu)\varepsilon_{yy} + \lambda\varepsilon_{xx} + (\mathcal{A} + 3\mathcal{B} + \mathcal{C})\varepsilon_{yy}^2 + (\mathcal{B} + \mathcal{C})\varepsilon_{xx}^2 + (\mathcal{A} + 2\mathcal{B})\varepsilon_{xy}^2 + (2\mathcal{B} + 2\mathcal{C})\varepsilon_{xx}\varepsilon_{yy} \\ \sigma_{xy} &= 2\mu\varepsilon_{xy} + (\mathcal{A} + 2\mathcal{B})\varepsilon_{xy}\varepsilon_{xx} + (\mathcal{A} + 2\mathcal{B})\varepsilon_{xy}\varepsilon_{yy} \end{aligned} \quad (2)$$

Equation (2), along with the momentum balance and the geometrical relationship for strains and displacements, has been implemented in two numerical models based on local formulations, i.e. the LISA and CAFE. The details on the formulations can be found in [10].

## 1.2 Numerical simulations based on LISA

The LISA method, originally proposed in [11], has been used extensively for Lamb wave propagation modelling in damage detection studies [12, 13]. The method relies on the Finite Difference (FD) approximation of the governing equations and on a top-down modelling approach. Since the explicit central difference is used for the time domain model, the method is well suited for parallel processing. The parallel algorithm of LISA - implemented on graphical cards - has been recently demonstrated in [7]. The LISA can be used for wave propagation in any heterogeneous, anisotropic and nonlinear material of arbitrary shape and complexity. The method discretizes structures under investigation into a regular grid of rectangular cells. Then material properties are assumed to be constant within each cell but may differ between cells. For the 2-D case, each nodal point belongs to four cells. For any particular point, before the equation for the displacement vector is established, solutions for each cell are written and treated as discontinuous. Then the so-called Sharp Interface Model (SIM) is used - to match displacements and stresses - and the LISA iteration equations are derived. The SIM leads to more accurate results when wave propagation problems in complex media with complex boundaries are investigated.

The LISA iteration equations for the 2-D case are obtained merely by substituting the linear constitutive relation by the nonlinear one. In other words, Equation (2), written in terms of the second-order Lamé constants,  $\lambda$  and  $\mu$ , and the third-order Landau constants  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{C}$  is used for modelling. The same nonlinear formulation can be used to establish stress continuity equations across all cells' interfaces. The resulting differential equation involves extra terms containing the third-order constants, *i.e.*  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{C}$ , and higher powers of strain components. Then finite difference formulas are used to obtain algebraic iteration equations, by analogy with the linear case.

## 1.3 Numerical simulations based on CAFE

In contrast to the LISA methodology, the CAFE's treatment of elastic wave propagation avoids the use of the classical elastodynamic wave equation. Instead the CAFE relies on a bottom-up formulation in which the Newton's laws are applied directly to discrete cells [8, 14]. Then the application of simple, local interaction rules lead to complex global behaviour termed 'emergent'. For 2-D domains, triangles (or rectangles) discretize the domain into state-holding cells. Each triangular cell is treated as an autonomous state machine storing pointers to its local neighbours' state, avoiding the need for global, or centralized, control. The original CAFE approach considered rectangular cells and detailed its application to domains with extreme heterogeneity [8]. A follow-on effort extended the approach to triangular cells [14]. The triangular approach, as applied to homogeneous media, is overviewed herein. State variables stored by each triangular automata include the displacement, velocity, and applied tractions. In addition, cells store parameters such as material properties ( $\rho$ ,  $\mu$ ,  $\lambda$ ) and cell geometry. The displacements and applied tractions are determined at each step of the simulation and updated based on a rule set using the previous state of each automaton and that of its neighbours. The rule sets are derived directly from the Newton's second law for each cell. While evaluating the forces acting on a single automaton, the stresses on the faces are calculated and stored, which are then multiplied by the area of each face. The procedure leads to first-order, semi-discrete differential equations governing the cell's velocity and displacement change. A suitable explicit temporal discretization of these equations yields the final rule set. The target cell's updated state is stored until all cells have also been updated in what amounts to a double buffering technique suitable for parallelization.

The CAFE approach for quadratic nonlinearity between stresses and strains differs very little from the linear case. The only difference is the computation of tensile and shear stresses according to the nonlinear constitutive relationship given by Equation (2). All computations leading up to these equations (namely strain computations) remain unchanged. The details of nonlinear implementation can be found in [10].

## 2. INTERNAL RESONANCE CONDITIONS FOR NUMERICAL MODELS

The generation of second harmonic - due to structural nonlinearities - has been extensively studied for the last few years. Previous studies show [6,9,15] that nonlinear wave propagation involves the energy transfer between the fundamental and secondary wave modes. This phenomenon drives high-order wave field properties. A number of detailed analytical studies has been carried out. The work presented in [9] shows that in order to achieve the highest energy transfer between two wave modes, the relationship between two symmetric Lamb wave modes  $S_1$  and  $S_2$  should be considered only. These modes should exhibit the same phase velocity at excitation frequency for  $S_1$  and double excitation frequency for  $S_2$ . As a result, the condition for phase velocity matching was proposed. Subsequently, the analytical solution based on perturbation theory, presented in [6], shows that the phase velocity matching between the fundamental and secondary wave modes should be accompanied by the group velocity matching in order to maximise the energy transfer. The analysis in [6] also demonstrates the dependence of the second harmonic wave amplitude on propagation distances and energy flux conditions. When phase velocities of both considered modes are matched, the amplitude of secondary mode increases linearly with propagation distance. For the case of non-phase velocity matching, the absolute-sine pattern of secondary wave amplitude was observed. It was further analysed and proved that the only possible energy transfer occurs from the fundamental symmetric and antisymmetric modes, to the secondary symmetric modes. This is due to the symmetry of the third-order elastic constants [16,17].

Many practically oriented applications have been proposed and analysed (e.g. [18,19]) since the above developments. These investigations frequently employ numerical modelling prior to experimental testing to predict responses and to enhance signal processing techniques used. However, it is important to note that dispersion properties from numerical models are different from analytical formulations due to discretization involved. Thus, all numerical simulation results should be carefully evaluated and the dispersion errors analysed. The work in [20] shows that errors associated with guided wave propagation in numerical models can be qualitatively and quantitatively examined. This approach is adopted here in order to calculate dispersion properties for the nonlinear case and to analyse the relevant energy flux conditions.

## 3. NUMERICAL SIMULATIONS

This section presents and discusses the results from numerical simulations based on the LISA and CAFE methods. First, dispersion characteristics are calculated and analysed. Subsequently, displacement patterns of waves propagating in a plate structure are presented with the explanation of wave modes and the second harmonics possible extraction. Next, the amplitude dependence of secondary wave mode on propagation distances and excitation frequencies is shown. The relevant numerical and analytical results are compared.

A 4 mm thick and 500 mm long aluminium plate was examined through the thickness. The following material properties were selected for numerical simulations: Young's modulus  $E = 72.321 \text{ GPa}$ , Poisson ratio  $\nu = 0.3393$ , density  $\rho = 2727 \frac{\text{kg}}{\text{m}^3}$ . The Landau's coefficients describing the quadratic strain-stress relationship were:  $\mathcal{A} = -320 \text{ GPa}$ ,  $\mathcal{B} = -200 \text{ GPa}$ ,  $\mathcal{C} = -190 \text{ GPa}$ . For such a geometrical and material configuration, the phase- and group-velocity dispersion curves were calculated analytically. The numerical models involved 16 elements through the plate thickness. The dispersion curves calculated analytically and based on the LISA methods are shown in Figure 1.

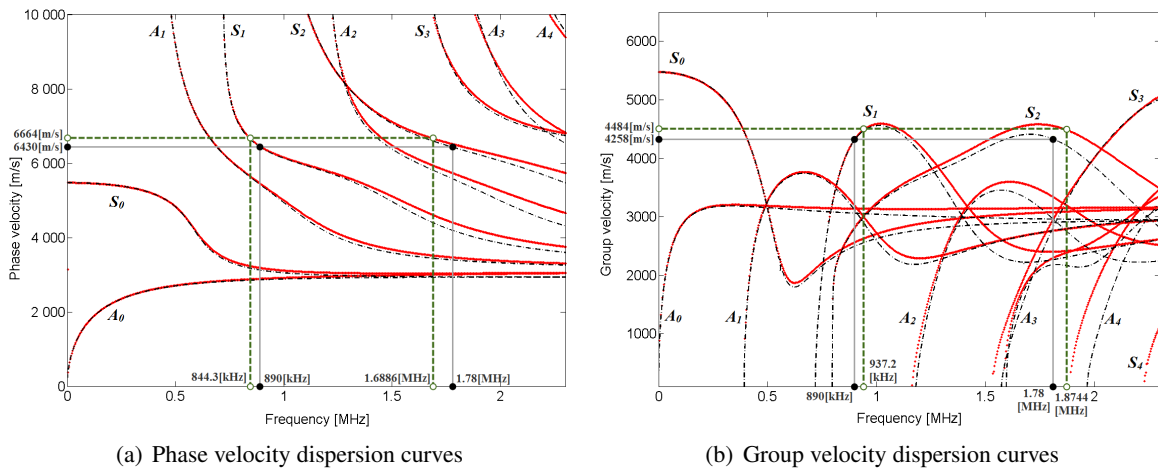


Figure 1: Dispersion curves for the 4 mm - thick plate : (---) - analytical solution; (●) - numerical solution using LISA with 16 elements throughout the plate thickness

Figure 1 shows that significant differences can be observed between the analytical and numerical formulations, particularly for the high frequency range. The mathematical model, indicates that in order to achieve the highest possible energy transfer between the primary and secondary modes, the excitation frequency equal to 890 kHz should be selected, as presented in Figure 1. This corresponds to phase- and group-velocity matching. However, since numerical models are used, the dispersion characteristics and the energy flux conditions are not satisfied. The results in Figure 1 show that the unified phase- and group-velocity matching does not exist for the numerical model. Instead, two distinct matching conditions for the excitation frequencies of 844.3 kHz and 937.2 kHz can be found based on the LISA dispersion characteristics. Clearly, discretization associated with numerical simulations leads to a significant deviation from the theoretical value of 890 kHz. This prevents the direct numerical and theoretical results evaluation.

In order to study wave propagation, particularly the internal resonance conditions, five excitation frequencies were used in the model based on the LISA numerical dispersion characteristics: 650, 844.3, 890, 937.2 and 1050 kHz. The third value (890 kHz) corresponds to the exact analytical matching. The second and fourth values (844.3 kHz and 937.2 kHz) correspond to numerical matching for the phase- and group-velocity in the LISA model, respectively. The remaining two frequencies (650 kHz and 1.05 MHz) were selected outside the matching range. For each model two excitation modes were considered, *i.e.* with +1 and -1 scaling factors for the amplitude. The excitation was applied as a prescribed displacement to a line of nodes along the 10 mm distance at the top surface of the far left end of the plate. The element size was set to 0.25 mm, leading to 16 elements through the thickness of the analysed plate. The time step - equal to 0.025 μs - was selected to ensure the stability of the explicit integration scheme. The results were acquired at the nodes on the top surface of the plate. Examples of the simulated wavefields are presented in Figure 2. The displacement field was further decomposed into symmetric and antisymmetric components and fundamental and high-order harmonics using the in- and out-of-phase calculations results.

Figure 2(a) shows the magnitude patterns for the total displacement. The decomposed symmetric and antisymmetric modes are given in Figures 2(b) and 2(c), respectively. The extracted 2<sup>nd</sup> harmonic symmetric and antisymmetric modes are given in Figures 2(d) and 2(e), respectively. The 2<sup>nd</sup> waves are presented for the excitation frequency corresponding to the numerical phase velocity matching.

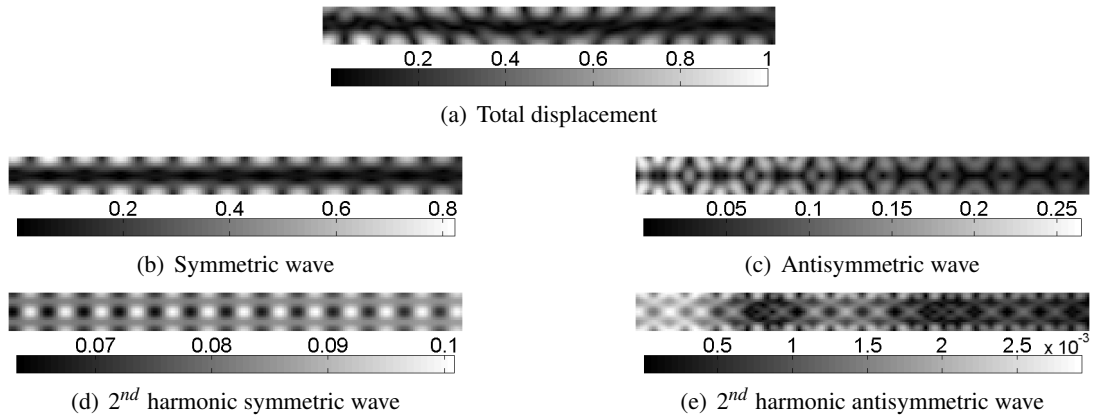
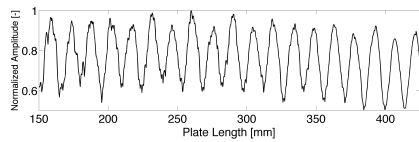


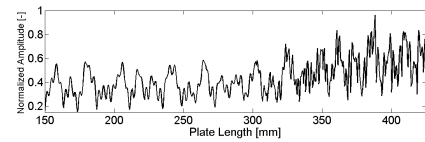
Figure 2: Normalized displacement magnitudes for simulated wave propagating from left to right at  $150\mu s$ . The responses were gathered between  $400$  and  $450mm$ , for the excitation frequency equal to  $f = 844.3kHz$ .

Interestingly, the results in Figure 2(a) show that despite the fact that the theoretical (analytical) condition has not been met, the energy transfer between the fundamental and higher harmonic antisymmetric modes can be observed. However, the amplitude of the 2<sup>nd</sup> harmonic antisymmetric wave decreases rapidly with the propagation distance. Conversely, the amplitude of the 2<sup>nd</sup> harmonic symmetric mode amplitude is increasing for the selected excitation frequency, as this frequency corresponds well to the phase matching condition.

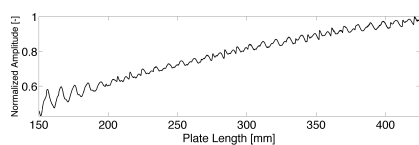
The second harmonic displacement patterns for various excitation frequencies are shown in Figure 3. Figure 3(e) and 3(f) demonstrates that the theoretically calculated matching condition does not apply to the numerical models. The second harmonic symmetric wave reveal sinusoidally changing or constant amplitude patterns for the LISA and the CAFE, respectively. A similar observation can be made for the numerical group velocity matching (Figure 3(g) and 3(h)). These results correspond to the auxiliary condition for increased energy transfer to the higher-harmonics. Clearly, an increasing amplitude can be observed for the excitation frequency  $f = 844.3kHz$  for the LISA model for which the numerical matching occurred. For the CAFE a slightly increasing trend can be observed, however it should be noted that the excitation frequency was not adjusted to exact spectral characteristics of the model in this case. For the remaining cases, namely the excitation frequency  $f = 650kHz$  and  $f = 1.05MHz$ , where neither the phase- nor group-velocity matching for the analysed model occur, the 2<sup>nd</sup> harmonic amplitude oscillates along plate's length dimension.



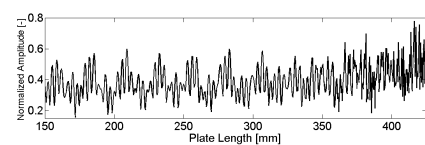
(a) The LISA results for  $f = 650kHz$



(b) The CAFE results for  $f = 650kHz$



(c) The LISA results for  $f = 844.3kHz$



(d) The CAFE results for  $f = 844.3kHz$

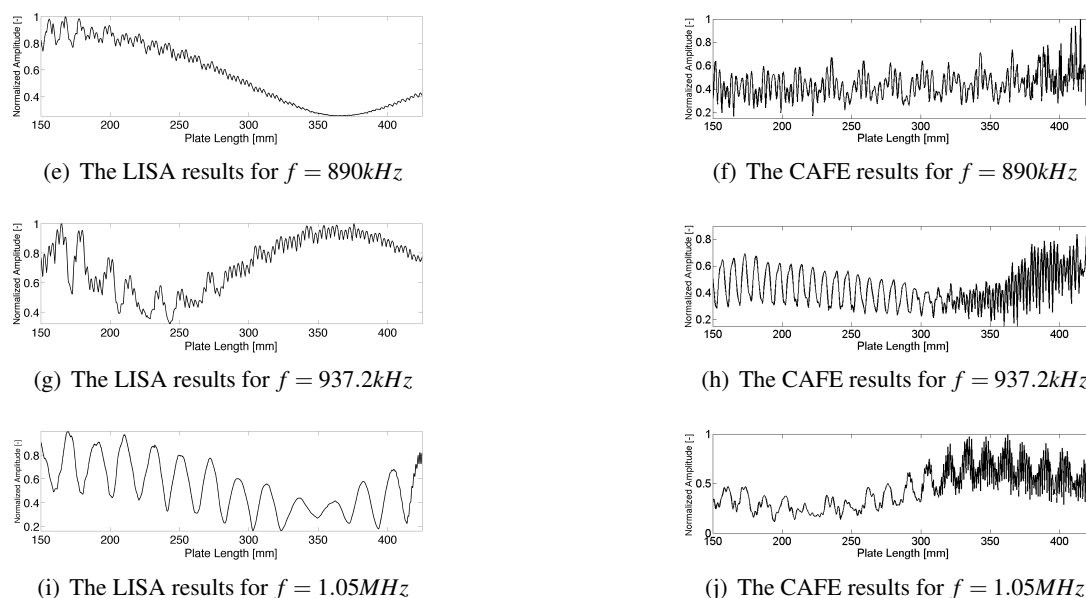


Figure 3: The 2<sup>nd</sup> harmonic amplitude along the plate's length. The LISA (left column) and CAFE (right column) numerical simulation results results for various excitation frequencies selected.

## CONCLUSIONS

Nonlinear guided wave propagation in thin plates has been investigated. Numerical simulations - based on two local approaches, i.e. LISA and CAFE - have been compared with analytical solutions.

The energy transfer from the fundamental to high-order modes has been observed, as expected. However, numerical simulations show that the effectiveness of the energy transfer depends on the excitation frequency. For certain values of frequency the internal resonance is satisfied and the continuous increase in amplitude of higher harmonics is noted. The internal resonance conditions can be calculated using the dispersion relationship for a particular set-up. It was shown that the exact spectral characteristics should be calculated individually for a particular numerical framework at hand.

The analysis presented also demonstrates that the simulated dispersion curves do not match the analytical characteristics, particularly in the high frequency range. Thus the analytical resonance conditions do not apply. The results shown that matching does not occur in numerical simulations for the theoretically calculated phase- and group-velocity matching frequency conditions. The unified internal resonance conditions do not exist for the numerical model. However, the dispersion curves for the model discretized using 16 elements through the plate thickness indicated separate conditions for the phase- and group-velocity matching. The analysis of 2<sup>nd</sup> harmonic amplitude confirmed the maximised energy transfer for the numerical phase matching. Although the theoretical and numerical conditions differ by approximately 50kHz (i.e. 5% of the excitation frequency value), the amplitude pattern of the 2<sup>nd</sup> harmonic is substantially different. Finally, contrary to the mathematical model, the energy transfer to the high-order antisymmetric modes is observed in the simulations.

## ACKNOWLEDGEMENT

The work presented in the paper was supported by the National Natural Science Foundation of China (Grant No. 51375414 and 11272272) and the Hong Kong Research Grants Council via a General Research Fund (GRF) (No. 523313).



## REFERENCES

- [1] T. Stepinski, T. Uhl, and W. J. Staszewski. *Advanced Structural Damage Detection: From Theory to Engineering Applications*. John Wiley & Sons Ltd, 2013.
- [2] K. Y. Jhang. Nonlinear Ultrasonic Techniques for Non-Destructive Assessment of Micro Damage in Material: A Review. *International Journal of Precision Engineering and Manufacturing*, 10(1):123–135, 2009.
- [3] D. Broda, W. J. Staszewski, A. Martowicz, T. Uhl, and V. V. Silberschmidt. Modelling of Nonlinear CrackWave Interactions for Damage Detection Based on Ultrasound A Review. *Journal of Sound and Vibration*, 333(14):10971118, 2014.
- [4] J. H. Cantrell and W. T. Yost. Nonlinear Ultrasonic Characterization of Fatigue Microstructures. *International Journal of Fatigue*, 23(1):487–490, 2001.
- [5] J.-Y. Kim, L. J. Jacobs, J. Qu, and J. W. Littles. Experimental Characterization of Fatigue Damage in a Nickel-Base Superalloy Using Nonlinear Ultrasonic Waves. *Journal of Acoustical Society of America*, 120(3):1266–1273, 2006.
- [6] M. F. Müller, J.-Y. Kim, J. Qu, and L. J. Jacobs. Characteristics of Second Harmonic Generation of Lamb Waves in Nonlinear Elastic Plates. *Journal of the Acoustical Society of America*, 127:2141–2152, 2010.
- [7] P. Packo, T. Bielak, A. B. Spencer, W. J. Staszewski, T. Uhl, and K. Worden. Lamb Wave Propagation Modelling and Simulation Using Parallel Processing Architecture and Graphical Cards. *Smart Materials and Structures*, 21, 2012.
- [8] M. J. Leamy. Application of Cellular Automata Modeling to Seismic Elastodynamics. *International Journal of Solids and Structures*, 47:4835–4849, 2008.
- [9] Y. Liu, V. K. Chillara, and C. J. Lissenden. On Selection of Primary Modes for Generation of Strong Internally Resonant Second Harmonics in Plate. *Journal of Sound and Vibration*, 332(19):4517–4528, 2013.
- [10] M. J. Leamy, T. B. Atrusson, W. J. Staszewski, T. Uhl, and P. Packo. Local Computational Strategies for Predicting Wave Propagation in Nonlinear Media. In T. Kundu, editor, *Health Monitoring of Structural and Biological Systems 2014*, volume 9064. SPIE, 2014.
- [11] P. P. Delsanto, R. S. Schechter, H. H. Chaskelis, R. B. Mignogna, and R. Kline. Connection Machine Simulation of Ultrasonic Wave Propagation in Materials. II: The Two-Dimensional Case. *Wave Motion*, 20(4):295–314, 1994.
- [12] B. C. Lee and W. J. Staszewski. Lamb Wave Propagation Modelling for Damage Detection: I. Two-Dimensional Analysis. *Smart Materials and Structures*, 16(2):249259, 2007.
- [13] B. C. Lee and W. J. Staszewski. Lamb Wave Propagation Modelling for Damage Detection: II. Damage Monitoring Strategy. *Smart Materials and Structures*, 16(2):260–274, 2007.
- [14] R. K. Hopman and M. J. Leamy. Triangular Cellular Automata for Computing Two-Dimensional Elastodynamic Response on Arbitrary Domains. *Journal of Applied Mechanics*, 78(2), 2011.
- [15] Ch. Zhou, M. Hong, Z. Su, Q. Wang, and L. Cheng. Evaluation of Fatigue Cracks Using Nonlinearities of Acousto-Ultrasonic Waves Acquired by an Active Sensor Network. *Smart Materials and Structures*, 22, 2013.
- [16] M. Deng. Cumulative Second-Harmonic Generation of Lamb-Mode Propagation in a Solid Plate. *Journal of Applied Physics*, 85(6):3051–3058, 1999.
- [17] T. B. Atrusson, K. G. Sabra, and M. J. Leamy. Reflection of Compressional and Rayleigh Waves on the Edges of an Elastic Plate with Quadratic Nonlinearity. *Journal of Acoustical Society of America*, 131(3):1928–1937, 2012.
- [18] A. Klepka, W. J. Staszewski, R. B. Jenal, M. Szewdo, J. Iwaniec, and T. Uhl. Nonlinear Acoustics for Fatigue Crack Detection Experimental Investigations of Vibro-Acoustic Wave Modulations. *Structural Health Monitoring*, 11(2):197–211, 2012.
- [19] P. Puthillatha, J. M. Galan, B. Ren, C. J. Lissenden, and J. L. Rose. Ultrasonic Guided Wave Propagation Across Waveguide Transitions: Energy Transfer and Mode Conversion. *Journal of Acoustical Society of America*, 133(5):2624–2633, 2013.
- [20] P. Packo, T. Uhl, and W. J. Staszewski. A General Semi-Analytical Finite Difference Method for Dispersion Curves Calculation and Numerical Dispersion Analysis for Lamb Waves. Submitted to *Journal of the Acoustical Society of America*.