

The maximum degree of a random Delaunay triangulation

Nicolas Broutin¹ Olivier Devillers² Ross Hemsley²

¹Inria Paris - Rocquencourt ²Inria Sophia Antipolis - Méditerranée

Objective

We give a new polylogarithmic bound on the maximum degree of a random Delaunay triangulation in a smooth convex, that holds with probability one as the number of points goes to infinity. In particular, our new bound holds even for points arbitrarily close to the boundary of the domain. *The proof can be found in [2].*

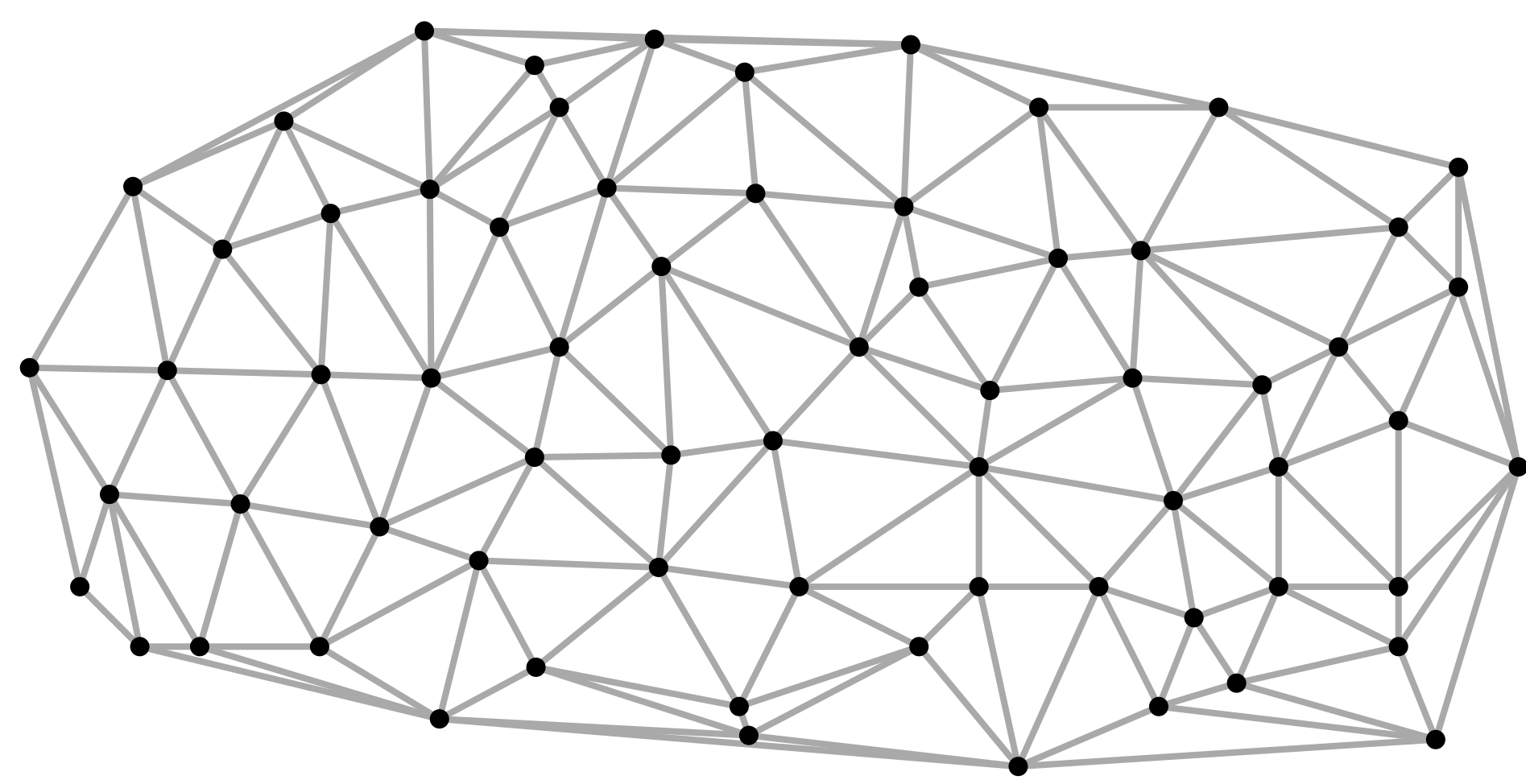


Figure 1: A Delaunay triangulation.

Introduction

The Delaunay triangulation is a fundamental data structure in computational geometry, with applications in shape reconstruction, interpolation, routing in networks and countless others. Many algorithms applied to the Delaunay triangulation are in some way dependent on the degree of the vertices, and so bounding the maximum degree of a triangulation implies useful bounds on the complexity of other geometric algorithms. For the Delaunay triangulation of n points, the max degree is easily seen to be $n - 1$ since, $n - 1$ points may be arranged in a circle around a central *hub*, as in Figure 3. For most applications, this bound is far too pessimistic, and so it is desirable to seek bounds which are more realistic.

Previous Results

For Γ a Poisson process on \mathbb{R}^2 with intensity 1, Bern et al. [1] bound the expected maximum degree of any vertex of the Delaunay triangulation $DT(\Gamma)$ falling within the box $[0, \sqrt{n}]^2$,

$$\mathbb{E} \left[\max_{x \in \Gamma \cap [0, \sqrt{n}]^2} \deg(x) \right] = \Theta \left(\frac{\log n}{\log \log n} \right) \quad (1)$$

This bound has a few properties that limit its utility in practice:

- ❶ It does not deal with ‘boundary effects’.
- ❷ It only gives the expectation.
- ❸ It is only given for a set of points generated by a Poisson process.

Contributions

We provide a new bound for the maximum degree for the Delaunay triangulation. Consider the set $\mathbf{X} := \{X_1, X_2, \dots, X_n\}$ of n points uniformly distributed in a smooth convex $\mathcal{D} \subset \mathbb{R}^2$, we demonstrate that for any $\xi > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\max_{x \in \mathbf{X}} \deg(x) \geq \log^{2+\xi} n \right) = 0 \quad (2)$$

This bound ensures that with high probability, no point has very high degree in the triangulation for n large enough, even near the border. We conjecture that this result could be improved to match Equation (1).

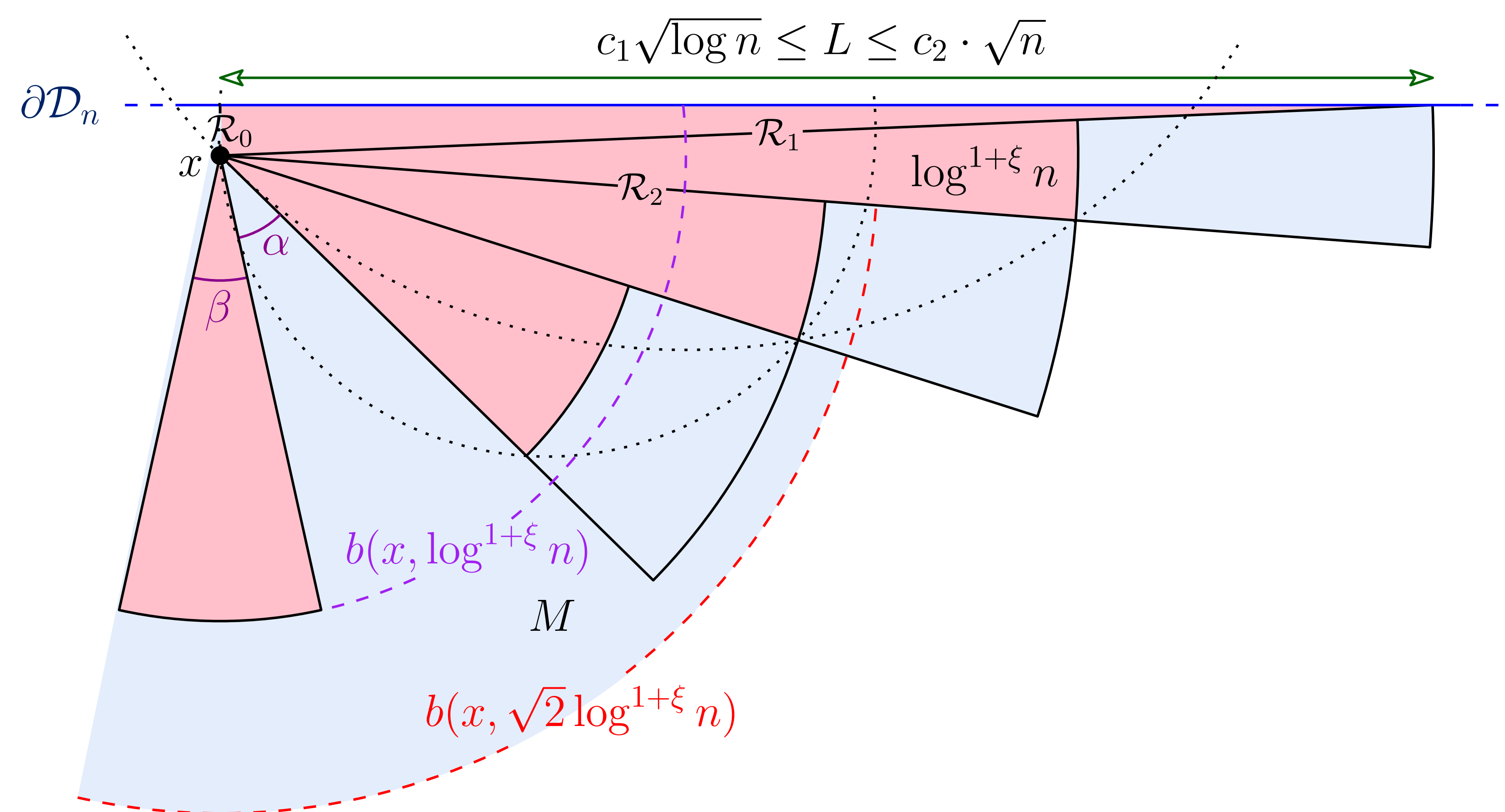


Figure 2: The proof works by computing the probability that carefully chosen regions about a point are empty. The details may be found in [2].

Application

Consider the *vertex deletion* operation, which removes a vertex, x , from $DT(\mathbf{X})$ and outputs $DT(\mathbf{X} \setminus x)$. It is well-known that this can be computed in $O(\deg(x))$ [3]. Unfortunately, the bound in Equation 1 is not sufficient to bound the worst-case run-time of this algorithm on a finite set of random points, since no information is given about points near the boundary. Our results guarantee that with probability tending to one, no vertex exists in the domain that can’t be removed in fewer than $O(\log^{2+\xi} n)$ steps.

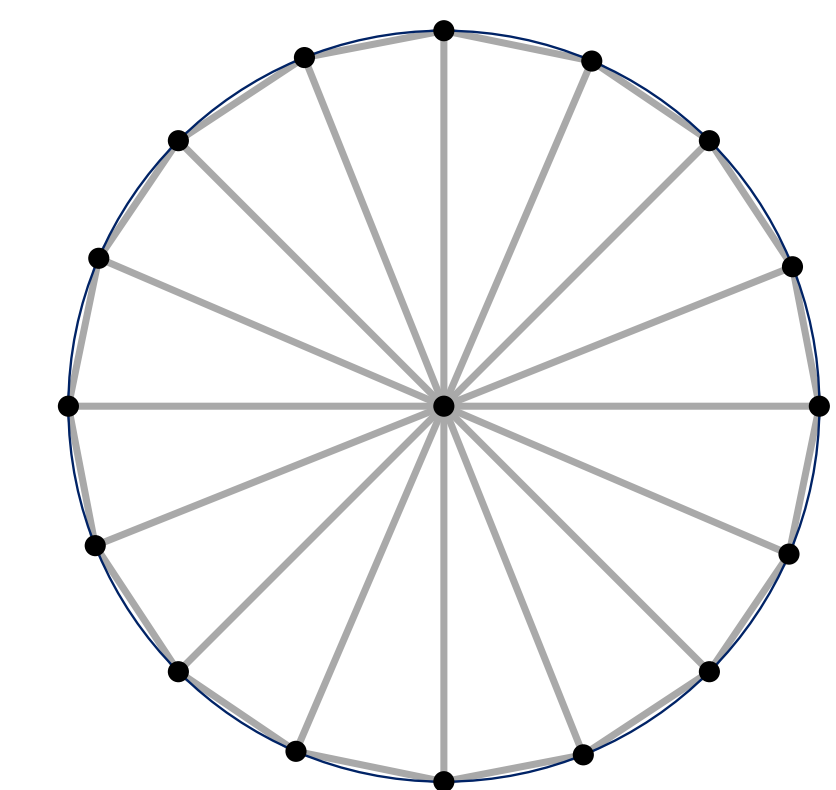


Figure 3: Worst case max degree for a Delaunay triangulation.

References

- [1] M. W. Bern, D. Eppstein, and F. F. Yao. The expected extremes in a Delaunay triangulation. In *ICALP*, pages 674–685, 1991.
- [2] N. Broutin, O. Devillers, and R. Hemsley. Efficiently navigating a random Delaunay triangulation. *ArXiv e-prints*, Feb. 2014.
- [3] O. Devillers. Vertex removal in two-dimensional Delaunay triangulation: Speed-up by low degrees optimization. *Comput. Geom. Theory Appl.*, 44(3):169–177, Apr. 2011.