

A chaotic random convex hull

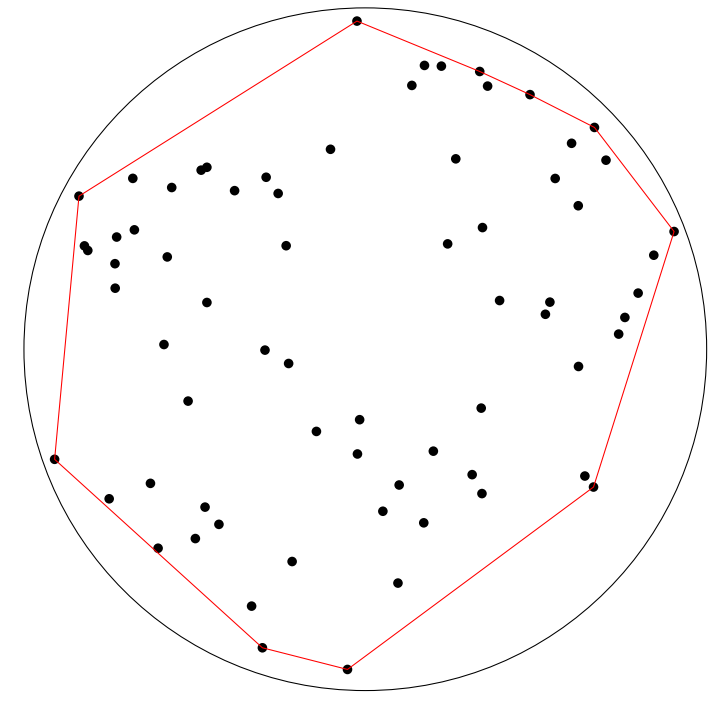
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Introduction

- ▶ $K \subset \mathbb{R}^d$ a convex body
- ▶ (x_1, \dots, x_n) a sample of n points uniformly chosen i.i.d. in K
- ▶ K_n the convex hull of these points
- ▶ $f_0(K_n)$ its number of vertices.



The expected value of $f_0(K_n)$ is well known [3]:

If K is a **polytope**:

$$\mathbb{E}f_0(K_n) = c_{d,K} \log^{d-1} n + o(\log^{d-1} n)$$

If K is a **smooth convex body** (\mathcal{C}^2 boundary, positive Gaussian curvature):

$$\mathbb{E}f_0(K_n) = c_{d,K} n^{\frac{d-1}{d+1}} + o(n^{\frac{d-1}{d+1}}).$$

- ▶ If the convex K is not a polytope or smooth: behavior **unpredictable** [1].
- ▶ Most convexes **oscillate between these two behaviors**.

Result

We propose a reasonably **constructive description of a convex body having an oscillating behavior**.

Approximations of convex bodies

Lemma 1

$K \subset \mathbb{R}^d$ a convex body, $\varepsilon > 0$, $N \in \mathbb{N}$.

There exist $p > N$ and $\alpha > 0$ such that:

\forall compact set $L \supset K$, $d_H(K, L) < \alpha$ (Hausdorff distance),

$$\frac{\mathbb{E}f_0(L_p)}{\mathbb{E}f_0(K_p)} \in [1 - \varepsilon, 1 + \varepsilon]$$

Idea:

- ▶ If L is very close to K , points chosen uniformly in L are in $L \setminus K$ very rarely.

Oscillations

Let $G : \mathbb{N}^* \rightarrow \mathbb{R}_+^*$ be an increasing function such that $G(n) \rightarrow_{n \rightarrow \infty} \infty$.

We can construct a convex body K such that:

$\forall N \in \mathbb{N}^*$, $\exists M_1, M_2 > N$, verifying

$$\mathbb{E}f_0(K_{M_1}) < G(M_1) \log^{d-1} M_1$$

and

$$\mathbb{E}f_0(K_{M_2}) > G(M_2)^{-1} M_2^{\frac{d-1}{d+1}}$$

Bibliography

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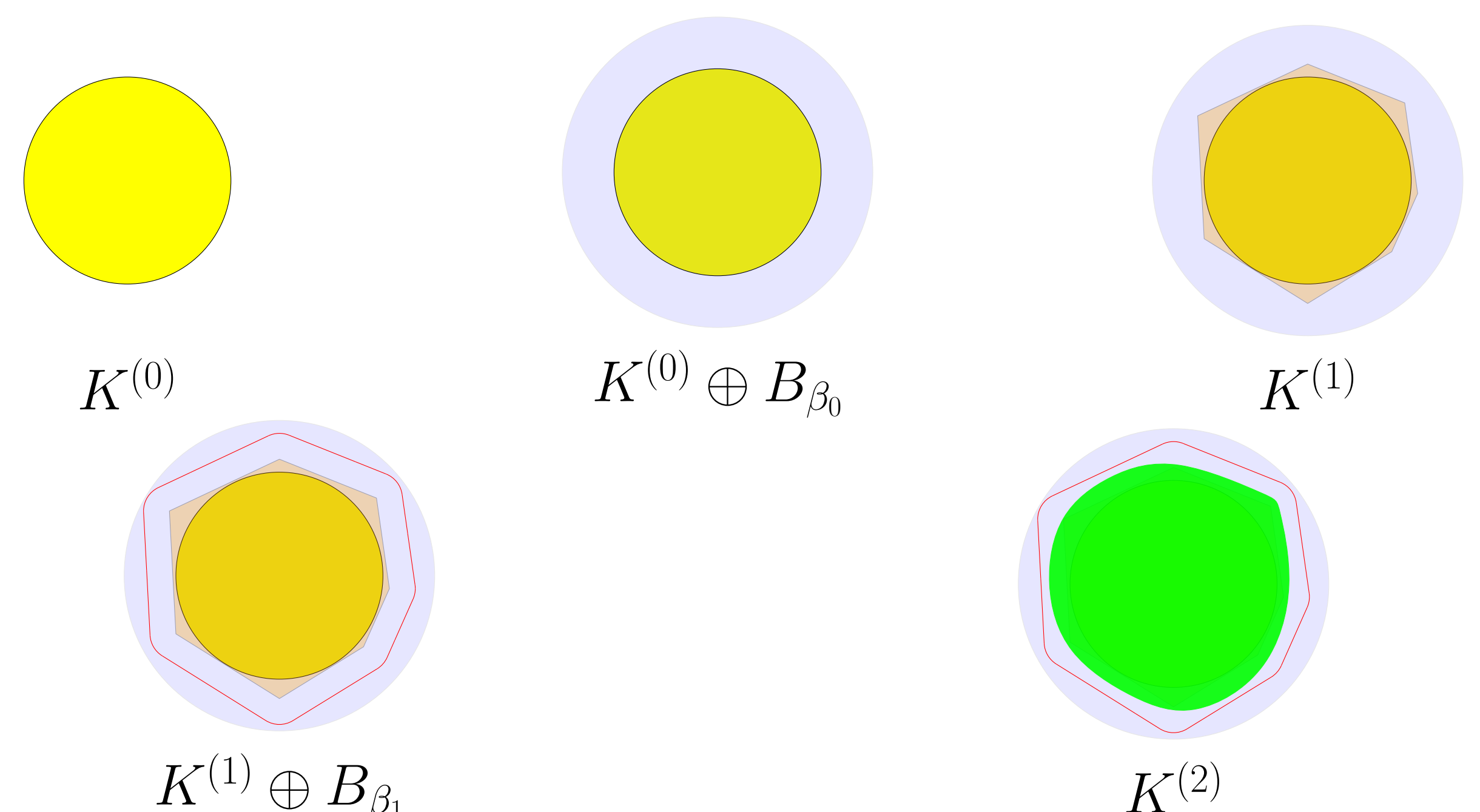
Iterative construction

Idea:

- ▶ Starting from a convex body $K^{(0)}$,
- ▶ Iterate smooth and polytopal approximations.
- ▶ Lemma 1 gives some range of n where the behavior of the random convex body will be:
 - ▶ very close to $n^{\frac{d-1}{d+1}}$ (**smooth convex body behavior**)
 - ▶ or very close to $\log^{d-1} n$ (**polytope behavior**).

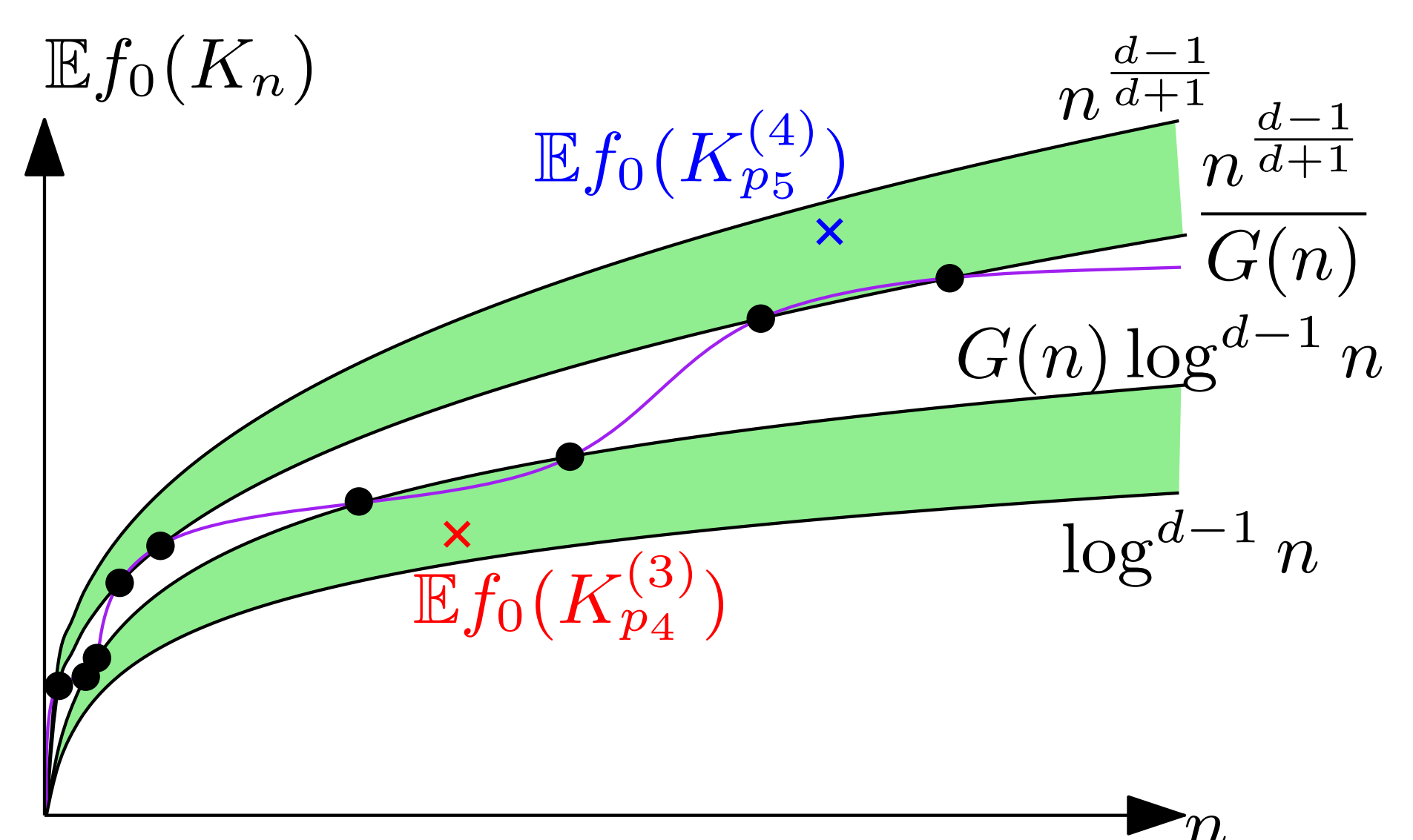
Increasing sequence of convex bodies

- ▶ Start from the unit ball, alternate **polytopal** and **smooth** approximations.
- ▶ $K^{(0)} := \text{unit ball}$, $p_0 := 1$.
- ▶ For all $i \in \mathbb{N}^*$, $K^{(i)}$ is an approximation of $K^{(i-1)}$ where $d_H(K^{(i-1)}, K^{(i)}) < \beta_{i-1}$, with $(\beta_i)_{i \in \mathbb{N}}$ some decreasing sequence.



- ▶ i **odd**, $\implies K^{(i-1)}$ smooth convex body.
- ▶ Lemma 1: $\exists \alpha_{i-1}, p_i > p_{i-1}$ such that $\forall L \supset K^{(i-1)}$ compact set, $d_H(K^{(i-1)}, L) < \alpha_{i-1}$:

$$\mathbb{E}f_0(L_{p_i}) > p_i^{\frac{d-1}{d+1}} G(p_i)^{-1}. \quad (1)$$
- ▶ $\beta_{i-1} := \min(\frac{\alpha_{i-1}}{2}, \frac{\beta_{i-2}}{2})$ if $i > 1$, $\beta_0 := \frac{\alpha_0}{2}$.
- ▶ We define $K^{(i)}$ as a convex polytope with $d_H(K^{(i-1)}, K^{(i)}) < \beta_{i-1}$, so (1) works for $L = K^{(i)}$.
- ▶ i **even** $\implies K^{(i-1)}$ convex polytope.
- ▶ $K^{(i)}$ a smooth approximation constructed in a similar manner.
- ▶ The convex body $K := \overline{\cup_{i=0}^{\infty} K^{(i)}}$ verifies the wanted properties [2].



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