



**HAL**  
open science

## A chaotic random convex hull

Olivier Devillers, Marc Glisse, Rémy Thomasse

► **To cite this version:**

Olivier Devillers, Marc Glisse, Rémy Thomasse. A chaotic random convex hull. AofA 2014- 25th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms, Jun 2014, Paris, France. hal-01015598

**HAL Id: hal-01015598**

**<https://inria.hal.science/hal-01015598v1>**

Submitted on 26 Jun 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A chaotic random convex hull<sup>†</sup>

Olivier Devillers<sup>1</sup> and Marc Glisse<sup>2</sup> and Rémy Thomasse<sup>1</sup>

<sup>1</sup>INRIA Sophia Antipolis - Méditerranée

<sup>2</sup>INRIA Saclay - Île de France

---

**Abstract.** The asymptotic behavior of the expected size of the convex hull of uniformly random points in a convex body in  $\mathbb{R}^d$  is polynomial for a smooth body and polylogarithmic for a polytope. We construct a body whose expected size of the convex hull oscillates between these two behaviors when the number of points increases.

**Keywords:** convex hull, random point distribution, geometric probability

---

## 1 Introduction

Consider a sequence of points in a convex body in dimension  $d$  whose convex hull is dynamically maintained when the points are inserted one by one, the convex hull size may increase, decrease, or remain constant when a new point is added. Studying the expected size of the convex hull when the points are evenly distributed in the convex is a classical problem of probabilistic geometry that yields some surprising facts. For example, although it seems quite natural to think that the expected size of the convex hull is increasing with  $n$  the number of points, this fact is only formally proven for  $n$  big enough Devillers et al. (2013). The asymptotic behavior of the expected size is known to be polylogarithmic for a polytopal body and polynomial for a smooth one. If for a polytope or a smooth body, the asymptotic behavior is *somehow* "nice", for "most" convex bodies the behavior is unpredictable (Bárány, 1989, corollary 3). It is possible to construct strange convex objects that have no such nice behaviors and this note exhibits a convex body, such that the behavior of the expected size of a random polytope oscillates between the polytopal and smooth behaviors when  $n$  increases.

More formally, let  $K$  be a convex body in  $\mathbb{R}^d$  and  $(x_1, \dots, x_n)$  a sample of  $n$  points chosen uniformly and independently at random in  $K$ . Let  $K_n$  be the convex hull of these points and  $f_0(K_n)$  the number of vertices of  $K_n$ .

It is well known Bárány (2008); Reitzner (2010) that if  $P$  is a polytope, then

$$\mathbb{E}f_0(P_n) = c_{d,P} \log^{d-1} n + o(\log^{d-1} n) \quad (1)$$

and if  $K$  is a smooth convex body (i.e with  $\mathcal{C}^2$  boundary with a positive Gaussian curvature), then

$$\mathbb{E}f_0(K_n) = c_{d,K} n^{\frac{d-1}{d+1}} + o(n^{\frac{d-1}{d+1}}) \quad (2)$$

---

<sup>†</sup>Supported by ANR blanc PRESAGE (ANR-11-BS02-003) and Région PACA

where  $c_{d,P}$  and  $c_{d,K}$  are constants depending only on  $d$  and on the convex body.

These are the two extreme behaviors : every random polytope of a convex body in  $\mathbb{R}^d$  has a behavior between (1) and (2) for  $n$  large enough (Bárány, 1989, corollary 3).

For general convex bodies, we cannot expect such a beautiful formula.

**Theorem 1 (Bárány-Larman, 1988)** *For any function  $G(n) \rightarrow_{n \rightarrow \infty} \infty$  and for most (in the Baire category sense) convex bodies  $K$  in  $\mathbb{R}^d$ ,*

$$G(n) \log^{d-1} n > \mathbb{E}f_0(K_n)$$

for infinitely many  $n$  and

$$G(n)^{-1} n^{\frac{d-1}{d+1}} < \mathbb{E}f_0(K_n)$$

for infinitely many  $n$ .

Note that this "most" does not contain convex polytopes and smooth convex bodies, which are the most used in practice.

In this paper, we present an explicit example of a convex body which has this chaotic behavior.

**Notations** Let's introduce some notations used in this paper:

- $\mathcal{V}(K)$  will denote the  $d$ -dimensional volume of the convex body  $K$ ;
- $K \oplus L$  will denote the Minkowski sum of  $K$  and  $L$ , defined as

$$J \oplus K = \{x + y \mid x \in J, y \in K\};$$

- $d_H(J, K)$  will denote the Hausdorff distance of the convex bodies  $K$  and  $L$ , defined as

$$d_H(J, K) = \min\{r \in \mathbb{R}^+ \mid J \subset K \oplus B_r, K \subset J \oplus B_r\}$$

where  $B_r$  is the Euclidean ball centered in 0 with radius  $r$  in  $\mathbb{R}^d$ .

## 2 Approximations of convex bodies

In this section, we present an intermediate lemma about random polytopes of close (in terms of Hausdorff distance) convex bodies. Let  $K$  be a convex body in  $\mathbb{R}^d$  and  $K_n$  be a random polytope in  $K$ . We want to show that if  $L$  is an approximation of  $K$  with small Hausdorff distance,  $L_n$  is approximating the asymptotic behavior of  $K_n$  for some value of  $n$ .

Let's assume that the expected size of  $K_n$  is in  $c_{d,K}g(n, d) + o(g(n, d))$ , where  $c_{d,K}$  is a constant and  $g$  some function.

Then, for every close-enough compact set  $L$  containing  $K$ ,  $L_n$  has an expected size as close as we want from  $c_{d,K}g(n, d)$  for values of  $n$  as big as we want.

The idea of this lemma is very simple: if  $L$  is very close to  $K$ , the volume in  $L \setminus K$  is small, and points chosen uniformly in  $L$  are very unlikely to be in  $L \setminus K$ . Then, even if asymptotically the expected size of  $L_n$  is different from the size of  $K_n$ , there exists some  $n$  where the expected size of  $L_n$  is as close as we want to  $c_{d,K}g(n, d)$ .

**Lemma 2** Let  $K$  be a convex body in  $\mathbb{R}^d$  such that

$$\mathbb{E}f_0(K_n) = c_{d,K}g(n, d) + o(g(n, d)). \quad (3)$$

Let  $\varepsilon > 0$  and  $N \in \mathbb{N}$ .

Then, there exist  $p > N$  and  $\alpha > 0$  such that for any compact set  $L$  containing  $K$  with  $d_H(K, L) < \alpha$ ,

$$\frac{\mathbb{E}f_0(L_p)}{c_{d,K}g(p, d)} \in [1 - \varepsilon, 1 + \varepsilon].$$

**Proof:** First, for all  $n \in \mathbb{N}^*$

$$\mathbb{E}f_0(L_n) = \mathbb{P}(L_n \subset K)\mathbb{E}(f_0(L_n)|L_n \subset K) + \mathbb{P}(L_n \not\subset K)\mathbb{E}(f_0(L_n)|L_n \not\subset K).$$

As the points are uniformly distributed,  $\mathbb{E}(f_0(L_n)|L_n \subset K) = \mathbb{E}(K_n)$ .

Using (3), let's choose  $p$  such that

$$\frac{\mathbb{E}f_0(K_p)}{c_{d,K}g(p, d)} \in \left[1 - \frac{\varepsilon}{2}, 1 + \frac{\varepsilon}{2}\right], \quad (4)$$

then

$$\frac{\mathbb{P}(L_p \subset K)\mathbb{E}(f_0(L_p)|L_p \subset K)}{c_{d,K}g(p, d)} \leq 1 + \frac{\varepsilon}{2}.$$

As

$$\mathbb{P}(L_p \not\subset K) = 1 - \mathbb{P}(L_p \subset K) = 1 - \left(\frac{\mathcal{V}(K)}{\mathcal{V}(L)}\right)^p$$

we get

$$\mathbb{P}(L_p \not\subset K)\mathbb{E}(f_0(L_p)|L_p \not\subset K) \leq \left(1 - \left(\frac{\mathcal{V}(K)}{\mathcal{V}(L)}\right)^p\right)p.$$

Now, as  $1 \geq \frac{\mathcal{V}(K)}{\mathcal{V}(L)} \geq \frac{\mathcal{V}(K)}{\mathcal{V}(K \oplus B_\alpha)} \rightarrow_{\alpha \rightarrow 0} 1$  we can choose  $\alpha$  such that

$$\left(\frac{\mathcal{V}(K)}{\mathcal{V}(L)}\right)^p \geq \max\left(1 - c_{d,K} \frac{g(p, d) \varepsilon}{p}, 1 - \frac{\varepsilon}{2}\right). \quad (5)$$

Finally,

$$\mathbb{E}(f_0(L_p)) \leq c_{d,K}g(p, d) \left(1 + \frac{\varepsilon}{2}\right) + c_{d,K}g(p, d) \frac{\varepsilon}{2} = c_{d,K}g(p, d)(1 + \varepsilon). \quad (6)$$

For the lower bound, using (4) and (5) we get

$$\begin{aligned} \mathbb{E}f_0(L_p) &\geq \mathbb{P}(L_p \subset K)\mathbb{E}(f_0(L_p)|L_p \subset K) \\ &\geq c_{d,K}g(p, d) \left[\left(\frac{\mathcal{V}(K)}{\mathcal{V}(L)}\right)^p \left(1 - \frac{\varepsilon}{2}\right)\right] \\ &\geq c_{d,K}g(p, d) \left[\left(\frac{\mathcal{V}(K)}{\mathcal{V}(L)}\right)^p - \frac{\varepsilon}{2}\right] \\ &\geq c_{d,K}g(p, d) \left(1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2}\right) \\ &= c_{d,K}g(p, d)(1 - \varepsilon). \end{aligned} \quad (7)$$

Inequalities (7) and (6) prove the lemma.  $\square$

### 3 Construction of the convex body

Given an increasing function  $G$ , we want to construct a convex body in  $\mathbb{R}^d$  where the size of a convex hull of random points has a chaotic behavior between  $\log^{d-1} n$  and  $n^{\frac{d-1}{d+1}}$  on some values arbitrarily big. More formally,

**Theorem 3** *Let  $G : \mathbb{N}^* \rightarrow \mathbb{R}_+^*$  an increasing function such that  $G(n) \rightarrow_{n \rightarrow \infty} \infty$ . We can construct a convex body  $K$  such that:*

*For all  $N \in \mathbb{N}^*$ , there exist  $M_1, M_2 > N$ , where*

$$\mathbb{E}f_0(K_{M_1}) < G(M_1) \log^{d-1} M_1$$

and

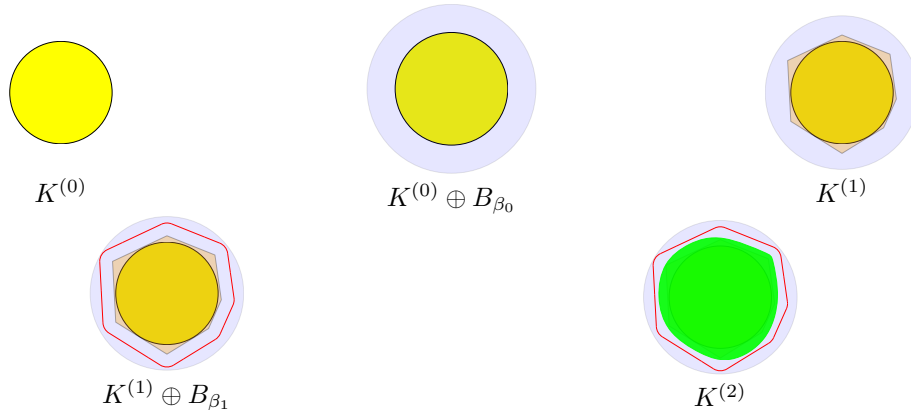
$$\mathbb{E}f_0(K_{M_2}) > G(M_2)^{-1} M_2^{\frac{d-1}{d+1}}.$$

**Proof:** The main idea of the proof is, starting from a convex body  $K^{(0)}$ , to iterate smooth and polytopal approximations. Lemma 2 will give us some number of points where the behavior of the random convex body will be very close to  $n^{\frac{d-1}{d+1}}$  (which is the behavior for smooth convex bodies) or very close to  $\log^{d-1} n$  (which is the behavior for polytopes).

**Iterations** We create an increasing sequence of convex bodies starting from the unit ball, made of polytopal and smooth approximations.

Let's define  $K^{(0)}$  as the unit ball.

For all  $n \in \mathbb{N}^*$ ,  $K^{(n)}$  is an approximation of  $K^{(n-1)}$  where  $d_H(K^{(n-1)}, K^{(n)}) < \beta_{n-1}$ , with  $(\beta_i)_{i \in \mathbb{N}}$  some decreasing sequence, as shown in Figure 1.



**Fig. 1:** Iterations are made of polygonal and smooth approximations

- If  $n$  is odd,  $K^{(n-1)}$  is a smooth convex body, so  $K^{(n)}$  is a convex polytope. Let's choose  $q_n > n$ , such that

$$\frac{\mathbb{E}f_0(K^{(n-1)})}{q_n^{\frac{d-1}{d+1}}} > \frac{2}{G(q_n)}.$$

We define

$$\varepsilon_n := 1 - \frac{2}{c_{d,K^{(n-1)}} G(q_n)}. \quad (8)$$

Using Lemma 2, with  $\varepsilon = \varepsilon_n$ , there exist  $\alpha_{n-1}$  and  $p_n > q_n$  such that for every compact set  $L$  containing  $K^{(n-1)}$  with  $d_H(K^{(n-1)}, L) < \alpha_{n-1}$ ,

$$\mathbb{E}f_0(L_{p_n}) > c_{d,K^{(n-1)}} p_n^{\frac{d-1}{d+1}} (1 - \varepsilon_n). \quad (9)$$

Therefore,

$$\begin{aligned} \mathbb{E}f_0(L_{p_n}) &> c_{d,K^{(n-1)}} p_n^{\frac{d-1}{d+1}} (1 - \varepsilon_n) \\ &> p_n^{\frac{d-1}{d+1}} G(q_n)^{-1} \\ &> p_n^{\frac{d-1}{d+1}} G(p_n)^{-1}. \end{aligned} \quad (10)$$

Now let's define  $\beta_{n-1} := \min(\frac{\alpha_{n-1}}{2}, \frac{\beta_{n-2}}{2})$  if  $n > 1$  and  $\beta_0 := \frac{\alpha_0}{2}$ . We define  $K^{(n)}$  as a convex polytope with  $d_H(K^{(n-1)}, K^{(n)}) < \beta_{n-1}$ , so (10) works for  $L = K^{(n)}$ .

- If  $n$  is even,  $K^{(n-1)}$  is a convex polytope, so  $K^{(n)}$  is a smooth approximation.

Let's choose  $q_n$  such that

$$\frac{\mathbb{E}f_0(K^{(n-1)})}{\log^{d-1} q_n} < \frac{G(q_n)}{2}$$

and define

$$\varepsilon_n := \frac{G(q_n)}{2c_{d,K^{(n-1)}}} - 1.$$

Using Lemma 2 with  $\varepsilon = \varepsilon_n$ , there exist  $\alpha_{n-1}$  and  $p_n > q_n$  such that for every compact set with  $d_H(K^{(n-1)}, L) < \alpha_{n-1}$ ,

$$\mathbb{E}f_0(L_{p_n}) < c_{d,K^{(n-1)}} \log^{d-1}(p_n)(1 + \varepsilon_n).$$

Finally,

$$\begin{aligned} \mathbb{E}f_0(L_{p_n}) &< G(q_n) \log^{d-1}(p_n) \\ &< G(p_n) \log^{d-1}(p_n). \end{aligned} \quad (11)$$

Again, we define  $\beta_{n-1} = \min(\frac{\alpha_{n-1}}{2}, \frac{\beta_{n-2}}{2})$ . We define  $K^{(n)}$  as a smooth approximation of  $K^{(n-1)}$  such that  $d_H(K^{(n-1)}, K^{(n)}) < \beta_{n-1}$ , so (11) works for  $L = K^{(n)}$ .

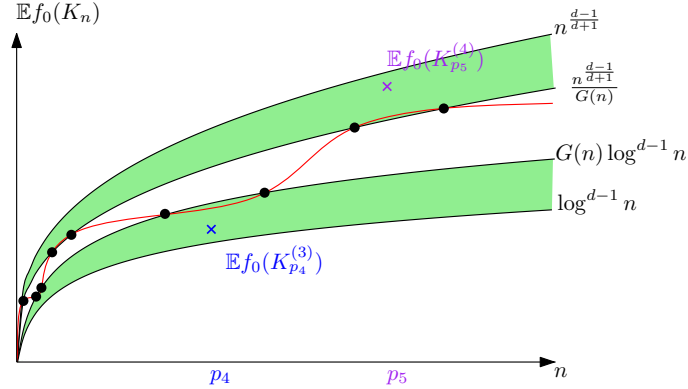
Note that for all  $m > n \in \mathbb{N}$ ,

$$\begin{aligned} d_H(K^{(n)}, K^{(m)}) &\leq \sum_{k=n}^{m-1} d_H(K^{(k)}, K^{(k+1)}) < \sum_{k=n}^{m-1} \beta_k \\ &\leq \sum_{k=0}^{m-n-1} \frac{\beta_n}{2^k} \leq 2\beta_n \leq \alpha_n. \end{aligned}$$

That means for all  $m > n$ , the property (10) or (11) (depending on the evenness of  $n$ ) are also true for  $K^{(m)}$ .

Now, defining  $K = \overline{\cup_{i=0}^{\infty} K^{(i)}}$ , the property (10) and (11) are true for arbitrary  $n \in \mathbb{N}$  with  $L = K$ , by considering  $K^{(n)}$  and  $K^{(n+1)}$ .

As we can choose  $q_n$  as big as we want for any  $n$  (it will just decrease  $\alpha_{n-1}$ ), we can choose this sequence to be increasing. As a result,  $\mathbb{E}f_0(K_n)$  will have a chaotic behavior within  $n^{\frac{d-1}{d+1}}/G(n)$  and  $G(n) \log^{d-1} n$ , as shown in Figure 2. □



**Fig. 2:** The expected size of the random polytope of  $K$ .

**Concluding remarks** We have constructed a convex body  $K$  such that the expected size of the convex hull of a random polytope in  $K$  has a chaotic behavior. This construction is the limit of a sequence of bodies  $(K^{(i)})$  that alternate polytopes and smooth shapes so it is difficult to provide an explicit description of  $K$ , in this note we just show that constructing such a sequence is possible by a repeated application of Lemma 2 but there is no obstacle, except long and painful computations, to a more constructive version with explicit description of the sequence. Notice that in such a case the complexity of  $K^{(i)}$  will be

increasing quite rapidly. Actually, since  $K^{(i)}$  is constrained in a slab of width  $\beta_i$  around  $K^{(i-1)}$ , the size of  $K^{(i)}$  can be lower bounded for polytopes, see Böröczky Jr. (2000):  $|K^{(i)}| = \Omega\left(\beta_i^{-\frac{d-1}{2}}\right)$  and since  $\beta_i < \frac{\alpha_0}{2^i}$  we get, at least, an exponential behavior for the size of  $K^{(i)}$ . Even with a constructive description of the  $K^{(i)}$ , the description of  $K$  as the limit of the  $K^{(i)}$  will remain quite abstract, but will allow to develop a membership test, given a point  $p$ ,  $p \in \text{int}(K)$  can be decided by computing the sequence  $K^{(i)}$  up to an index where  $p \in K^{(i)}$  or  $p \notin K^{(i)} \oplus B_{\beta_i}$ .

**Acknowledgments** The authors wish to thank Xavier Goaoc for proposing this problem and helpful discussions.

## References

- I. Bárány. Intrinsic volumes and  $f$ -vectors of random polytopes. *Mathematische Annalen*, 285(4):671–699, 1989.
- I. Bárány. Random points and lattice points in convex bodies. *Bull. Amer. Math. Soc. (N.S.)*, 45(3):339–365, 2008.
- K. Böröczky Jr. Polytopal approximation bounding the number of  $k$ -faces. *Journal of Approximation Theory*, 102(2):263 – 285, 2000. ISSN 0021-9045. doi:10.1006/jath.1999.3413. URL <http://www.sciencedirect.com/science/article/pii/S0021904599934135>.
- O. Devillers, M. Glisse, X. Goaoc, G. Moroz, and M. Reitzner. The monotonicity of  $f$ -vectors of random polytopes. *Electron. Commun. Probab.*, 18:no. 23, 1–8, 2013. ISSN 1083-589X. doi:10.1214/ECP.v18-2469. URL <http://ecp.ejpecp.org/article/view/2469>.
- M. Reitzner. Random polytopes. In *New perspectives in stochastic geometry*. Oxford Univ. Press, 2010.