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A chaotic random convex hull

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Abstract. The asymptotic behavior of the expected size of the convex hull of uniformly random points in a convex body in $\mathbb{R}^d$ is polynomial for a smooth body and polylogarithmic for a polytope. We construct a body whose expected size of the convex hull oscillates between these two behaviors when the number of points increases.

Keywords: convex hull, random point distribution, geometric probability

1 Introduction

Consider a sequence of points in a convex body in dimension $d$ whose convex hull is dynamically maintained when the points are inserted one by one, the convex hull size may increase, decrease, or remain constant when a new point is added. Studying the expected size of the convex hull when the points are evenly distributed in the convex is a classical problem of probabilistic geometry that yields some surprising facts. For example, although it seems quite natural to think that the expected size of the convex hull is increasing with $n$ the number of points, this fact is only formally proven for $n$ big enough [Devillers et al. (2013)]. The asymptotic behavior of the expected size is known to be polylogarithmic for a polytopal body and polynomial for a smooth one. If for a polytope or a smooth body, the asymptotic behavior is somehow "nice", for "most" convex bodies the behavior is unpredictable [ Bárány (1989) corollary 3]. It is possible to construct strange convex objects that have no such nice behaviors and this note exhibits a convex body, such that the behavior of the expected size of a random polytope oscillates between the polytopal and smooth behaviors when $n$ increases.

More formally, let $K$ be a convex body in $\mathbb{R}^d$ and $(x_1, \ldots, x_n)$ a sample of $n$ points chosen uniformly and independently at random in $K$. Let $K_n$ be the convex hull of these points and $f_0(K_n)$ the number of vertices of $K_n$.

It is well known [ Bárány (2008), Reitzner (2010)] that if $P$ is a polytope, then

$$
E f_0(P_n) = c_{d,P} \log^{d-1} n + o(\log^{d-1} n)
$$

and if $K$ is a smooth convex body (i.e with $C^2$ boundary with a positive Gaussian curvature), then

$$
E f_0(K_n) = c_{d,K} n^{\frac{d-1}{d+1}} + o(n^{\frac{d-1}{d+1}})
$$

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where \( c_{d,P} \) and \( c_{d,K} \) are constants depending only on \( d \) and on the convex body. These are the two extreme behaviors: every random polytope of a convex body in \( \mathbb{R}^d \) has a behavior between (1) and (2) for \( n \) large enough (Bárány [1989] corollary 3). For general convex bodies, we cannot expect such a beautiful formula.

**Theorem 1 (Bárány-Larman, 1988)** For any function \( G(n) \to_{n\to\infty} \infty \) and for most (in the Baire category sense) convex bodies \( K \) in \( \mathbb{R}^d \),

\[
G(n) \log^{d-1} n > \mathbb{E} f_0(K_n)
\]

for infinitely many \( n \) and

\[
G(n)^{-1} n^{d-1} < \mathbb{E} f_0(K_n)
\]

for infinitely many \( n \).

Note that this "most" does not contain convex polytopes and smooth convex bodies, which are the most used in practice.

In this paper, we present an explicit example of a convex body which has this chaotic behavior.

**Notations** Let’s introduce some notations used in this paper:

- \( V(K) \) will denote the \( d \)-dimensional volume of the convex body \( K \);
- \( K \oplus L \) will denote the Minkowski sum of \( K \) and \( L \), defined as
  \[
  J \oplus K = \{ x + y \mid x \in J, y \in K \};
  \]
- \( d_H(J, K) \) will denote the Hausdorff distance of the convex bodies \( K \) and \( L \), defined as
  \[
  d_H(J, K) = \min\{ r \in \mathbb{R}^+ \mid J \subset K \oplus B_r, K \subset J \oplus B_r \}
  \]
  where \( B_r \) is the Euclidean ball centered in 0 with radius \( r \) in \( \mathbb{R}^d \).

## 2 Approximations of convex bodies

In this section, we present an intermediate lemma about random polytopes of close (in terms of Hausdorff distance) convex bodies. Let \( K \) be a convex body in \( \mathbb{R}^d \) and \( K_n \) be a random polytope in \( K \). We want to show that if \( L \) is an approximation of \( K \) with small Hausdorff distance, \( L_n \) is approximating the asymptotic behavior of \( K_n \) for some value of \( n \).

Let’s assume that the expected size of \( K_n \) is in \( c_{d,K} g(n, d) + o(g(n, d)) \), where \( c_{d,K} \) is a constant and \( g \) some function.

Then, for every close-enough compact set \( L \) containing \( K \), \( L_n \) has an expected size as close as we want from \( c_{d,K} g(n, d) \) for values of \( n \) as big as we want.

The idea of this lemma is very simple: if \( L \) is very close to \( K \), the volume in \( L \setminus K \) is small, and points chosen uniformly in \( L \) are very unlikely to be in \( L \setminus K \). Then, even if asymptotically the expected size of \( L_n \) is different from the size of \( K_n \), there exists some \( n \) where the expected size of \( L_n \) is as close as we want to \( c_{d,K} g(n, d) \).
Lemma 2 Let $K$ be a convex body in $\mathbb{R}^d$ such that

$$\mathbb{E}f_0(K_n) = c_{d,K}g(n,d) + o(g(n,d)).$$ (3)

Let $\varepsilon > 0$ and $N \in \mathbb{N}$.

Then, there exist $p > N$ and $\alpha > 0$ such that for any compact set $L$ containing $K$ with $d_H(K,L) < \alpha$,

$$\frac{\mathbb{E}f_0(L_p)}{c_{d,K}g(p,d)} \in [1 - \varepsilon, 1 + \varepsilon].$$

Proof: First, for all $n \in \mathbb{N}^*$

$$\mathbb{E}f_0(L_n) = \mathbb{P}(L_n \subset K)\mathbb{E}(f_0(L_n)|L_n \subset K) + \mathbb{P}(L_n \not\subset K)\mathbb{E}(f_0(L_N)|L_n \not\subset K).$$

As the points are uniformly distributed, $\mathbb{E}(f_0(L_n)|L_n \subset K) = \mathbb{E}(K_n)$.

Using (3), let’s choose $p$ such that

$$\frac{\mathbb{E}f_0(K_p)}{c_{d,K}g(p,d)} \in [1 - \frac{\varepsilon}{2}, 1 + \frac{\varepsilon}{2}],$$ (4)

then

$$\frac{\mathbb{P}(L_p \subset K)\mathbb{E}(f_0(L_p)|L_p \subset K)}{c_{d,K}g(p,d)} \leq 1 + \frac{\varepsilon}{2}.$$ As

$$\mathbb{P}(L_p \not\subset K) = 1 - \mathbb{P}(L_p \subset K) = 1 - \left(\frac{V(K)}{V(L)}\right)^p$$

we get

$$\mathbb{P}(L_p \not\subset K)\mathbb{E}(f_0(L_p)|L_p \not\subset K) \leq \left(1 - \left(\frac{V(K)}{V(L)}\right)^p\right) p.$$ Now, as $1 \geq \frac{V(K)}{V(L)} \geq \frac{V(K)}{V(K \oplus B_n)} \to_{\alpha \to 0} 1$ we can choose $\alpha$ such that

$$\left(\frac{V(K)}{V(L)}\right)^p \geq \max\left(1 - c_{d,K}g(p,d)\frac{\varepsilon}{p}, 1 - \frac{\varepsilon}{2}\right).$$ (5)

Finally,

$$\mathbb{E}(f_0(L_p)) \leq c_{d,K}g(p,d) \left(1 + \frac{\varepsilon}{2}\right) + c_{d,K}g(p,d)\frac{\varepsilon}{2} = c_{d,K}g(p,d)(1 + \varepsilon).$$ (6)

For the lower bound, using (4) and (5) we get

$$\mathbb{E}f_0(L_p) \geq \mathbb{P}(L_p \subset K)\mathbb{E}(f_0(L_n)|L_n \subset K)$$

$$\geq c_{d,K}g(p,d)\left(\frac{V(K)}{V(L)}\right)^p(1 - \frac{\varepsilon}{2})$$

$$\geq c_{d,K}g(p,d)\left(\frac{V(K)}{V(L)}\right)^p - \frac{\varepsilon}{2}$$

$$\geq c_{d,K}g(p,d)\left(1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2}\right)$$

$$= c_{d,K}g(p,d)(1 - \varepsilon).$$ (7)

Inequalities (7) and (6) prove the lemma. \qed
3 Construction of the convex body

Given an increasing function $G$, we want to construct a convex body in $\mathbb{R}^d$ where the size of a convex hull of random points has a chaotic behavior between $\log^{d-1} n$ and $n^{\frac{d}{d+1}}$ on some values arbitrarily big. More formally, 

**Theorem 3** Let $G : \mathbb{N}^* \rightarrow \mathbb{R}_+^*$ an increasing function such that $G(n) \rightarrow_{n \rightarrow \infty} \infty$. We can construct a convex body $K$ such that:

For all $N \in \mathbb{N}^*$, there exist $M_1, M_2 > N$, where

$$
E f_0(K_{M_1}) < G(M_1) \log^{d-1} M_1
$$

and

$$
E f_0(K_{M_2}) > G(M_2)^{-1} M_2^{d-1}.
$$

**Proof:** The main idea of the proof is, starting from a convex body $K^{(0)}$, to iterate smooth and polytopal approximations. Lemma 2 will give us some number of points where the behavior of the random convex body will be very close to $n^{\frac{d-1}{d+1}}$ (which is the behavior for smooth convex bodies) or very close to $\log^{d-1} n$ (which is the behavior for polytopes).

**Iterations** We create an increasing sequence of convex bodies starting from the unit ball, made of polytopal and smooth approximations.

Let’s define $K^{(0)}$ as the unit ball.

For all $n \in \mathbb{N}^*$, $K^{(n)}$ is an approximation of $K^{(n-1)}$ where $d_H(K^{(n-1)}, K^{(n)}) < \beta_{n-1}$, with $(\beta_i)_{i \in \mathbb{N}}$ some decreasing sequence, as shown in Figure 1.

![Fig. 1: Iterations are made of polygonal and smooth approximations](image)

- If $n$ is odd, $K^{(n-1)}$ is a smooth convex body, so $K^{(n)}$ is a convex polytope.

Let’s choose $q_n > n$, such that

$$
\frac{E f_0(K^{(n-1)})}{q_n^{\frac{d-1}{d+1}}} > \frac{2}{G(q_n)}.
$$


A chaotic random convex hull

We define
\[ \varepsilon_n := 1 - \frac{2}{c_{d,K^{(n-1)}} G(q_n)}. \] (8)

Using Lemma 2 with \( \varepsilon = \varepsilon_n \), there exist \( \alpha_{n-1} \) and \( p_n > q_n \) such that for every compact set \( L \) containing \( K^{(n-1)} \) with \( d_H(K^{(n-1)}, L) < \alpha_{n-1} \),
\[ E f_0(L_{p_n}) > c_{d,K^{(n-1)}} p_n \frac{d-1}{d+1} (1 - \varepsilon_n). \] (9)

Therefore,
\[ E f_0(L_{p_n}) > c_{d,K^{(n-1)}} p_n \frac{d-1}{d+1} (1 - \varepsilon_n) \]
\[ > p_n \frac{d-1}{d+1} G(q_n)^{-1} \]
\[ > p_n \frac{d-1}{d+1} G(p_n)^{-1}. \] (10)

Now let’s define \( \beta_{n-1} := \min\left(\frac{\alpha_{n-1}}{2}, \frac{\beta_{n-2}}{2}\right) \) if \( n > 1 \) and \( \beta_0 := \frac{\alpha_0}{2} \). We define \( K^{(n)} \) as a convex polytope with \( d_H(K^{(n-1)}, K^{(n)}) < \beta_{n-1} \), so (10) works for \( L = K^{(n)} \).

- If \( n \) is even, \( K^{(n-1)} \) is a convex polytope, so \( K^{(n)} \) is a smooth approximation.

Let’s choose \( q_n \) such that
\[ \frac{E f_0(K^{(n-1)})}{\log^{d-1} q_n} < \frac{G(q_n)}{2} \]
and define
\[ \varepsilon_n := \frac{G(q_n)}{2c_{d,K^{(n-1)}}} - 1. \]

Using Lemma 2 with \( \varepsilon = \varepsilon_n \), there exist \( \alpha_{n-1} \) and \( p_n > q_n \) such that for every compact set with \( d_H(K^{(n-1)}, L) < \alpha_{n-1} \),
\[ E f_0(L_{p_n}) < c_{d,K^{(n-1)}} \log^{d-1}(p_n)(1 + \varepsilon_n). \]

Finally,
\[ E f_0(L_{p_n}) < G(q_n) \log^{d-1}(p_n) \]
\[ < G(p_n) \log^{d-1}(p_n). \] (11)

Again, we define \( \beta_{n-1} = \min\left(\frac{\alpha_{n-1}}{2}, \frac{\beta_{n-2}}{2}\right) \). We define \( K^{(n)} \) as a smooth approximation of \( K^{(n-1)} \) such that \( d_H(K^{(n-1)}, K^{(n)}) < \beta_{n-1} \), so (11) works for \( L = K^{(n)} \).
Note that for all $m > n \in \mathbb{N}$,

$$d_H(K^{(n)}, K^{(m)}) \leq \sum_{k=n}^{m-1} d_H(K^{(k)}, K^{(k+1)}) < \sum_{k=n}^{m-1} \beta_k$$

$$\leq \sum_{k=0}^{m-n-1} \frac{\beta_n}{2^k} \leq 2\beta_n \leq \alpha_n.$$

That means for all $m > n$, the property (10) or (11) (depending on the evenness of $n$) are also true for $K^{(m)}$.

Now, defining $K = \bigcup_{i=0}^{\infty} K^{(i)}$, the property (10) and (11) are true for arbitrary $n \in \mathbb{N}$ with $L = K$, by considering $K^{(n)}$ and $K^{(n+1)}$.

As we can choose $q_n$ as big as we want for any $n$ (it will just decrease $\alpha_{n-1}$), we can choose this sequence to be increasing. As a result, $E f_0(K_n)$ will have a chaotic behavior within $n^{d-\frac{1}{d+1}}/G(n)$ and $G(n) \log^{d-1} n$, as shown in Figure 2.

**Fig. 2:** The expected size of the random polytope of $K$.

**Concluding remarks** We have constructed a convex body $K$ such that the expected size of the convex hull of a random polytope in $K$ has a chaotic behavior. This construction is the limit of a sequence of bodies $(K^{(i)})$ that alternate polytopes and smooth shapes so it is difficult to provide an explicit description of $K$, in this note we just show that constructing such a sequence is possible by a repeated application of Lemma 2 but there is no obstacle, except long and painful computations, to a more constructive version with explicit description of the sequence. Notice that in such a case the complexity of $K^{(i)}$ will be...
A chaotic random convex hull

increasing quite rapidly. Actually, since \( K^{(i)} \) is constrained in a slab of width \( \beta_i \) around \( K^{(i-1)} \), the size of \( K^{(i)} \) can be lower bounded for polytopes, see Böröczky Jr. (2000): \( |K^{(i)}| = \Omega \left( \beta_i^{\frac{d+1}{2}} \right) \) and since \( \beta_i < \frac{\alpha_0}{2i} \) we get, at least, an exponential behavior for the size of \( K^{(i)} \). Even with a constructive description of the \( K^{(i)} \), the description of \( K \) as the limit of the \( K^{(i)} \) will remain quite abstract, but will allow to develop a membership test, given a point \( p \), \( p \in \text{int}(K) \) can be decided by computing the sequence \( K^{(i)} \) up to an index where \( p \in K^{(i)} \) or \( p \notin K^{(i)} \oplus B_{\beta_i} \).

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References


