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Learning computationally efficient dictionaries and their implementation as fast transforms

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Supplementary material

A. Anonymous

1 Projection operator proof

We want to find the projection operator onto the following set:

$$\mathcal{E} := \{\mathbf{A} \in \mathbb{R}^{n \times n} : \|\mathbf{A}\|_0 \leq p, \|\mathbf{A}\|_F = 1\}, \quad (1)$$

with $p \in \mathbb{N}^*$. We are interested in the projection of some matrix $\mathbf{S} \in \mathbb{R}^{n \times n}$ onto the set \mathcal{E} , namely we want to find \mathbf{S}^* such that:

$$\mathbf{S}^* = P_{\mathcal{E}}(\mathbf{S}) \in \arg \min_{\mathbf{U}} \{\|\mathbf{U} - \mathbf{S}\|_F^2 : \mathbf{U} \in \mathcal{E}\} \quad (2)$$

Proposition 1.1. *Projection operator formula.*

$$P_{\mathcal{E}}(\mathbf{S}) = B_1(T_p(\mathbf{S}))$$

where

$$T_p(\mathbf{S}) := \arg \min_{\mathbf{V}} \{\|\mathbf{V} - \mathbf{S}\|_F^2 : \|\mathbf{V}\|_0 \leq p\}$$

and

$$B_1(\mathbf{R}) := \arg \min_{\mathbf{W}} \{\|\mathbf{W} - \mathbf{R}\|_F^2 : \|\mathbf{W}\|_F = 1\}$$

Proof. Let us denote by P the set of indices corresponding to the p greatest entries (in absolute value) of \mathbf{S} , and by \mathbf{S}_P the matrix we get by setting all the other entries of \mathbf{S} to 0. Let us also introduce $\mathbf{S}_{\bar{P}} = \mathbf{S} - \mathbf{S}_P$, the matrix we get by keeping only the $n^2 - p$ littlest entries. We have:

$$\|\mathbf{S}\|_F^2 = \|\mathbf{S}_P\|_F^2 + \|\mathbf{S}_{\bar{P}}\|_F^2 \quad (3)$$

and

$$\mathbf{S}_P = \arg \min_{\mathbf{V}} \{\|\mathbf{V} - \mathbf{S}\|_F^2 : \|\mathbf{V}\|_0 \leq p\} = T_p(\mathbf{S}) \quad (4)$$

and

$$P = \arg \max_J \{\|\mathbf{S}_J\|_F : \text{card}(J) \leq p\} \quad (5)$$

The operator B_1 is simply the projection onto the Euclidean sphere of radius 1, that is:

$$B_1(\mathbf{X}) = \frac{1}{\|\mathbf{X}\|_F} \mathbf{X}. \quad (6)$$

Let us now denote by I the support of any $\mathbf{U} \in \mathcal{E}$, we have:

$$\|\mathbf{U} - \mathbf{S}\|_F^2 = \|\mathbf{S}_{\bar{I}}\|_F^2 + \|\mathbf{U} - \mathbf{S}_I\|_F^2 \quad (7)$$

so the problem can be rewritten:

$$(\mathbf{S}^*, I^*) = \arg \min_{(\mathbf{U}, I)} \{\|\mathbf{S}_{\bar{I}}\|_F^2 + \|\mathbf{U} - \mathbf{S}_I\|_F^2 : \|\mathbf{U}\|_F = 1, \text{card}(I) \leq p\} \quad (8)$$

Isolating \mathbf{U} we have:

$$\mathbf{S}^* = \arg \min_{\mathbf{U}} \{\|\mathbf{U} - \mathbf{S}_{I^*}\|_F^2 : \|\mathbf{U}\|_F \leq q\} = B_1(\mathbf{S}_{I^*}) \quad (9)$$

with:

$$\begin{aligned} I^* &= \arg \min_I \{\|\mathbf{S}_{\bar{I}}\|_F^2 + \inf_{\mathbf{U}} \{\|\mathbf{U} - \mathbf{S}_I\|_F^2 : \|\mathbf{U}\|_F = 1\} : \text{card}(I) \leq p\} \\ I^* &= \arg \min_I \{\|\mathbf{S}_{\bar{I}}\|_F^2 + \|B_1(\mathbf{S}_I) - \mathbf{S}_I\|_F^2 : \text{card}(I) \leq p\} \\ I^* &= \arg \min_I \{\|\mathbf{S}_{\bar{I}}\|_F^2 + (1 - \frac{1}{\|\mathbf{S}_I\|_F})^2 \|\mathbf{S}_I\|_F^2 : \text{card}(I) \leq p\} \\ I^* &= \arg \min_I \{\underbrace{\|\mathbf{S}_{\bar{I}}\|_F^2 + \|\mathbf{S}_I\|_F^2}_{\|\mathbf{S}\|_F^2} - 2\|\mathbf{S}_I\|_F : \text{card}(I) \leq p\} \\ I^* &= \arg \max_I \{\|\mathbf{S}_I\|_F : \text{card}(I) \leq p\} \\ I^* &= P \end{aligned} \quad (10)$$

So we have: $\mathbf{S}^* = B_1(\mathbf{S}_P) = B_1(T_p(\mathbf{S}))$. \square

2 Lipschitz moduli proof

Let us look at the Lipschitz moduli of the gradient of the smooth part of the objective:

$$\begin{aligned} & \left\| \nabla_{\mathbf{S}_j} H(\mathbf{S}_1^{i+1} \dots \mathbf{S}_1 \dots \mathbf{S}_p^i, \lambda^i) - \nabla_{\mathbf{S}_j} H(\mathbf{S}_1^{i+1} \dots \mathbf{S}_2 \dots \mathbf{S}_p^i, \lambda^i) \right\|_F \\ &= \left\| \lambda^i \mathbf{L}^T (\lambda^i \mathbf{L} \mathbf{S}_1 \mathbf{R} - \mathbf{X}) \mathbf{R}^T - \lambda^i \mathbf{L}^T (\lambda^i \mathbf{L} \mathbf{S}_2 \mathbf{R} - \mathbf{X}) \mathbf{R}^T \right\|_F \\ &= (\lambda^i)^2 \left\| \mathbf{L}^T \mathbf{L} (\mathbf{S}_1 - \mathbf{S}_2) \mathbf{R} \mathbf{R}^T \right\|_F \\ &= (\lambda^i)^2 \left\| (\mathbf{R} \mathbf{R}^T) \otimes (\mathbf{L}^T \mathbf{L}) \cdot \text{vec}(\mathbf{S}_1 - \mathbf{S}_2) \right\|_2 \\ &\leq (\lambda^i)^2 \left\| (\mathbf{R} \mathbf{R}^T) \otimes (\mathbf{L}^T \mathbf{L}) \right\|_2 \|\mathbf{S}_1 - \mathbf{S}_2\|_F \\ &= (\lambda^i)^2 \|\mathbf{R}\|_2^2 \cdot \|\mathbf{L}\|_2^2 \|\mathbf{S}_1 - \mathbf{S}_2\|_F. \end{aligned} \quad (11)$$

So we can say that the following quantity is a Lipschitz modulus: $L_j(\mathbf{L}, \mathbf{R}, \lambda^i) = (\lambda^i)^2 \|\mathbf{R}\|_2^2 \cdot \|\mathbf{L}\|_2^2$.