

# Flat extensions in \*-algebras

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## Abstract

The main result of the paper is a flat extension theorem for positive linear functionals on \*-algebras. The theorem is applied to truncated moment problems on cylinder sets, on matrices of polynomials and on enveloping algebras of Lie algebras.

**Keywords:** \*-algebras; flat extension; hermitian linear form; GNS construction; truncated moment problem; enveloping algebra;

## 1 Introduction

Given real numbers  $s_\alpha$ , where  $\alpha \in \mathbb{N}_0^d$ ,  $|\alpha| \leq 2n$ , the (real) truncated moment problem [1, 2] asks: When does there exist a positive Borel measure  $\mu$  on  $\mathbb{R}^d$  such that  $s_\alpha = \int x^\alpha d\mu$  for all  $\alpha \in \mathbb{N}_0^d$ ,  $|\alpha| \leq 2n$ ? A fundamental result on this problem is the flat extension theorem due to R. Curto and L. Fialkow [2], see e.g. [4] for a nice exposition and [5] for another proof. The flat extension theorem extends the corresponding linear functional on the polynomials of degree at most  $2n$  to a flat positive linear functional on the whole polynomial algebra  $\mathbb{C}[x_1, \dots, x_d]$ . Then the extended functional is represented by a positive measure. For this it is crucial that the extension is also flat, because then the GNS representation acts on a finite dimensional space and the existence of the measure  $\mu$  can be derived (for instance) from the finite dimensional spectral theorem. The aim of this paper is to extend the flat extension theorem to complex unital \*-algebras.

Suppose that  $\mathcal{A}$  is a unital complex \*-algebra. Let  $\pi$  be a \*-representation of  $\mathcal{A}$  on a unitary space  $(V, \langle \cdot, \cdot \rangle)$ . For any  $v \in V$ , there is a positive linear functional  $L_v$  on  $\mathcal{A}$  defined by

$$L_v(a) = \langle \pi(a)v, v \rangle, \quad a \in \mathcal{A}.$$

Functionals of this form are called *vector functionals* in the representation  $\pi$ .

Let  $L$  be a linear functional on a \*-invariant linear subspace  $\mathcal{B}$  of  $\mathcal{A}$ . A natural question is when  $L$  has an extension to a positive linear functional  $\tilde{L}$  on the whole \*-algebra  $\mathcal{A}$ . By the GNS construction (see e.g. [9, Section 8.6]) this holds if and only if  $L$  is the restriction of a vector functional  $L_v$  in some representation  $\pi$ . Furthermore, the extension  $\tilde{L}$  should be "nice" in order to represent it in some "well-behaved" representation  $\pi$ . The latter is the counter-part for requiring that the extended functional can be given by some positive measure. In general, it seems that these problems are difficult and no simple answers can be expected.

The main result of this paper is a general flat extension theorem for hermitean linear functionals. Roughly speaking and omitting technical details and assumptions, it says that if  $L$  is a hermitean linear functional on a subspace which is flat with respect to some appropriate smaller subspace, then  $L$  has a unique flat extension  $\tilde{L}$  to  $\mathcal{A}$ . Furthermore, if  $L$  is positive, so is  $\tilde{L}$ .

This theorem contains the Curto-Fialkow theorem as a special case when  $\mathcal{A}$  is the polynomial algebra. But it applies also to certain noncommutative \*-algebras and it allows one

to treat noncommutative truncated moment problems. We mention three applications, the first concerns truncated moment problems on cylinder sets, the second is about truncated moment problems for matrices over polynomial algebras, and the third deals with enveloping algebras of finite dimensional Lie algebras.

Flat extensions of positive functionals on path  $*$ -algebras have been considered in [7].

## 2 Basic definitions and preliminaries

First we prove a proposition which contains a reformulation of flatness for hermitean matrices. It gives the justification for the flatness Definition 2.3 below.

Let  $A \in M_n(\mathbb{C})$ ,  $B \in M_{n,k}(\mathbb{C})$ ,  $C \in M_k(\mathbb{C})$ , where  $A = A^*$  and  $C = C^*$ , and consider the hermitean block matrix  $X$  acting on the vector space  $\mathbb{C}^{n+k}$ :

$$X = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}.$$

Then the matrix  $X$  is called a *flat extension* of the matrix  $A$  if  $\text{rank} A = \text{rank} X$ .

**Proposition 2.1** *The following statements are equivalent:*

- (i)  $X$  is a flat extension of  $A$ .
- (ii)  $\mathbb{C}^{n+k} = (\mathbb{C}^n, 0) + \ker X$ .

**Proof.** (i)→(ii): Since  $X$  is a flat extension of  $A$ , a classical result of Shmuljan [11] states that there exist a matrix  $W \in M_{n,k}(\mathbb{C})$  such that  $B = AW$  and  $C = W^*AW$ . Let  $x \in \mathbb{C}^n$  and  $y \in \mathbb{C}^k$ . Then we compute  $X(-Wy, y) = 0$ , so that  $(x, y) = (x + Wy, 0) + (-Wy, y) \in \mathbb{C}^{n+k} + \ker X$  which proves (ii).

(ii)→(i): By reordering the canonical basis of  $\mathbb{C}^n$ , we may assume that the first  $n'$  canonical basis elements of  $\mathbb{C}^n$  are linearly independent of  $\ker X$  and that, together with a basis of  $\ker X$ , they form a basis of  $\mathbb{C}^{n+k}$ . This implies that  $\mathbb{C}^{n+k} = (\mathbb{C}^{n'}, 0) \oplus \ker X$  and we can take  $n' = n$  in order to prove that (ii) implies (i).

The matrix of change of bases between the canonical basis of  $\mathbb{C}^{n+k}$  and this basis is of the form

$$P = \begin{pmatrix} I & -W \\ 0 & I \end{pmatrix}$$

and the matrix of the symmetric operator in this basis is

$$\begin{aligned} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ -W^* & I \end{pmatrix} \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \begin{pmatrix} I & -W \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} A & B - AW \\ B^* - W^*A & C - W^*B - B^*W + W^*AW \end{pmatrix} \end{aligned}$$

This implies that  $B = AW$ ,  $B^* = W^*A$  and  $C = W^*AW$ . Thus  $X$  is a flat extension of  $A$ .  $\square$

Throughout this paper we assume that  $\mathcal{A}$  is a complex unital  $*$ -algebra. The involution of  $\mathcal{A}$  is denoted by  $*$  and the unit element by 1.

Let  $\mathcal{C}$  be a  $*$ -invariant linear subspace of  $\mathcal{A}$ , which contains the unit element 1 of  $\mathcal{A}$ . A linear functional  $L$  on  $\mathcal{C}$  is called *hermitian* if  $L(b^*) = \overline{L(b)}$  for all  $b \in \mathcal{C}$ . Set

$$\mathcal{C}^2 := \text{Lin} \{ab; a, b \in \mathcal{C}\}.$$

Now suppose that  $L$  is a hermitian linear functional on  $\mathcal{C}^2$ . Then

$$\langle a, b \rangle_L := L(b^*a), a, b \in \mathcal{C},$$

defines a hermitian sesquilinear form  $\langle \cdot, \cdot \rangle_L$  acting on the vector space  $\mathcal{C}$ . We denote by  $K_L(\mathcal{C})$  the vector space

$$K_L(\mathcal{C}) := \{a \in \mathcal{C} : \langle a, b \rangle_L = 0, \forall b \in \mathcal{C}\}.$$

We verify that  $K_L(\mathcal{C})$  is the kernel of the Hankel operator  $H_L : \mathcal{C} \rightarrow \mathcal{C}^*$  defined by

$$H_L : a \mapsto a \star L, \quad \text{where} \quad a \star L : b \in \mathcal{C} \mapsto L(b^*a) \in \mathbb{C}.$$

Further, for  $a \in \mathcal{C}$ , we define

$$\mathcal{D}_{\mathcal{C}}(a) = \{b \in \mathcal{C}^2 : ab \in \mathcal{C}^2\}, \quad \rho(a)b = ab \quad \text{for} \quad b \in \mathcal{D}_{\mathcal{C}}(a).$$

Then  $\rho(a)$  is a linear operator with domain  $\mathcal{D}_{\mathcal{C}}(a)$  on  $\mathcal{C}$ . It is easily verified that

$$\begin{aligned} \rho(ab)c &= \rho(a)\rho(b)c \quad \text{if} \quad abc \in \mathcal{C}^2, \\ \rho(1)b &= b \quad \text{for} \quad b \in \mathcal{C}^2, \\ L(c^*ab) &= \langle \rho(a)b, c \rangle_L = \langle b, \rho(a^*)c \rangle_L \quad \text{if} \quad c^*ab \in \mathcal{C}^2. \end{aligned}$$

A linear functional  $L$  on  $\mathcal{C}^2$  is called *positive* if  $L(a^*a) \geq 0$  for all  $a \in \mathcal{C}$ .

**Lemma 2.2** *If a functional  $L$  on  $\mathcal{C}^2$  is positive, then  $K_L(\mathcal{C}) = \{a \in \mathcal{C} : L(a^*a) = 0\}$ .*

**Proof.** If  $a \in K_L(\mathcal{C})$ , then in particular  $L(a^*a) = \langle a, a \rangle_L = 0$ . Conversely, suppose that  $L(a^*a) = 0$ . Since  $L$  is positive, we have the Cauchy-Schwarz inequality

$$|L(b^*a)|^2 \leq L(a^*a)L(b^*b)$$

for  $b \in \mathcal{C}$ , which implies that  $\langle a, b \rangle_L = L(b^*a) = 0$ , so that  $a \in K_L(\mathcal{C})$ .  $\square$

The following definition contains the main concept of this paper. It is suggested by the equivalence of conditions (i) and (ii) in Proposition 2.1.

**Definition 2.3** *Let  $\mathcal{B}$  and  $\mathcal{C}$  be  $*$ -invariant linear subspaces of  $\mathcal{A}$  such that  $1 \in \mathcal{B}$  and  $\mathcal{B} \subseteq \mathcal{C}$ . A hermitian linear functional  $L$  on  $\mathcal{C}^2$  is called a flat extension with respect to  $\mathcal{B}$  if*

$$\mathcal{C} = \mathcal{B} + K_L(\mathcal{C}).$$

Another important notion (see e.e. [9, Definition 8.1.9]) is the following.

**Definition 2.4** *Let  $V$  be a vector space equipped with a hermitian sesquilinear form  $\langle \cdot, \cdot \rangle$ . A  $*$ -representation of  $\mathcal{A}$  on  $(V, \langle \cdot, \cdot \rangle)$  is an algebra homomorphism  $\rho$  of  $\mathcal{A}$  into the algebra of linear operators on  $V$  such that  $\rho(1)v = v$  and  $\langle \rho(a)v, w \rangle = \langle v, \rho(a^*)w \rangle$  for  $v, w \in V$ .*

Now suppose that  $L$  is a hermitian linear functional defined on the whole  $*$ -algebra  $\mathcal{A}$ . Then it is easily checked that  $K_L(\mathcal{A})$  is a left ideal. Therefore, we can quotient out the kernel  $\mathcal{I} = K_L(\mathcal{A})$  and get a *non-degenerate* hermitian form on  $\mathcal{A}/\mathcal{I}$ , denoted again by  $\langle \cdot, \cdot \rangle_L$  by some abuse of notation, and a  $*$ -representation  $\rho$  on  $(\mathcal{A}/\mathcal{I}, \langle \cdot, \cdot \rangle_L)$  such that

$$L(a) = \langle \rho(a)1, 1 \rangle_L \quad \text{for} \quad a \in \mathcal{A}.$$

If the functional  $L$  is positive, then this procedure coincides with the "ordinary" GNS construction, see e.g. [9, Theorem 8.6.2].

For the formulation of our main theorem the following concept is convenient.

**Definition 2.5** *Let  $\{a_i, i \in I\}$  be a fixed set of generators of the algebra  $\mathcal{A}$ . For a vector space  $V \subset \mathcal{A}$ , let  $V^+ = \text{Lin}\{a_i v | i \in I, v \in V\}$  be the prolongation of  $V$ . We shall say that  $V$  is connected to 1 if  $1 \in V$  and there exists a increasing sequence of vector spaces*

$$V_0 = \mathbb{C} \cdot 1 \subset V_1 \subset \cdots \subset V_l \subset \cdots \subset V$$

*such that  $\cup_{l \in \mathbb{N}} V_l = V$  and  $V_{l+1} \subset V_l^+$ . For an element  $v \in V$ , the  $V$ -index of  $v$  is the smallest  $l \in \mathbb{N}$  such that  $v \in V_l$ . Let  $V^{[0]} = V$  and for  $l \in \mathbb{N}$ , we define  $V^{[l+1]} = (V^{[l]})^+$ . The index of an arbitrary element  $a \in \mathcal{A}$  is its  $\mathbb{C} \cdot 1$ -index.*

### 3 The extension theorem

Throughout this section we suppose that  $\{a_i; i \in I\}$  is a fixed set of hermitean generators of the unital complex  $*$ -algebra  $\mathcal{A}$ , such that  $\forall i \in I$  there exists  $j \in I$  such that  $a_i^* = a_j$ .

Let  $\mathcal{F} = \mathbb{C}\langle x_i, i \in I \rangle$  denote the free complex unital  $*$ -algebra in variables  $x_i, i \in I$ . Then there is a  $*$ -homomorphism  $\sigma : \mathcal{F} \rightarrow \mathcal{A}$  such that  $\sigma(x_i) = a_i, i \in I$ . The kernel  $\mathcal{I}$  of  $\sigma$  is a two-sided  $*$ -ideal of  $\mathcal{F}$  and we have  $\mathcal{A} = \mathcal{F}/\mathcal{I}$ . For any set of generators  $S$  of the two-sided ideal  $\mathcal{I}$ , we denote by  $\delta(S) \in \mathbb{N} \cup \{\infty\}$  the maximal index of the elements of  $S$ . Let  $\delta(\mathcal{A})$  be the lowest  $\delta(S)$  of all sets  $S$  of generators of  $\mathcal{I}$ .

**Theorem 3.1** *Let  $\mathcal{B}$  and  $\mathcal{C}$  be  $*$ -invariant linear subspaces of  $\mathcal{A}$ , connected to 1 such that  $\mathcal{B}^{[m]} \subseteq \mathcal{C}$  with  $2m \geq \delta(\mathcal{A})$ . Let  $L$  be a hermitian linear functional on  $\mathcal{C}^2$  which is a flat extension with respect to  $\mathcal{B}$ . Then there exists a unique extension of  $L$  to a hermitian linear functional  $\mathcal{L}$  on  $\mathcal{A}$  which is a flat extension with respect to  $\mathcal{C}$ .*

Let us choose a linear subspace  $\mathcal{B}'$  of  $\mathcal{B}$  such that  $1 \in \mathcal{B}'$  and  $\mathcal{C}$  is the direct sum of the vector spaces  $\mathcal{B}'$  and  $K_L(\mathcal{C})$ . The form  $\langle \cdot, \cdot \rangle_{\mathcal{L}}$  is non-degenerate on  $\mathcal{B}'$  and the projection of  $\mathcal{C}$  on  $\mathcal{B}'$  along  $K_L(\mathcal{C})$  extends to a projection  $\pi_{\mathcal{L}} : \mathcal{A} \rightarrow \mathcal{B}'$ . The  $*$ -representation  $\rho_{\mathcal{L}}$  of  $\mathcal{A}$  associated with  $\mathcal{L}$  acts on  $(\mathcal{B}', \langle \cdot, \cdot \rangle_{\mathcal{L}})$  by  $\rho_{\mathcal{L}}(a)b = \pi_{\mathcal{L}}(ab), a \in \mathcal{A}, b \in \mathcal{B}'$ .

Further, if  $L$  is a positive linear functional on  $\mathcal{C}^2$ , then  $\mathcal{L}$  is a positive linear functional on  $\mathcal{A}$  and  $\langle \cdot, \cdot \rangle_{\mathcal{L}}$  is a scalar product on  $\mathcal{B}'$ .

The remaining part of this section, is devoted to the proof of this theorem. Let us recall the main assumptions. Since the vector space  $\mathcal{C}$  is connected to 1, there exists a sequence of vector spaces  $\mathcal{C}_0 \subset \mathcal{C}_1 \subset \dots \subset \mathcal{C}_i \subset \dots \subset \mathcal{C}$  such that  $\mathcal{C}_{i+1} \subset \mathcal{C}_i^+$  and  $\cup_i \mathcal{C}_i = \mathcal{C}$ . That the hermitean linear functional  $L$  on  $\mathcal{C}^2$  is a flat extension means that

$$\mathcal{C} = \mathcal{B} + K_L(\mathcal{C}).$$

A first consequence of these hypotheses is the following property:

$$a \in K_L(\mathcal{C}) \Leftrightarrow \forall b \in \mathcal{C}, L(b^*a) = 0 \quad (1)$$

$$\Leftrightarrow \forall b \in \mathcal{B}, L(b^*a) = 0 \quad (2)$$

#### Coherence

The flat extension property induces a structure of  $K_L(\mathcal{C})$ , which we are going to analyze. We first show that  $K_L(\mathcal{C})$  is stable by prolongation and intersection with  $\mathcal{C}$ .

**Lemma 3.2**  $K_L(\mathcal{C})^+ \cap \mathcal{C} \subset K_L(\mathcal{C})$ .

**Proof.** For any element  $\kappa \in K_L(\mathcal{C})^+$ , we have

$$\kappa = \sum_{j \in J} a_j \kappa'_j,$$

with  $J \subset I, \kappa'_j \in K_L(\mathcal{C})$ . For any  $b \in \mathcal{B}$ ,

$$L(b^* \kappa) = \sum_{j \in J} L(b^* a_j \kappa'_j) = 0$$

since  $b^* a_j \in \mathcal{B}^+ \subset \mathcal{C}$  and  $\kappa'_j \in K_L(\mathcal{C})$ . The relation (2) implies that  $\kappa \in K_L(\mathcal{C})$ .  $\square$

**Remark 3.3** There is no loss of generality to assume that  $1 \in \mathcal{B}'$ . Otherwise  $1 \in K_L(\mathcal{C})$ . By Lemma 3.2,  $\mathcal{C}_1 \subset (\mathcal{C}_0)^+ \cap \mathcal{C} \subset K_L(\mathcal{C})$ . By induction, we deduce that  $\forall l \in \mathbb{N}, \mathcal{C}_l \subset K_L(\mathcal{C})$ , which implies that  $\mathcal{C} = \cup \mathcal{C}_l = K_L(\mathcal{C})$ . Consequently,  $L \equiv 0$ . But in this case the assertion of Theorem 3.1 is trivial.

The next lemma shows that  $K_L(\mathcal{C})$  contains iterated prolongations of  $\mathcal{K}$  intersected with  $\mathcal{C}$ . As we will see,  $\mathcal{K}$  generates the left ideal  $K_L(\mathcal{C})$  of the extension  $\mathcal{L}$ .

**Lemma 3.4**  $\mathcal{K}^{[2m-1]} \cap \mathcal{C} \subset K_L(\mathcal{C})$ .

**Proof.** Let  $\kappa \in \mathcal{K}^{[2m-1]} \cap \mathcal{C}$ . It is a sum of terms of the form  $a_{i_1} \cdots a_{i_l} \kappa'$  with  $l \leq 2m-1$ ,  $i_k \in I$ ,  $\kappa' \in \mathcal{K}$ . We decompose  $a_{i_1} \cdots a_{i_l} = \mathfrak{a}_1 \mathfrak{a}_2$  as a product of a term  $\mathfrak{a}_1$  of index  $\leq m$  and a term  $\mathfrak{a}_2$  of index  $m-1$ . Then, we have  $\forall \mathfrak{b} \in \mathcal{B}$ ,  $\langle \mathfrak{a} \kappa' | \mathfrak{b} \rangle = \langle \mathfrak{a}_2 \kappa' | \mathfrak{a}_1^* \mathfrak{b} \rangle$ .

By hypothesis,  $\mathcal{B}^{[m]} \subset \mathcal{C}$  and  $\mathcal{K} \subset \mathcal{B}^{[1]}$ , and  $\mathfrak{a}_2 \kappa' \in \mathcal{K}^{[m-1]} \subset \mathcal{B}^{[m]} \subset \mathcal{C}$ .

Let us prove that  $\mathfrak{a}_2 \kappa' \in K_L(\mathcal{C})$ :  $\forall \mathfrak{b}' \in \mathcal{B}$ ,  $\mathfrak{a}_2^* \mathfrak{b}' \in \mathcal{B}^{[m]} \subset \mathcal{C}$ . Relation (1) implies that

$$\langle \mathfrak{a}_2 \kappa' | \mathfrak{b}' \rangle = \langle \kappa' | \mathfrak{a}_2^* \mathfrak{b}' \rangle = 0.$$

Relation (2) implies that  $\mathfrak{a}_2 \kappa' \in K_L(\mathcal{C})$ .

By hypothesis, we also have  $\forall \mathfrak{b} \in \mathcal{B}$ ,  $\mathfrak{a}_1^* \mathfrak{b} \in \mathcal{B}^{[m]} \subset \mathcal{C}$  and by relation (1),

$$\langle \mathfrak{a}_2 \kappa' | \mathfrak{a}_1^* \mathfrak{b} \rangle = 0 = \langle \mathfrak{a} \kappa' | \mathfrak{b} \rangle.$$

We deduce that  $\forall \mathfrak{b} \in \mathcal{B}$ ,  $\langle \kappa | \mathfrak{b} \rangle = 0$ , which implies, by relation (2), that  $\kappa \in K_L(\mathcal{C})$ . This ends the proof of this Lemma.  $\square$

## The GNS construction

We describe now a GNS-type construction, which allows to extend  $L$  on  $\mathcal{A}$ . For  $i \in I$ , we define the following operator

$$\begin{aligned} X_i : \mathcal{B} &\rightarrow \mathcal{B} \\ b &\mapsto \pi(a_i b) \end{aligned}$$

and the map

$$\begin{aligned} \phi : \mathcal{F} &\rightarrow \mathcal{B} \\ p &\mapsto p(X)(1). \end{aligned}$$

The kernel of  $\phi$  is denoted  $\mathcal{J}$ . We easily check that  $\mathcal{J}$  is a left ideal of  $\mathcal{F} = \mathbb{C}\langle x_i, i \in I \rangle$ . Let  $B, C \subset \mathcal{F}$  be  $*$ -invariant linear subspaces of  $\mathcal{F}$ , connected to 1 such that  $\sigma(B) = \mathcal{B}$  and  $\sigma(C) = \mathcal{C}$ . We denote by  $S$  a generating set of the two-sided ideal  $\mathcal{I} = \ker \sigma$  such that  $\delta(S) = \delta(\mathcal{A})$  is minimal.

**Lemma 3.5**  $\forall p, q \in C$ ,  $\phi(pq) = \sigma(p)\phi(q) + \kappa_1\phi(q) + \kappa_0$  with  $\kappa_0 \in \mathcal{K} \subset K_L(\mathcal{C})$ ,  $\kappa_1 \in K_L(\mathcal{C})$ . If  $l$  is the index of  $p$  then  $\phi(pq) - \sigma(p)\phi(q) \in \mathcal{K}^{[l-1]}$ .

**Proof.** We prove the property by induction on the index  $l$  of  $p$ .

The property is obviously true for  $l = 0$ , in which case we can take  $p = 1$  and  $\phi(1) = \sigma(1) = 1$ .

Let us assume that it is true for  $l \in \mathbb{N}$  and let  $p \in C$  of index  $l+1$ . Then  $\sigma(p) = \sum_{j \in J} a_j \sigma(p'_j)$  with  $p'_j \in C$  of index  $\leq l$ . By induction hypothesis, we have

$$\begin{aligned} \phi(pq) &= \sum_{j \in J} \pi(a_j \phi(p'_j q)) \\ &= \sum_{j \in J} a_j \phi(p'_j q) + \kappa_0 \\ &= \sum_{j \in J} a_j (\sigma(p'_j) \phi(q) + \kappa'_j \phi(q)) + \kappa_0 \\ &= \sigma(p) \phi(q) + \kappa_1 \phi(q) + \kappa_0 \end{aligned}$$

with  $\kappa_0 \in \mathcal{K} \subset K_L(\mathcal{C})$ ,  $\kappa'_j \in K_L(\mathcal{C})$  and  $\kappa_1 = \sum_{j \in J} a_j \kappa'_j \in K_L(\mathcal{C})^+ \cap \mathcal{C}$ . By Lemma 3.2,  $\kappa_1 \in K_L(\mathcal{C})$ . By induction hypothesis,  $\phi(p'_j q) - \sigma(p'_j) \phi(q) \in \mathcal{K}^{[l-1]}$ . Therefore

$$\phi(pq) - \sigma(p) \phi(q) = \sum_{j \in J} a_j (\phi(p'_j q) - \sigma(p'_j) \phi(q)) + \kappa_0 \in (\mathcal{K}^{[l-1]})^+ = \mathcal{K}^{[l]}$$

which proves the induction hypothesis for  $l+1$  and concludes the proof of this Lemma.  $\square$

**Proposition 3.6**  $\forall g \in S, \forall b, b' \in B, \phi(bgb') = 0$ .

**Proof.** By Lemma 3.5,  $\phi(gb') = \sigma(g) \phi(b') + \kappa = \kappa$  with  $\kappa \in \mathcal{K}^{[2m-1]}$ . As  $\phi(gb') \in \mathcal{B}' \subset \mathcal{B}$ , by Lemma 3.4,  $\kappa \in K_L(\mathcal{C}) \cap \mathcal{B}' = \{0\}$ . This implies that  $\phi(gb') = 0$ . Consequently,  $\phi(bgb') = \phi(b(X) \phi(gb')) = 0$   $\square$

**Proposition 3.7**  $\forall p \in C, \phi(p) = \pi(\sigma(p))$ .

**Proof.** Let  $p \in C$ . Applying Lemma 3.5 with  $q = 1$ , we have  $\phi(p) = \sigma(p) + \kappa$ ,  $\kappa \in K_L(\mathcal{C})$ .

As  $\sigma(p) \in \mathcal{C}$ ,  $\phi(p) \in \mathcal{B}'$  and  $\kappa \in K_L(\mathcal{C})$ ,  $\phi(p)$  is the projection of  $\sigma(p)$  on  $\mathcal{B}'$  along  $K_L(\mathcal{C})$ . In other words,  $\phi(p) = \pi(\sigma(p))$ , which proves the proposition.  $\square$

As  $\mathcal{I} = \ker \sigma$  is the left-right ideal generated by  $S$ , this proposition shows that  $\phi$  factors through a map  $\bar{\phi}$  on  $\mathcal{A} = \mathcal{F}/\mathcal{I}$ . We prove now that  $\ker \bar{\phi}$  is the left ideal generated by  $\mathcal{K}$ .

**Proposition 3.8**  $\sigma(\ker \phi) = \mathcal{A} \cdot \mathcal{K}$ .

**Proof.** Let us prove by induction on the index  $l$  of  $p$  that  $\forall p \in A, \phi(p) = \sigma(p) + \kappa$  with  $\kappa \in \mathcal{A} \cdot \mathcal{K}$ .

If  $p$  is of index 0, then  $p = 1$  and  $\phi(1) = 1 = \sigma(1) = 1$ .

Let us assume that it is true for  $l \in \mathbb{N}$  and let  $p \in A$  of index  $l+1$ . Then  $p = \sum_{j \in J} x_j p'_j$  with  $p'_j \in A$  of index  $\leq l$ . Then we have,

$$\begin{aligned} \phi(p) &= \sum_{j \in J} \pi(a_j \phi(p'_j)) \\ &= \sum_{j \in J} a_j \phi(p'_j) + \kappa_0 \\ &= \sum_{j \in J} a_j (\sigma(p'_j) + \kappa'_j) + \kappa_0 \\ &= \sigma(p) + \kappa \end{aligned}$$

with  $\kappa_0 \in \mathcal{K}$  and  $\kappa = \sum_{j \in J} a_j \kappa'_j + \kappa_0 \in \mathcal{A} \cdot \mathcal{K}$ . This proves the induction hypothesis for  $l+1$ .

We deduce that if  $a \in \ker \phi$ , then  $\phi(a) = 0$  and  $\sigma(a) \in \mathcal{A} \cdot \mathcal{K}$ , so that  $\sigma(\ker \phi) \subset \mathcal{A} \cdot \mathcal{K}$ .

Conversely,  $\forall a \in \mathcal{A}, \forall \kappa \in \mathcal{K}$ , there exists  $a \in \mathcal{F}$ ,  $k \in C$  such that  $\sigma(a) = a$  and  $\sigma(k) = \kappa$ . By Proposition 3.7,  $\phi(k) = \pi(\kappa) = 0$  and  $\phi(ak) = a(X) \phi(k) = 0$ . This shows that  $ak \in \ker \phi$  and  $a\kappa = \sigma(ak) \in \sigma(\ker \phi)$ . Therefore  $\mathcal{A} \cdot \mathcal{K} \subset \sigma(\ker \phi)$ .  $\square$

## The extension of $L$

We define the extension  $\mathcal{L}$  of  $L$  on  $\mathcal{F}$  by

$$\begin{aligned} \mathcal{L} : \mathcal{F} &\rightarrow \mathbb{C} \\ a &\mapsto L(\phi(a)). \end{aligned} \tag{3}$$

We recall that  $\mathcal{A} = \mathcal{F}/\mathcal{I}$  where  $\mathcal{I}$  is the left and right ideal degenerated by the elements of  $S$ . By Proposition 3.6,  $\phi(\mathcal{I}) = 0$  and  $\mathcal{L}$  induces a linear form on  $\mathcal{A} = \mathcal{F}/\mathcal{I}$ , that we still denote  $\mathcal{L}$ . If  $\mathfrak{a} = \sigma(a)$  with  $a \in \mathcal{F}$ , then  $\mathcal{L}(\mathfrak{a}) = L(\phi(a)) = L(a(X)(1))$ .

We first check that  $\mathcal{L}$  is an extension of  $L$ .

**Proposition 3.9**  $\forall \mathfrak{a}, \mathfrak{b} \in \mathcal{C}, \mathcal{L}(\mathfrak{a}\mathfrak{b}) = L(\mathfrak{a}\mathfrak{b})$ .

**Proof.** Let  $\mathfrak{a} = \sigma(a), \mathfrak{b} = \sigma(b) \in \mathcal{C}$  with  $a, b \in C$ . By Proposition 3.7,

$$\begin{aligned} \mathfrak{a} &= \pi(a) + \kappa = \phi(a) + \kappa, \\ \mathfrak{b} &= \pi(b) + \kappa' = \phi(b) + \kappa' \end{aligned}$$

with  $\kappa, \kappa' \in K_L(\mathcal{C})$ . Relation (1) implies have

$$L(\mathfrak{a}\mathfrak{b}) = L((\phi(a) + \kappa)(\phi(b) + \kappa')) = L(\phi(a)\phi(b)).$$

We can therefore assume that  $\mathfrak{a} = \sigma(a) = \phi(a), \mathfrak{b} = \sigma(b) = \phi(b) \in \mathcal{B}$  with  $a, b \in B$  in order to prove that  $\mathcal{L}(\mathfrak{a}\mathfrak{b}) = L(\phi(ab)) = L(\mathfrak{a}\mathfrak{b})$ .

By Lemma 3.5,  $\phi(ab) = \mathfrak{a}\phi(b) + \kappa_1\phi(b) + \kappa_0$  with  $\kappa_0, \kappa_1 \in K_L(\mathcal{C})$ . By Proposition 3.7,  $\phi(b) = \pi(b) = \mathfrak{b} + \kappa'$  with  $\kappa' \in K_L(\mathcal{C})$ . We deduce that

$$\phi(ab) = \mathfrak{a}\mathfrak{b} + \mathfrak{a}\kappa' + \kappa_1\phi(b) + \kappa_0.$$

The relation (1) implies that  $L(\mathfrak{a}\kappa' + \kappa_1\phi(b) + \kappa_0) = 0$ , which proves that  $\mathcal{L}(\mathfrak{a}\mathfrak{b}) = L(\phi(ab)) = L(\mathfrak{a}\mathfrak{b})$ .  $\square$

As a consequence of the previous results, we deduce that  $\mathcal{L}$  is a flat extension on  $\mathcal{A}$  with respect to  $\mathcal{B}$ .

**Proposition 3.10**  $K_{\mathcal{L}}(\mathcal{A}) = \mathcal{A} \cdot \mathcal{K}$  and  $\mathcal{A} = \mathcal{B}' \oplus K_{\mathcal{L}}(\mathcal{A})$ .

**Proof.** Let  $\kappa \in \mathcal{K}$ .  $\forall \mathfrak{a} \in \mathcal{A}$ , there exists  $a \in \mathcal{F}$ , such that  $\sigma(a) = \mathfrak{a}$ . We deduce that

$$\mathcal{L}(\mathfrak{a}\kappa) = L(\tilde{\phi}(\mathfrak{a}\kappa)) = L(a(X)\tilde{\phi}(\kappa)) = 0$$

since  $\tilde{\phi}(\kappa) = \pi(\kappa) = 0$  by Proposition 3.7. We deduce that  $\kappa \in K_{\mathcal{L}}(\mathcal{A})$ . As  $K_{\mathcal{L}}(\mathcal{A})$  is a left ideal, we have  $\mathcal{A} \cdot \mathcal{K} \subset K_{\mathcal{L}}(\mathcal{A})$ .

Conversely, let  $\kappa \in K_{\mathcal{L}}(\mathcal{A})$ . Then  $\forall \mathfrak{a} \in \mathcal{C}$ , there exists  $a \in \mathcal{C}$ , such that  $\sigma(a) = \mathfrak{a}$ . By proposition 3.7,  $\tilde{\phi} \circ \tilde{\phi}(\kappa) = \pi(\tilde{\phi}(\kappa)) = \tilde{\phi}(\kappa)$  and by Proposition 3.9, we have

$$\mathcal{L}(\mathfrak{a}\kappa) = L(a(X)\tilde{\phi}(\kappa)) = L(\tilde{\phi}(\mathfrak{a}\tilde{\phi}(\kappa))) = \mathcal{L}(\mathfrak{a}\phi(\kappa)) = L(\mathfrak{a}\phi(\kappa)) = 0.$$

This implies that  $\phi(\kappa) \in K_L(\mathcal{C}) \cap \mathcal{B}' = \{0\}$ . We deduce that  $\kappa \in \ker \tilde{\phi} = \mathcal{A} \cdot \mathcal{K}$  (by Proposition 3.8). This proves the reverse inclusion  $K_{\mathcal{L}}(\mathcal{A}) \subset \mathcal{A} \cdot \mathcal{K}$ .

Let  $\mathfrak{a} \in \mathcal{A}$ . It can be decomposed as

$$\mathfrak{a} = \tilde{\phi}(\mathfrak{a}) + \mathfrak{a} - \tilde{\phi}(\mathfrak{a}),$$

with  $\tilde{\phi}(\mathfrak{a}) \in \mathcal{B}'$  and  $\mathfrak{a} - \tilde{\phi}(\mathfrak{a}) \in \ker \tilde{\phi} = \mathcal{A} \cdot \mathcal{K} = K_{\mathcal{L}}(\mathcal{A})$ . This shows that  $\mathcal{A} = \mathcal{B}' + K_{\mathcal{L}}(\mathcal{A})$ .

To prove that the sum is direct, we consider an element  $\mathfrak{b} \in \mathcal{B}' \cap K_{\mathcal{L}}(\mathcal{A})$ . Then  $\tilde{\phi}(\mathfrak{b}) = \mathfrak{b}$  by Proposition 3.7 and  $\tilde{\phi}(\mathfrak{b}) = 0$  since  $\mathfrak{b} \in K_{\mathcal{L}}(\mathcal{A}) = \mathcal{A} \cdot \mathcal{K} = \ker \tilde{\phi}$ . We deduce that  $\mathfrak{b} = 0$  and therefore that  $\mathcal{A} = \mathcal{B}' \oplus K_{\mathcal{L}}(\mathcal{A})$ .  $\square$

The next proposition shows the uniqueness of the extension.

**Proposition 3.11** *There exists a unique hermitian linear form  $\mathcal{L} \in \mathcal{A}^*$  which extends  $L$  and is a flat extension with respect to  $\mathcal{B}$ .*

**Proof.** If  $\mathcal{L}$  extends  $L$  and is a flat extension with respect to  $\mathcal{B}$  then  $\mathcal{A} = \mathcal{B} + K_{\mathcal{L}}(\mathcal{A})$ . As  $\mathcal{L}$  is hermitian,  $K_{\mathcal{L}}(\mathcal{A})^* = K_{\mathcal{L}}(\mathcal{A})$ . Then  $\forall \kappa \in \mathcal{K}, \forall \mathbf{a} \in \mathcal{A}, \mathbf{a} = \mathbf{b} + \kappa'$  with  $\kappa' \in K_{\mathcal{L}}(\mathcal{A})$  and

$$\mathcal{L}(\mathbf{a}\kappa) = \mathcal{L}(\mathbf{b}\kappa) + \mathcal{L}(\kappa'\kappa) = \mathcal{L}(\mathbf{b}\kappa) + \overline{\mathcal{L}(\kappa^* \kappa'^*)} = L(\mathbf{b}\kappa) = 0,$$

since  $\kappa'^* \in K_{\mathcal{L}}(\mathcal{A})$ ,  $\mathbf{b}\kappa \in \mathcal{C}^2$  and  $\kappa \in K_L(\mathcal{C})$ . This proves that  $\mathcal{A} \cdot \mathcal{K} \subset K_{\mathcal{L}}(\mathcal{A})$ .

For any  $\mathbf{a} \in \mathcal{A}$ , let  $\mathbf{b} = \tilde{\phi}(\mathbf{a}) \in \mathcal{B}$  so that  $\mathbf{a} - \mathbf{b} \in \mathcal{A} \cdot \mathcal{K} \subset K_{\mathcal{L}}(\mathcal{A})$ . This implies that

$$\mathcal{L}(\mathbf{a}) = \mathcal{L}(\mathbf{b}) = L(\mathbf{b}) = L(\tilde{\phi}(\mathbf{a})).$$

This shows that  $\mathcal{L}$  coincides with the linear form defined in (3) and thus that the extension  $\mathcal{L}$  is unique.  $\square$

Finally, we show that the positivity of  $L$  can also be extended to  $\mathcal{L}$ .

**Proposition 3.12** *If  $\forall \mathbf{a} \in \mathcal{C}, L(\mathbf{a}^* \mathbf{a}) \geq 0$ , then  $\forall \mathbf{a} \in \mathcal{A}, \mathcal{L}(\mathbf{a}^* \mathbf{a}) \geq 0$ .*

**Proof.**  $\forall \mathbf{a} \in \mathcal{A}$ , let  $\mathbf{b} = \tilde{\phi}(\mathbf{a})$  so that  $\mathbf{a} - \mathbf{b} \in \ker \tilde{\phi} = \mathcal{A} \cdot \mathcal{K} = K_{\mathcal{L}}(\mathcal{A})$ . As  $\tilde{\phi}(\mathbf{a}^*) = \tilde{\phi}(\mathbf{a})^*$  and  $\forall \mathbf{b} \in \mathcal{A}, \tilde{\phi}(\mathbf{a}\mathbf{b})^* = \tilde{\phi}(\mathbf{b}^* \mathbf{a}^*)$ , we deduce from Proposition 3.9 that

$$\begin{aligned} \mathcal{L}(\mathbf{a}^* \mathbf{a}) &= \mathcal{L}((\mathbf{b}^* + (\mathbf{a}^* - \mathbf{b}^*))(\mathbf{b} + (\mathbf{a} - \mathbf{b}))) = \mathcal{L}(\mathbf{b}^* \mathbf{b}) + \mathcal{L}((\mathbf{a}^* - \mathbf{b}^*)\mathbf{b}) \\ &= \mathcal{L}(\mathbf{b}^* \mathbf{b}) + \overline{\mathcal{L}(\mathbf{b}^*(\mathbf{a} - \mathbf{b}))} = \mathcal{L}(\mathbf{b}^* \mathbf{b}) = L(\mathbf{b}^* \mathbf{b}). \end{aligned}$$

As  $L$  is positive,  $\mathcal{L}(\mathbf{a}^* \mathbf{a}) = L(\mathbf{b}^* \mathbf{b}) \geq 0$  so that  $\mathcal{L}$  is also positive.  $\square$

## Representation of $\mathcal{A}$

We can now construct the representation of  $\mathcal{A}$  on the vector space  $\mathcal{B}'$ , using the decomposition of Proposition 3.10. Let denote by  $\pi_{\mathcal{L}}$  the projection operator of  $\mathcal{A}$  on  $\mathcal{B}'$  along  $K_{\mathcal{L}}(\mathcal{A})$ . By definition of  $K_{\mathcal{L}}(\mathcal{A})$ , for all  $\mathbf{a}, \mathbf{b} \in \mathcal{A}$ , we have  $\langle \pi_{\mathcal{L}}(\mathbf{a}), \pi_{\mathcal{L}}(\mathbf{b}) \rangle_{\mathcal{L}} = \mathcal{L}(\mathbf{b}^* \mathbf{a})$ . For  $\mathbf{a} \in \mathcal{A}$ , let  $\rho_{\mathcal{L}}(\mathbf{a})$  denote the linear operator on  $\mathcal{B}'$  defined by

$$\rho_{\mathcal{L}}(\mathbf{a})\mathbf{b} = \pi_{\mathcal{L}}(\mathbf{a}\mathbf{b}), \quad \text{where } \mathbf{a} \in \mathcal{A}, \mathbf{b} \in \mathcal{B}'.$$

**Proposition 3.13**  *$\rho_{\mathcal{L}}$  is the  $*$ -representation of  $\mathcal{A}$  on  $(\mathcal{B}', \langle \cdot, \cdot \rangle_{\mathcal{L}})$  which is associated with the functional  $\mathcal{L}$ .*

**Proof.** By construction,  $\forall \mathbf{a} \in \mathcal{A}, \rho_{\mathcal{L}}(\mathbf{a})$  is a linear operator of  $\mathcal{B}'$ . We have to prove first that  $\forall \mathbf{a}, \mathbf{b} \in \mathcal{A}, \forall \mathbf{c} \in \mathcal{B}', \rho_{\mathcal{L}}(\mathbf{a}\mathbf{b})(\mathbf{c}) = \rho_{\mathcal{L}}(\mathbf{a}) \circ \rho_{\mathcal{L}}(\mathbf{b})(\mathbf{c})$ :

$$\begin{aligned} \rho_{\mathcal{L}}(\mathbf{a}\mathbf{b})(\mathbf{c}) &= \pi_{\mathcal{L}}(\mathbf{a}\mathbf{b}\mathbf{c}) = \pi_{\mathcal{L}}(\mathbf{a}\pi_{\mathcal{L}}(\mathbf{b}\mathbf{c})) + \pi_{\mathcal{L}}(\mathbf{a}\kappa) \quad (\kappa \in \mathcal{A} \cdot \mathcal{K}) \\ &= \pi_{\mathcal{L}}(\mathbf{a}\pi_{\mathcal{L}}(\mathbf{b}\mathbf{c})) = \rho_{\mathcal{L}}(\mathbf{a}) \circ \rho_{\mathcal{L}}(\mathbf{b})(\mathbf{c}) \end{aligned}$$

since  $\mathbf{a}\kappa \in \mathcal{A} \cdot \mathcal{K} = \ker \pi_{\mathcal{L}}$  (Proposition 3.10).

We have to prove secondly that  $\forall \mathbf{a} \in \mathcal{A}, \forall \mathbf{b}, \mathbf{c} \in \mathcal{B}', \langle \rho_{\mathcal{L}}(\mathbf{a})\mathbf{b}, \mathbf{c} \rangle_{\mathcal{L}} = \langle \mathbf{b}, \rho_{\mathcal{L}}(\mathbf{a}^*)\mathbf{c} \rangle_{\mathcal{L}}$ :

$$\begin{aligned} \langle \rho_{\mathcal{L}}(\mathbf{a})\mathbf{b}, \mathbf{c} \rangle_{\mathcal{L}} &= \mathcal{L}(\mathbf{c}^* \pi_{\mathcal{L}}(\mathbf{a}\mathbf{b})) \\ &= \mathcal{L}(\mathbf{c}^* \mathbf{a}\mathbf{b}) + \mathcal{L}(\mathbf{c}^* \kappa) = \mathcal{L}(\mathbf{c}^* \mathbf{a}\mathbf{b}) \quad (\kappa \in K_{\mathcal{L}}(\mathcal{A})) \\ &= \overline{\mathcal{L}(\mathbf{b}^* \mathbf{a}^* \mathbf{c})} = \overline{\mathcal{L}(\mathbf{b}^* \pi_{\mathcal{L}}(\mathbf{a}^* \mathbf{c}))} + \overline{\mathcal{L}(\mathbf{b}^* \kappa')} \quad (\kappa' \in K_{\mathcal{L}}(\mathcal{A})) \\ &= \overline{\mathcal{L}(\mathbf{b}^* \rho_{\mathcal{L}}(\mathbf{a}^*)\mathbf{c})} = \langle \rho_{\mathcal{L}}(\mathbf{a}^*)\mathbf{c}, \mathbf{b} \rangle_{\mathcal{L}} = \langle \mathbf{b}, \rho_{\mathcal{L}}(\mathbf{a}^*)\mathbf{c} \rangle_{\mathcal{L}}. \end{aligned}$$

This concludes the proof that  $\rho_{\mathcal{L}}$  is a  $*$ -representation of  $\mathcal{A}$  on  $\mathcal{B}'$ . Since  $\mathcal{L}(\mathbf{a}) = \langle \rho_{\mathcal{L}}(\mathbf{a})1, 1 \rangle_{\mathcal{L}}$  for all  $\mathbf{a} \in \mathcal{A}$ ,  $\rho_{\mathcal{L}}$  is the  $*$ -representation associated with  $\mathcal{L}$ .  $\square$

## 4 Applications

Let  $\pi$  be a  $*$ -representation of  $\mathcal{A}$  on a unitary space  $(V, \langle \cdot, \cdot \rangle)$ . For any  $v \in V$ , the linear functional  $L_v$  on  $\mathcal{A}$  defined by  $L_v(\cdot) = \langle \pi(\cdot)v, v \rangle$  is positive and  $\mathcal{K}_v := \{a \in \mathcal{A} : \pi(a)v = 0\}$  is a two-sided  $*$ -ideal of  $\mathcal{A}$ . Let  $\mathcal{B}$  and  $\mathcal{C}$  be  $*$ -invariant linear subspaces of  $\mathcal{A}$  such that  $1 \in \mathcal{B}$  and  $\mathcal{B} \subseteq \mathcal{C}$ . Clearly,  $\mathcal{K}_L(\mathcal{C}) = \mathcal{K}_v \cap \mathcal{C}$ . Then the restriction of  $L_v$  to  $\mathcal{C}^2$  is a flat extension with respect to  $\mathcal{B}$  if and only if  $\mathcal{C} = \mathcal{B} + \mathcal{K}_v \cap \mathcal{C}$ .

Suppose in addition that  $V_v := \pi(\mathcal{A})v$  is finite dimensional. Then  $\mathcal{A}/\mathcal{K}_v$  is finite dimensional, so we can choose a finite dimensional  $*$ -invariant subspace  $\mathcal{B}$  containing 1 such that  $\mathcal{A} = \mathcal{B} + \mathcal{K}_v$ . Then, for any  $*$ -invariant subspace  $\mathcal{C}$  of  $\mathcal{A}$  which contains  $\mathcal{B}$ , the restriction of  $L_v$  to  $\mathcal{C}^2$  is a flat extension with respect to  $\mathcal{B}$ . This provides a large class of examples of flat extensions.

We now develop the three applications of Theorem 3.1 mentioned in the Introduction.

### 4.1 Truncated moment problem on cylinder sets

Let  $\mathcal{A}$  be the polynomial  $*$ -algebra  $\mathbb{C}[x_1, \dots, x_d, y]$  in  $d+1$  hermitean variables  $x_1, \dots, x_d, y$ . The algebra  $\mathcal{A}$  is the quotient of the free  $*$ -algebra  $\mathcal{F} = \mathbb{C}\langle x_1, \dots, x_d, y \rangle$  by the commutation relations  $x_i x_j - x_j x_i = 0$ ,  $x_i y - y x_i = 0$ , so that  $\delta(\mathcal{A}) = 2$ . For  $k \in \mathbb{N}$ , let  $\mathcal{A}_k$  be the linear span of  $x^\alpha \mathbb{C}[y]$ , where  $\alpha \in \mathbb{N}_0^d$ ,  $|\alpha| \leq k$ .

**Proposition 4.1** *Let  $m \in \mathbb{N}$ ,  $\mathcal{B} = \mathcal{A}_m$  and  $\mathcal{C} = \mathcal{A}_{m+1}$ . Suppose that  $L$  is a positive linear functional on  $\mathcal{C}^2$  which is a flat extension with respect to  $\mathcal{B}$ . Then there exist finitely points  $t_1, \dots, t_k \in \mathbb{R}^d$ , where  $k \leq \binom{d+1+m}{m}$ , and a positive Borel measure  $\mu$  on  $\mathbb{R}^{d+1}$  supported by the set  $\cup_{j=1}^k t_j \times \mathbb{R}$  such that*

$$L(p) = \int_{\mathbb{R}^d \times \mathbb{R}} p(x, y) d\mu(x, y) \quad \text{for } p \in \mathcal{C}^2 = \mathcal{A}_{2m+2}.$$

**Proof.** Since  $\delta(\mathcal{A}) = 2$  and  $\mathcal{B}^{[1]} = \mathcal{A}_m^{[1]} = \mathcal{A}_{m+1} = \mathcal{C}$ , the assumptions of Theorem 3.1 are fulfilled. Hence  $L$  has an extension to a positive linear functional  $\tilde{L}$  on  $\mathcal{A}$  which is a flat extension with respect to  $\mathcal{B}$ . Let  $L_0$  and  $\tilde{L}_0$  denote the restrictions of  $L$  and  $\tilde{L}$ , respectively, to the  $*$ -subalgebra  $\mathbb{C}[x_1, \dots, x_d]$  of  $\mathcal{A}$ . Then  $\tilde{L}_0$  is a flat extension of  $L_0$  with respect to  $\mathcal{B} \cap \mathbb{C}[x_1, \dots, x_d]$ . Therefore, by the theorem of Curto and Fialkow [2],  $\tilde{L}_0$  is given by a  $k$ -atomic measure  $\mu_0$  on  $\mathbb{R}^d$ , where  $k \leq \binom{d+1+m}{m}$ . Let  $t_1, \dots, t_k$  denote the atoms of  $\mu_0$ . For  $f \in \mathcal{A}$ , it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} |\tilde{L}(\|x - t_1\|^2 \cdots \|x - t_k\|^2 f)|^2 &\leq \tilde{L}(\|x - t_1\|^4 \cdots \|x - t_k\|^4) \tilde{L}(f^2) \\ &= \tilde{L}_0(\|x - t_1\|^4 \cdots \|x - t_k\|^4) \tilde{L}(f^2) = 0. \end{aligned}$$

Hence  $\tilde{L}$  vanishes on the ideal  $\mathcal{J}$  generated by the polynomial  $\|x - t_1\|^2 \cdots \|x - t_k\|^2$ . The zero set of the ideal  $\mathcal{J}$  is the set  $\{t_1, \dots, t_k\}$ . Hence (by [6, Note 2.1.8])  $\mathcal{J} + \sum \mathcal{A}^2$  is the preorder of the semi-algebraic set  $K := \cup_{j=1}^k t_j \times \mathbb{R}$  in  $\mathbb{R}^{d+1}$ . Since  $K$  is a cylinder with finite, hence compact, base set  $\cup_{j=1}^k t_j$  in  $\mathbb{R}^d$ , it follows from Corollary 10 in [10] (see also [3]) that  $K$  has property (SMP), that is,  $\tilde{L}$  is given by a positive Borel measure  $\mu$  supported by  $K$ .  $\square$

### 4.2 Truncated moment problem for matrices of polynomials

Let  $\mathcal{A}$  be the unital  $*$ -algebra of  $n \times n$  matrices with entries from  $\mathbb{C}[x_1, \dots, x_d]$ . Let  $e_{ij}$  denote the corresponding matrix units. We have  $e_{ij}^* = e_{ji}$ . The algebra  $\mathcal{A}$  is the quotient of the free unital  $*$ -algebra  $\mathcal{F} = \mathbb{C}\langle x_1, \dots, x_d, e_{11}, \dots, e_{nn} \rangle$  by the ideal generated by the commutation relations between the variables  $x_i$ , the commutation relations  $x_i e_{jk} - e_{jk} x_i = 0$  and the product relations  $e_{ij} e_{jl} = e_{il}$ ,  $e_{ij} e_{kl} = 0$  if  $k \neq l$ . This implies that  $\delta(\mathcal{A}) = 2$ .

First we prove a simple well-known lemma.

**Lemma 4.2** *Let  $\rho$  be an irreducible  $*$ -representation of  $\mathcal{A}$  on a finite dimensional unitary space  $V$ . Then there exist a unitary operator  $U$  of  $\mathbb{C}^d$  onto  $V$  and a point  $t_0 \in \mathbb{R}^d$  such that  $\rho((p_{jk})) = U((p_{jk}(t_0))U^{-1}$  for all matrices  $(p_{jk}) \in \mathcal{A}$ .*

**Proof.** Since the unit matrix  $E$  is the unit element of  $\mathcal{A}$ ,  $\rho(E) = I$  by Definition 2.4. Hence, for any  $p \in \mathbb{C}[x_1, \dots, x_d]$  the operator  $\rho(pE)$  belongs to the commutant of  $\rho(\mathcal{A})$ . Therefore, since  $\rho$  is irreducible, there is a complex number  $\chi(p)$  such that  $\rho(pE) = \chi(p)I$ . Clearly,  $p \rightarrow \rho(pE)$  is a  $*$ -homomorphism. Hence the map  $p \rightarrow \chi(p)$  is a character on the  $*$ -algebra  $\mathbb{C}[x_1, \dots, x_d]$ , so there is a point  $t_0 \in \mathbb{R}^d$  such that  $\chi(p) = p(t_0)$  for  $p \in \mathbb{C}[x_1, \dots, x_d]$ .

Let  $M_d(\mathbb{C})$  denote the  $*$ -subalgebra of constant matrices in  $\mathcal{A}$ . For  $(p_{jk}) \in \mathcal{A}$  we have  $\rho((p_{jk})) = \sum_{j,k} \rho(p_{jk}E)\rho(e_{jk})$ . This implies that  $\rho(\mathcal{A})$  and  $\rho(M_d(\mathbb{C}))$  have the same commutants. Therefore,  $\rho|_{M_d(\mathbb{C})}$  is an irreducible  $*$ -representation of the matrix algebra  $M_d(\mathbb{C})$  and hence unitarily equivalent to the identity representation. That is, there is a unitary  $U$  of  $\mathbb{C}^d$  onto  $V$  such that  $\rho(e_{jk}) = Ue_{jk}U^{-1}$  for  $j, k = 1, \dots, d$ . Then, for  $(p_{jk}) \in \mathcal{A}$ , we derive

$$\begin{aligned} \rho((p_{jk})) &= \sum_{j,k=1}^d \rho(p_{jk}E)\rho(e_{jk}) = \sum_{j,k=1}^d p_{jk}(t_0)Ue_{jk}U^{-1} \\ &= U\left(\sum_{j,k=1}^d p_{jk}(t_0)e_{jk}\right)U^{-1} = U(p_{jk}(t_0))U^{-1}. \end{aligned}$$

□

Fixe  $k \in \mathbb{N}$ . We denote by  $\mathcal{A}_k$  the span of all elements  $x^\alpha e_{ij}$ , where  $|\alpha| \leq k$  and  $i, j = 1, \dots, n$ . Here  $x^\alpha e_{ij}$  denotes the matrix with entry  $x^\alpha$  at the  $(i, j)$ -place and zero otherwise.

**Proposition 4.3** *Suppose that  $m \in \mathbb{N}$ ,  $\mathcal{B} = \mathcal{A}_m$ ,  $\mathcal{C} = \mathcal{A}_{m+1}$ , and  $L$  is a positive linear functional on  $\mathcal{C}^2$  which is a flat extension with respect to  $\mathcal{B}$ . Then there exist points  $t_i \in \mathbb{R}^d$  and vectors  $u_i = (u_{li}) \in \mathbb{C}^d$ ,  $i = 1, \dots, r$ ,  $r \in \mathbb{N}$ , such that*

$$L((p_{jk})) = \sum_{j,k=1}^d \sum_{i=1}^r p_{jk}(t_i)u_{ki}\overline{u_{ji}} \quad \text{for } (p_{jk}) \in \mathcal{C}^2 = \mathcal{A}_{2m+2}. \quad (4)$$

**Proof.** As  $\delta(\mathcal{A}) = 2$  and  $\mathcal{B}^{[1]} = \mathcal{A}_m^{[1]} = \mathcal{A}_{m+1} = \mathcal{C}$ , by Theorem 3.1,  $L$  has a flat extension to a positive linear functional  $\tilde{L}$  on the whole algebra  $\mathcal{A}$ . Let  $\rho_{\tilde{L}}$  denote the GNS representation of  $\tilde{L}$ . Since  $\rho_{\tilde{L}}$  acts on a subspace of the finite dimensional space  $\mathcal{B}$ , it is an orthogonal direct sum of finite dimensional irreducible  $*$ -representations  $\rho_i$  acting on unitary spaces  $V_i$ ,  $i = 1, \dots, r$ . Each representation  $\rho_i$  is of the form described in Lemma 4.2. For  $(p_{jk}) \in \mathcal{A}$ , we then obtain

$$\tilde{L}((p_{jk})) = \langle \rho_{\tilde{L}}((p_{jk}))v, v \rangle_V = \sum_{i=1}^r \langle \rho_i((p_{jk}))v_i, v_i \rangle_{V_i} = \sum_{i=1}^r \sum_{j,k=1}^d p_{jk}(t_i)u_{ki}\overline{u_{ji}}$$

which implies (4), since  $\tilde{L}$  is an extension of  $L$ . Here  $v$  is the direct sum of vectors  $v_i$ . Further,  $u_i = U_i^{-1}v_i$  and  $t_i$  denotes the point and  $U_i$  is the unitary from Lemma 4.2 applied to the irreducible representation  $\rho_i$ . □

### 4.3 Truncated moment problem for enveloping algebras

Let  $\mathfrak{g}$  be real finite dimensional Lie algebra. Recall that the universal enveloping algebra  $\mathcal{A} := \mathcal{E}(\mathfrak{g})$  of  $\mathfrak{g}$  is a complex unital  $*$ -algebra with involution determined by the requirement  $(iy)^* = iy$  for  $y \in \mathfrak{g}$ . Fix a basis  $\{y_1, \dots, y_d\}$  of the real vector space  $\mathfrak{g}$ . Then there are real numbers  $c_{jkl}$ ,  $j, k, l = 1, \dots, d$ , the structure constants of the Lie algebra  $\mathfrak{g}$ , such that

$$[y_j, y_k] = \sum_l c_{jkl} y_l.$$

The  $*$ -algebra  $\mathcal{E}(\mathfrak{g})$  is the quotient algebra of the free  $*$ -algebra  $\mathcal{F}(x_1, \dots, x_d)$  by the two-sided  $*$ -ideal  $\mathcal{I}$  generated by the set  $\mathcal{I}_G$  of elements

$$x_j x_k - x_k x_j - \sum_l c_{jkl} i x_l, \quad l = 1, \dots, d, \quad (5)$$

where the quotient map  $\sigma$  is given by  $\sigma(x_j) = iy_j$ ,  $j = 1, \dots, d$ . We deduce that  $\delta(\mathcal{A}) \leq 2$ . Further, by the Poincare-Birkhoff-Witt theorem, the set

$$\{y_1^{n_1} \dots y_d^{n_d}; (n_1, \dots, n_d) \in N_0^d\}$$

is a vector space basis of  $\mathcal{E}(\mathfrak{g})$ . For  $m \in \mathbb{N}$ , let  $\mathcal{A}_m$  denote the linear span of elements  $y_1^{n_1} \dots y_d^{n_d}$ , where  $n_1 + \dots + n_d \leq m$ .

**Proposition 4.4** *Let  $\mathcal{B} = \mathcal{A}_m$  and  $\mathcal{C} = \mathcal{A}_{m+1}$ . Suppose that  $L$  is a positive linear functional on  $\mathcal{C}^2$  which is a flat extension with respect to  $\mathcal{B}$ . Then  $L$  has an extension to a positive linear functional  $\tilde{L}$  on  $\mathcal{A}$  and  $\tilde{L}$  is a flat extension with respect to  $\mathcal{B}$ .*

**Proof.** Since the ideal  $\mathcal{I}$  is generated the quadratic elements in (5),  $\delta(\mathcal{A}) \leq 2$  and  $\mathcal{B}^{[1]} = \mathcal{A}_m^{[1]} = \mathcal{A}_{m+1} = \mathcal{C}$ , the assumptions of Theorem 3.1 are fulfilled which gives the assertion.  $\square$

Let  $\mathcal{R}_{\text{fin}}(\mathfrak{g})$  denote the family of all  $*$ -representations of the  $*$ -algebra  $\mathcal{E}(\mathfrak{g})$  acting on finite dimensional Hilbert spaces. By Theorem 3.1, the GNS representation  $\rho_{\tilde{L}}$  associated with the positive functional  $\tilde{L}$  on  $\mathcal{E}(\mathfrak{g})$  acts on a subspace of the finite dimensional space  $\mathcal{B}$ . Hence  $\rho_{\tilde{L}}$  belongs to  $\mathcal{R}_{\text{fin}}(\mathfrak{g})$  and there is a vector  $v$  in the representation space of  $\rho_{\tilde{L}}$  such that

$$L(a) = \langle \rho_{\tilde{L}}(a)v, v \rangle \quad \text{for } a \in \mathcal{C}^2 = \mathcal{A}_{2m+2}. \quad (6)$$

Let  $G$  be the simply connected Lie group which has the Lie algebra  $\mathfrak{g}$ . Then the  $*$ -representation  $\rho_{\tilde{L}}$  of  $\mathcal{E}(\mathfrak{g})$  exponentiates to a unitary representation  $U$  of  $G$ . That is, we have  $\rho_{\tilde{L}} = dU$  and equation (6) can be considered as a solution of a truncated non-commutative moment problem for the enveloping algebra  $\mathcal{E}(\mathfrak{g})$ , see e.g. [8]. Note that it may happen that there is no non-trivial finite dimensional  $*$ -representation of  $\mathcal{E}(\mathfrak{g})$ . However, if the Lie group  $G$  is *compact*, all irreducible unitary representations of  $G$  are finite dimensional, so there is a rich theory of truncated non-commutative moment problems for  $\mathcal{E}(\mathfrak{g})$ .

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