HAL open science

SINR in wireless networks and the two-parameter Poisson-Dirichlet process

Holger Paul Keeler, Bartlomiej Blaszczyszyn

To cite this version:

Holger Paul Keeler, Bartlomiej Blaszczyszyn. SINR in wireless networks and the two-parameter Poisson-Dirichlet process. IEEE Wireless Communications Letters, 2014, 3 (5), pp.525-528. 10.1109/LWC.2014.2345691 . hal-01002977v5

HAL Id: hal-01002977

https://inria.hal.science/hal-01002977v5

Submitted on 2 Aug 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

SINR in wireless networks and the two-parameter Poisson-Dirichlet process

Holger Paul Keeler and Bartłomiej Błaszczyszyn

***Abstract*—Stochastic geometry models of wireless networks based on Poisson point processes are increasingly being developed with a focus on studying various signal-to-interference-plus-noise ratio (SINR) values. We show that the SINR values experienced by a typical user with respect to different base stations of a Poissonian cellular network are related to a specific instance of the so-called two-parameter Poisson-Dirichlet process. This process has many interesting properties as well as applications in various fields. We give examples of several results proved for this process that are of immediate or potential interest in the development of analytic tools for cellular networks. Some of them simplify or are akin to certain results that are being developed in the network literature. By doing this we hope to motivate further research and use of Poisson-Dirichlet processes in this new setting.**

***Index Terms*—SINR process, Poisson-Dirichlet process, factorial moment measures.**

I. Introduction

To derive accurate analytic tools of cellular networks, stochastic geometry models have been developed with the almost standard assumption that the network base stations are positioned according to a Poisson point process. The aim of these models is often to derive distributional characteristics of various signal-to-interference-plus-noise ratio (SINR) values, which are, due to information theoretic arguments, related to network performance characteristics and user quality of service metrics. Besides tractability and 'worst-case' arguments, the Poisson assumption is justified by a recent convergence result [1] showing that a large class of stationary network configurations give results for functions of incoming signal strengths, such as the SINR, as though the placement of the base stations is a Poisson process when sufficiently large log-normal shadowing is incorporated into the model [1].

A stochastic process known as the two-parameter Poisson-Dirichlet process has been thoroughly studied over the years owing to the discovery of its many interesting properties and relations to other random structures and applications in various fields such as population genetics, number theory, Bayesian statistics and economics [3, 4]. In this letter we detail how a specific case of this well-studied process is equivalent to (what we call) the process of *signal-to-total-interference-ratio (STIR)* values experienced by a typical user with respect to different base stations of a Poissonian cellular network. The STIR process is trivially related to the signal-to-interference-ratio (SIR) process and further to the SINR one. We then list and apply results that have been derived in different settings and suggest results that may be useful in the future.

INRIA-ENS, 23 Avenue d'Italie, 75214 Paris, France
Bartek.Blaszczyszyn@ens.fr, keeler@ens.fr

[1]This result is actually more general and holds under log-normal-type distributions, e.g. the Suzuki model [2].

For related work, there is a number of Poisson-based models with a focus on calculating the distribution of the SINR; see [5, 6] and references therein. Błaszczyszyn and Keeler [6] characterized the SINR process by obtaining its *factorial moment measures*. The densities of these measure lead to the joint probability density of the order statistics of the SINR process, which can be used to calculate the coverage probability under some signal combination and interference cancellation models [6]. Invariance properties of Poisson models have been investigated in connection to general random marks [7] and the special case of log-normal shadowing marks [2]. Equivalent results in relation to the Sherrington-Kirkpatrick spin glass model have been derived independently in physics, as detailed by Panchenko [8].

The two-parameter Poisson-Dirichlet process is examined by Pitman and Yor [3], hence it is also called the Pitman-Yor process, though it was introduced earlier by Pearman, Pitman, and Yor [9]. Handa [10] derived the *factorial moment density* (or *correlation function*) of the process and other useful results. Kingman [11] covers the Poisson point process and its relationships to subordinators and the original (i.e. one-parameter or Kingman's) Poisson-Dirichlet process. In physics a related but different one-parameter process is sometimes also called the Poisson-Dirichlet process [8] [2], which is exactly our STIR process. This process and Kingman's one are both special cases of the two-parameter Poisson-Dirichlet process.

We believe that we are the first to illustrate these connections and in doing so it is our hope that certain results on the two-parameter Poisson-Dirichlet process will be adopted and used to develop analytic tools for studying SINR-based characteristics in communication networks.

II. Network model and quantities of interest

We consider the "typical user" approach where one assumes a typical user is located at the origin. On $\mathbb{R}^2$, we model the base stations with a homogeneous or stationary Poisson point process $\Phi = \{X\}$ with density λ. Define the path-loss function as $\ell(|x|) = (K|x|)^\beta$, with constants $K > 0$ and $2 < \beta < \infty$ assumed henceforth. Given Φ, let $\{S_X\}_{X\in\Phi}$ be a collection of independent and identically and *arbitrarily* distributed random variables representing the random *propagation effects* (i.e fading and/or shadowing) from the origin to X. Let S be equal in distribution to S_X. In this paper we will always (tacitly) require the moment condition $\mathbf{E}(S^{\frac{2}{\beta}}) < \infty$.

We define the propagation (loss) process [3], considered as a

[2]It appears as the thermodynamic (large system) limit in the low temperature regime of Derrida's random energy model and a key component of the so called Ruelle probability cascades, which are used to represent the thermodynamic limit of the Sherrington-Kirkpatrick model for spin glasses.

[3]We introduce the propagation process for historical reasons, in particular to be consistent with [6] and papers cited therein. Otherwise the process of received powers can be considered.

point process on the positive half-line $\mathbb{R}^+$, as

$$\Theta = \{Y\} := \left\{ \frac{\ell(|X|)}{S_X} : X \in \Phi \right\}. \tag{1}$$

Lemma 1. *The propagation process $\{Y\}$ is an inhomogeneous Poisson point process with intensity measure $\Lambda_\Theta([0,t)) = at^{\frac{2}{\beta}}$, where*

$$a := \frac{\lambda \pi \mathbf{E}[S^{\frac{2}{\beta}}]}{K^2}. \tag{2}$$

This invariance result [4] has been observed a number of times; see, e.g. [1] for a proof, [6] for related work, and [2, 7, 8] for generalizations with random marks .

We define the *SINR process* on the positive half-line $\mathbb{R}^+$ as

$$\Psi = \{Z\} := \left\{ \frac{Y^{-1}}{W + (I - Y^{-1})} : Y \in \Theta \right\}, \tag{3}$$

where the constant $W \geq 0$ is the additive noise power, and

$$I = \sum_{Y \in \Theta} Y^{-1}, \tag{4}$$

is the power received from the entire network (so that $I - Y^{-1}$ is the interference).

To represent the *signal-to-total-received-power-and-noise ratio*, we define the STINR process on $(0,1]$ as

$$\Psi' = \{Z'\} := \left\{ \frac{Y^{-1}}{W + I} : Y \in \Theta \right\}. \tag{5}$$

Working with Ψ' is algebraically simpler and information on it gives information on Ψ by the relation $Z = Z'/(1-Z')$ and $Z' = Z/(1+Z)$.

For $n \geq 1$, we define the factorial moment measure $M'^{(n)}(t'_1, \ldots, t'_n) = M'^{(n)}((t'_1, 1] \times \cdots \times (t'_n, 1])$ of the STINR process $\{Z'\}$ as

$$M'^{(n)}(t'_1, \ldots, t'_n) = \mathbf{E}\left(\sum_{\substack{(Z'_1, \ldots, Z'_n) \in (\Psi')^{\times n} \\ \text{distinct}}} \prod_{j=1}^{n} \mathbb{1}(Z'_j > t'_j) \right), \tag{6}$$

where $\mathbb{1}$ is an indicator function. The equivalent measure of $\Psi = \{Z\}$ is defined by analogy but in relation to the rectangle $(t_1, \infty] \times \cdots \times (t_n, \infty]$. Both measures require two integrals. For $x \geq 0$ define

$$\mathcal{I}_{n,\beta}(x) = \frac{2^n \int_0^\infty u^{2n-1} e^{-u^2 - u^\beta x \Gamma(1-2/\beta)^{-\beta/2}} du}{\beta^{n-1} (C'(\beta))^n (n-1)!} \tag{7}$$

where

$$C'(\beta) := 2\pi/(\beta \sin(2\pi/\beta)) = \Gamma(1+2/\beta)\Gamma(1-2/\beta), \tag{8}$$

and Γ is the gamma function. Note that $\mathcal{I}_{n,\beta}(0) = 2^{n-1}/(\beta^{n-1}[C'(\beta)]^n)$. For all $x_i \geq 0$ define

$$\mathcal{J}_{n,\beta}(x_1, \ldots, x_n) = \frac{(1 + \sum_{j=1}^n x_j)}{n} \times \int_{[0,1]^{n-1}} \frac{\prod_{i=1}^{n-1} v_i^{i(2/\beta+1)-1} (1-v_i)^{2/\beta}}{\prod_{i=1}^n (x_i + \eta_i)} dv_1 \ldots dv_{n-1}, \tag{9}$$

[4] One can assume arbitrary propagation effects by setting, e.g. $S = 1$, and replacing λ with $\lambda' = \lambda \mathbf{E}(S^{\frac{2}{\beta}})$. A further generalization exists: an isotropic power-law base station density in d dimensions; see, e.g. [6, Footnote 6] or [12, Lemma 1], where the propagation process is called "path-loss process with fading".

where

$$\eta_1 = v_1 v_2 \ldots v_{n-1}, \quad \eta_2 = (1-v_1) v_2 \ldots v_{n-1},$$
$$\eta_3 = (1-v_2) v_3 \ldots v_{n-1}, \quad \cdots, \quad \eta_n = 1 - v_{n-1}. \tag{10}$$

For reasonably low n (i.e. $n \leq 20$), both these integrals are numerically tractable [13]. Let $\hat{t}_i = \hat{t}_i(t'_1, \ldots, t'_n) := t'_i/(1 - \sum_{j=1}^n t'_j)$ and the define the n-dimensional unit simple

$$\Delta_n = \{(t'_1, \ldots t'_n) : t'_1, \ldots t'_n \geq 0,\ t'_1 + \cdots + t'_n \leq 1\},$$

and $\mathbb{1}_{\Delta_n}$ denotes the corresponding indicator function. We now present the factorial moment measures [6].

Proposition 2. *For $t'_i \in (0,1]$, the factorial moment measure of order $n \geq 1$ of the STINR process (5) satisfies*

$$M'^{(n)}(t'_1, \ldots t'_n) = n! \left(\prod_{i=1}^n \hat{t}_i^{-2/\beta} \right) \times \mathcal{I}_{n,\beta}(Wa^{-\beta/2}) \mathcal{J}_{n,\beta}(\hat{t}_1, \ldots, \hat{t}_n) \mathbb{1}_{\Delta_n}(t'_1, \ldots, t'_n). \tag{11}$$

Furthermore, for $t_i \in (0, \infty)$ the SINR process (3) has the moment measure

$$M^{(n)}(t_1, \ldots t_n,) = M'^{(n)}(t'_1, \ldots, t'_n), \tag{12}$$

where $t_i = t'_i/(1-t'_i)$ and $t'_i = t_i/(1+t_i)$.

Let $M_0'^{(n)}$ and $M_0^{(n)}$ respectively denote the factorial moment measures of the STIR and SIR processes, i.e., $M'^{(n)}$ and $M^{(n)}$ with $W = 0$. Hence

$$M'^{(n)}(\cdot) = \bar{\mathcal{I}}_{n,\beta}(Wa^{-\beta/2}) M_0'^{(n)}(\cdot), \tag{13}$$

$$M^{(n)}(\cdot) = \bar{\mathcal{I}}_{n,\beta}(Wa^{-\beta/2}) M_0^{(n)}(\cdot), \tag{14}$$

where

$$\bar{\mathcal{I}}_{n,\beta}(Wa^{-\beta/2}) = \frac{\mathcal{I}_{n,\beta}(Wa^{-\beta/2})}{\mathcal{I}_{n,\beta}(0)}. \tag{15}$$

This ability to separate the noise term in the factorial moment measures (and densities) is convenient and is reminiscent of factoring out the noise term in the distribution of the SINR under Rayleigh fading, an assumption that is not required, however, in our present setting.

III. Two-parameter Poisson-Dirichlet process

One way to define the two-parameter Poisson-Dirichlet process [3] for two given parameters $0 \leq \alpha < 1$ and $\theta > -\alpha$ is to first introduce a sequence of random variables $\{\tilde{V}_i\}$ by

$$\tilde{V}_1 = U_1, \quad \tilde{V}_i = (1-U_1)\ldots(1-U_{i-1})U_i, \quad i \geq 2, \tag{16}$$

where $U_1, U_2, \ldots$ are independent beta variables such that each U_i has $B(1-\alpha, \theta + i\alpha)$ distribution. Note that $\sum_{i=1}^\infty \tilde{V}_i = 1$ with probability one. Denote the decreasing order statistics $\{\tilde{V}_{(i)}\}$ of $\{\tilde{V}_i\}$ by $\{V_i\}$ ($V_1 \geq V_2 \geq \ldots$), then define the two-parameter Poisson-Dirichlet distribution with parameters α and θ, abbreviated as PD(α, θ), to be the distribution of $\{V_i\}$. By considering $\{V_i\}$ (or equivalenently $\{\tilde{V}_i\}$) as atoms of a point process, we see PD(α, θ) as a distribution of a point process. The above approach of defining the PD(α, θ) distribution is related to problems on so-called size-biased sampling and stick-breaking or the residual allocation model, where PD(α, θ) plays a central role. In fact, the distribution of $\{\tilde{V}_i\}$ coincides with that of the size-biased permutation of $\{V_i\}$ [14].

Another way to define a Poisson-Dirichlet process [3], more aligned with our setting, is to use the concept of a subordinator

having almost surely increasing trajectories. For $s \geq 0$, let σ_s be a subordinator, and, assuming it has zero drift, its Laplace transform is

$$\mathbf{E}[\exp(-z\sigma_s)] = \exp\left[-s\int_0^\infty (1-e^{-zr})\Lambda(dr)\right], \qquad (17)$$

where Λ is a measure on $(0,\infty)$, called the Lévy measure, characterizing the subordinator without drift. When multiplied by s, this measure $(s\Lambda(dr))$ can be identified with the intensity measure of the Poisson point process of jumps the subordinator makes in the interval $(0,s)$. Let us order and denote these jumps by $V_1(\sigma_s) \geq V_2(\sigma_s) \geq \dots$. Clearly $\sigma_s = \sum_{i=1}^\infty V_i(\sigma_s)$.

Let $0<\alpha<1$, and then σ_s is called an α-stable subordinator if $\Lambda(dr) = Dr^{-\alpha-1}\,dr$ for some constant $D>0$, which implies $\mathbf{E}[\exp(-z\sigma_s)] = \exp[-sD\Gamma(1-\alpha)z^\alpha]$. A crucial observation [3, Proposition 6] says that for any $s>0$ the sequence $\{V_1(\sigma_s)/\sigma_s, V_2(\sigma_s)/\sigma_s, \dots\}$ has PD$(\alpha,0)$ distribution [5].

Set $s=1$ and the constants $\alpha = 2/\beta$ and $D=a$, where a is given by (2). Then we see that the jumps of the subordinator in the interval $(0,s)$ can be identified with the power values of the signals from all the base stations or, equivalently, the inverse values of the propagation process Θ which, in view of Lemma 1, is an inhomogeneous Poisson process with intensity measure $(2a/\beta)t^{-1-2/\beta}dt$. Consequently, σ_1 represents the interference in our Poisson network model, and its Laplace transform is $\mathbf{E}[\exp(-z\sigma_1)] = \mathbf{E}[\exp(-zI)] = \exp[-a\Gamma(1-2/\beta)z^{2/\beta}]$.

In other words, the subordinator representation of the Poisson-Dirichlet process PD$(\alpha,0)$ ([3, Proposition 6]) relates this process to our STIR process. More precisely, denote the *increasing* orders statistics of $\{Y\}$ by $\{Y_{(i)}\}$, such that $Y_{(1)} \leq Y_{(2)} \leq \dots$, and the *decreasing* order statistics of $\{Z'\}$ by $\{Z'_{(i)}\}$. Then we have the following relation, which is a key observation of this letter.

Proposition 3. *Assume $W=0$. Then the sequence $\{Z'_{(i)}\}$ is equal in distribution to $\{V_i\}$ for $\alpha = 2/\beta$ and $\theta = 0$. In other words, the STIR process Ψ' is a PD$(2/\beta,0)$ point process.*

The fact that $\{\tilde{V}_i\}$, defined in (16), form a size-biased permutation of $\{V_i\}$ can be interpreted as follows regarding our STIR process.

Remark 4. Assume when the typical user is choosing its serving base station that, instead of looking for the strongest received signal $Y_{(1)}^{-1}$, it makes a randomized decision, picking a base station i with a bias proportional to Y_i^{-1} (hence stronger stations have more chance to be selected). Then its STIR, with respect to the chosen station, has the distribution of $\tilde{V}_1$, i.e. $B(1-2/\beta, 2/\beta)$. Suppose now that another user positioned with the typical one and subject to the same propagation effects makes its choice of the serving base station by applying the same randomized procedure but excluding the station already selected by the first user. Then the join distribution of the STIR's experienced by these two users is equal to that of the random pair $(\tilde{V}_1, \tilde{V}_2)$, which can be easily derived from (16). The above *randomized access policy*, and the corresponding evaluation of the STIR values, which can be extended to an arbitrary number of users, is of potential interest for managing user hotspots.

[5] To define the original Kingman's Poisson-Dirichlet process with PD$(0,\theta)$ distribution, a subordinator known as the Moran [11] or Gamma subordinator [3] is used. A subtle combination of this and the α-stable subordinator is used to define the PD(α,θ) process, but we omit the details for brevity.

IV. Some useful results

Appropriately adapted for this setting, we list some interesting results of the PD$(2/\beta,0)$ distribution. The first result [3, Proposition 8] applied here shows that the ratio of successive STINR values have beta distributions.

Proposition 5. *For the STINR process Ψ' ($W \geq 0$), the random variables*

$$R_i := \frac{Z'_{(i+1)}}{Z'_{(i)}} = \frac{Y_{(i)}}{Y_{(i+1)}} \qquad (18)$$

have, respectively, $B(2i/\beta,1)$ distributions such that $\mathbf{P}(R_i \leq r) = r^{i2/\beta}$ (for $0 \leq r \leq 1$). Moreover, $\{R_i\}$ are mutually independent.

The fact that each R_i is a ratio of $Y_{(i)}$ values indicates that this result (proved in [3] under assumption $W=0$) is invariant of the noise term W. This applies also to the next result, which involves the following variables

$$A_i := \frac{Z'_{(1)}+\dots+Z'_{(i)}}{Z'_{(i+1)}} = \frac{Y_{(1)}^{-1}+\dots+Y_{(i)}^{-1}}{Y_{(i+1)}^{-1}}. \qquad (19)$$

$$\Sigma_i := \frac{Z'_{(i+1)}+Z'_{i+2}+\dots}{Z'_i} = \frac{Y_{(i+1)}^{-1}+Y_{(i+2)}^{-1}+\cdots}{Y_{(i)}^{-1}} \qquad (20)$$

defined for $i=1,2,\dots$. For $\gamma \geq 0$ let

$$\phi_\beta(\gamma) := \frac{2}{\beta}\int_1^\infty e^{-\gamma x}x^{-2/\beta-1}dx, \qquad (21)$$

$$\psi_\beta(\gamma) := \Gamma(1-2/\beta)\gamma^{2/\beta} + \phi_\beta(\gamma). \qquad (22)$$

The next proposition follows [3, Proposition 11].

Proposition 6. *Consider the STINR process Ψ' ($W \geq 0$). Then $1/Z'_{(i)} = 1 + A_{i-1} + \Sigma_i$ where A_{i-1} is distributed as the sum of $i-1$ independent copies of A_1, with the characteristic function $\mathbf{E}[e^{-\gamma A_{i-1}}] = (\phi_\beta(\gamma))^{i-1}$; Σ_i is distributed as the sum of i independent copies of Σ_1, with the characteristic function $\mathbf{E}[e^{-\gamma\Sigma_i}] = (\psi_\beta(\gamma))^{-i}$; and A_{i-1} and Σ_i are independent.*

Remark 7. By observing that $\Sigma_i^{-1} = Y_{(i)}^{-1}/(Y_{(i+1)}^{-1} + Y_{(i+2)}^{-1} + \dots)$, the above result in the setting of *successive-interference cancellation* (with no noise, $W=0$) can be compared to a result [12, Theorem 1] and its generalization [6, Proposition 21] on the ratio of the kth strongest propagation process and a successively reduced interference term. Moreover, the ratio of independent random variables $(1+A_{i-1})/\Sigma_i = (Y_{(1)}^{-1}+\dots+Y_{(i)}^{-1})/(Y_{(i+1)}^{-1}+Y_{(i+2)}^{-1}+\cdots)$ relates the above result to a recent *signal combination* model in the STIR ($W=0$) scenario [6]. The difference between the STINR and STIR results suggests that the noise term W can add a significant layer of complexity to the models.

Proposition 6 leads to a Laplace transform result (cf [3, Corollary 12]), which can be compared to a previous observation [6, Remark 18].

Corollary 8. *The inverse of the ith strongest STIR ($W=0$) value, $1/Z'_{(i)}$, has the Laplace transform*

$$\mathbf{E}[e^{-\gamma/Z'_{(i)}}] = e^{-\gamma}(\phi_\beta(\gamma))^{i-1}(\psi_\beta(\gamma))^{-i}. \qquad (23)$$

Furthermore, a previous result [15, Corollary 7] gives an expression for the tail of the distribution function of the ith strongest STINR ($W \geq 0$) value.

Remarkably, the interference I, as defined by (4), and noise W can be recovered from the STINR processes. Indeed, the first statement of the following result is trivial, while the second one can be proved using the same arguments as [3, Proposition 10].

Proposition 9. *For the STINR process ($W \geq 0$), $W/I = \left(\sum_{i=1}^{\infty} Z'_{(i)}\right)^{-1} - 1$, and $W + I = (L/a)^{-\beta/2}$, where the limit $L := \lim_{i\to\infty} i(Z'_{(i)})^{2/\beta}$, both exists almost surely and for all p-means with $p \geq 1$.*

The densities of the factorial moment measures $M'^{(n)}$ of the STINR process can be used to find an expression for the joint probability density of the order statistics of the STINR process [6, Proposition 20]. But, the expression on the right-hand-side of (11) appears too unwieldy to differentiate with respect to more than a couple variables. However, using the representation (16), the factorial moment density of the PD(α, θ) process was derived in closed-form [10, Theorem 2.1], which implies the following new result for our STINR process. For $n \geq 0$ denote $c_{n,\alpha,\theta} = \prod_{i=1}^{n} \Gamma(\theta + 1 + (i-1)\alpha)/(\Gamma((1-\alpha)\Gamma(\theta + i\alpha))$; in particular $c_{n,2/\beta,0} = (2/\beta)^{n-1}\Gamma(n)/(\Gamma(2n/\beta)\Gamma(1-2/\beta)^n)$.

Proposition 10. *For the STINR process Ψ' ($W \geq 0$), the nth factorial moment density is given by*

$$\mu^{(n)}(t'_1, \dots t'_n) := (-1)^n \frac{\partial^n M'^{(n)}(t'_1, \dots t'_n)}{\partial t'_1 \dots \partial t'_n} \tag{24}$$
$$= c_{n,2/\beta,0}\, \bar{\mathcal{I}}_{n,\beta}(Wa^{-\beta/2}) \Big(\prod_{i=1}^{n} t_i'^{-(2/\beta+1)}\Big)\Big(1 - \sum_{j=1}^{n} t'_j\Big)^{2/\beta n - 1}$$

for $(t'_1, \dots, t'_n)$ in Δ_n and 0 otherwise.

This result follows from including the noise term W, via (13), and using [10, Theorem 2.1]. We are unaware of anybody showing the equivalence of Propositions 2 and 10, either by differentiating the measure (11) or integrating the density (24).

Another way for calculating the joint density of the order statistics of the STIR process is offered by the result [10, Theorem 5.4], where the two-parameter Dickman function was introduced as

$$\rho_{\alpha,\theta}(s) := \mathbf{P}(V_1 < 1/s) \tag{25}$$

where V_1 is the largest value of the PD(α, θ) process, which can be computed as follows:

$$\rho_{\alpha,\theta}(s) := \sum_{n=0}^{\infty} \frac{(-1)^n c_{n,\alpha,\theta}}{n!} I_{n,\alpha,\theta}(s), \tag{26}$$

where
$$I_{n,\alpha,\theta}(s) = \int_{\Delta_n} \prod_{i=1}^{n} \frac{\mathbb{1}_{[1,\infty)}(st'_i)}{t_i'^{\alpha+1}} \Big(1 - \sum_{j=1}^{n} t'_j\Big)^{\theta+\alpha n-1} dt'_1 \dots dt'_n, \tag{27}$$

and for $n = 1, 2, \dots$, with $I_{n,\alpha,\theta}(s) = 0$ whenever $n > s$, which makes that the right-hand-side of (26) is actually a finite sum; cf [10, Section 4].

Remark 11. Recall that when $\alpha = 2/\beta$ and $\theta = 0$, V_1 is equal in distribution to the strongest STIR value $Z'_{(1)}$ and thus (26) should be compared to [15, Corollary 7, with $k = 1$, valid for $W \geq 0$] or [16, Theorem 1, taken for single-tier network, valid for SINR values greater than one].

Proposition 12. *For the STIR process Ψ' ($W = 0$) and for each $m = 1, 2, \dots$, the joint probability density of $\{Z'_{(1)}, \dots, Z'_{(m)}\}$ is given by*

$$f_{m,\beta}(t'_1, \dots t'_m) := c_{m,2/\beta,0}\Big(\prod_{i=1}^{m} t_i'^{-(2/\beta+1)}\Big)\Big(1 - \sum_{j=1}^{m} t'_j\Big)^{2m/\beta-1}$$
$$\times\, \rho_{2/\beta,2m/\beta}\Big(\frac{1 - \sum_{j=1}^{m} t'_j}{t'_m}\Big) \mathbb{1}_{\Delta_m}(t'_1, \dots, t'_m) \mathbb{1}_{\{t'_1 > \dots > t'_n\}}. \tag{28}$$

The two non-zero parameters of the Dickman function explains why PD(α, θ) is needed in (25), and not just PD$(\alpha, 0)$.

V. Conclusion

We showed the relationship between the SINR process, which is an important object in the study of the performance of cellular networks, and the two-parameter Poisson-Dirichlet process. We presented some results recently proved for the former process, which have interesting interpretation in terms of the STINR process (easily related to the SINR one). Our goal is to encourage further research aimed at building bridges between these two, until now, separate research areas.

Acknowledgements: We thank M.K. Karray for carefully proofreading the manuscript.

References

[1] B. Błaszczyszyn, M. K. Karray, and H. P. Keeler, "Using Poisson processes to model lattice cellular networks," in *INFOCOM, 2013 Proceedings IEEE*, 2013, pp. 773–781.

[2] ——, "Wireless networks appear Poissonian due to strong shadowing," In preparation, 2014.

[3] J. Pitman and M. Yor, "The two-parameter Poisson–Dirichlet distribution derived from a stable subordinator," *Ann. Prob.*, vol. 25, no. 2, pp. 855–900, 1997.

[4] S. Feng, *The Poisson-Dirichlet Distribution and Related Topics: Models and Asymptotic Behaviors*. Springer, 2010.

[5] S. Mukherjee, *Analytical Modeling of Heterogeneous Cellular Networks*. Cambridge University Press, 2013.

[6] B. Blaszczyszyn and H. P. Keeler, "Studying the SINR process of the typical user in Poisson networks by using its factorial moment measures," *arXiv preprint arXiv:1401.4005*, 2014.

[7] B. Błaszczyszyn and H. Keeler, "Equivalence and comparison of heterogeneous cellular networks," in *Proc. of PIMRC Workshops*, 2013, pp. 153–157.

[8] D. Panchenko, *The Sherrington-Kirkpatrick model*. Springer, 2013.

[9] M. Perman, J. Pitman, and M. Yor, "Size-biased sampling of poisson point processes and excursions," *Probability Theory and Related Fields*, vol. 92, no. 1, pp. 21–39, 1992.

[10] K. Handa, "The two-parameter Poisson–Dirichlet point process," *Bernoulli*, vol. 15, no. 4, pp. 1082–1116, 2009.

[11] J. F. C. Kingman, *Poisson Processes*, 1st ed. Oxford University Press, 1993.

[12] X. Zhang and M. Haenggi, "On decoding the kth strongest user in Poisson networks with arbitrary fading distribution," in *Proc. of Asilomar'13)*, Pacific Grove, CA, Nov. 2013.

[13] H. P. Keeler, "Studying the SINR process in Poisson networks by using its factorial moment measures," MATLAB Central File Exchange, 2014. [Online]. Available: http://www.mathworks.com.au/matlabcentral/fileexchange/45299

[14] J. Pitman, "Random discrete distributions invariant under size-biased permutation," *Adv. Appl. Probab.*, vol. 28, no. 2, pp. 525–539, 1996.

[15] H. P. Keeler, B. Błaszczyszyn, and M. K. Karray, "SINR-based k-coverage probability in cellular networks with arbitrary shadowing," in *Proc. of ISIT*, 2013, pp. 1167–1171.

[16] H. S. Dhillon, R. K. Ganti, F. Baccelli, and J. G. Andrews, "Modeling and analysis of k-tier downlink heterogeneous cellular networks," *IEEE JSAC*, vol. 30, no. 3, pp. 550–560, 2012.