HAL
open science

Persistently damped transport on a network of circles

Yacine Chitour, Guilherme Mazanti, Mario Sigalotti

To cite this version:

Yacine Chitour, Guilherme Mazanti, Mario Sigalotti. Persistently damped transport on a network of circles. Transactions of the American Mathematical Society, 2017, 369 (6), pp.3841-3881. 10.1090/tran/6778 . hal-00999743

HAL Id: hal-00999743

https://inria.hal.science/hal-00999743v1

Submitted on 23 Nov 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Persistently damped transport on a network of circles*

Yacine Chitour[†], Guilherme Mazanti[‡ §], Mario Sigalotti[§ ‡]

June 29, 2015

Abstract

In this paper we address the exponential stability of a system of transport equations with intermittent damping on a network of $N \geq 2$ circles intersecting at a single point O. The N equations are coupled through a linear mixing of their values at O, described by a matrix M. The activity of the intermittent damping is determined by persistently exciting signals, all belonging to a fixed class. The main result is that, under suitable hypotheses on M and on the rationality of the ratios between the lengths of the circles, such a system is exponentially stable, uniformly with respect to the persistently exciting signals. The proof relies on an explicit formula for the solutions of this system, which allows one to track down the effects of the intermittent damping.

Contents

1 Introduction **2**

2 Notations, definitions and preliminary facts **4**
2.1 Formulation and well-posedness of the Cauchy problem 6
2.2 Some examples of asymptotic behavior 7
2.3 Discussion on the hypotheses of Theorem 1.2 8

3 Explicit solution **9**
3.1 The undamped system 9
3.2 Formula for the explicit solution in the general case 11
3.3 Recursive formula for the coefficients 13

4 Proof of the main result **15**
4.1 Convergence of the coefficients implies convergence of the solution 15
4.2 Preliminary estimates of $\vartheta^{(i)}_{j,\mathrm{n},x,t}$ 17
4.3 Exponential decay of $\vartheta^{(i)}_{j,\mathrm{n},x,t}$ in $\mathfrak{N}_b(\rho)$ 19
4.4 Exponential decay of $\vartheta^{(i)}_{j,\mathrm{n},x,t}$ in $\mathfrak{N}_c(\rho)$ 19
4.5 Exponential convergence of the solutions 25

*This research was partially supported by the iCODE Institute, research project of the IDEX Paris-Saclay, and by the Hadamard Mathematics LabEx (LMH) through the grant number ANR-11-LABX-0056-LMH in the "Programme des Investissements d'Avenir".

Keywords: transport equation, persistent excitation, exponential stability, multistructures

Mathematics Subject Classification: 35R02, 35B35, 35C05, 35L40, 93C20

[†]Laboratoire des Signaux et Systèmes, Supélec, and Université Paris Sud, Orsay, France.

[‡]CMAP, École Polytechnique, Palaiseau, France.

[§]Team GECO, Inria Saclay, Palaiseau, France.

A Well-posedness of the Cauchy problems (2.5)**,** (2.6) **and** (2.7) **25**
A.1 Preliminaries 26
A.2 Proof of Theorem 2.5 27
A.3 Proof of Theorem 2.4 28

B Asymptotic behavior of (2.9) **29**
B.1 LaSalle Principle in a Banach space 29
B.2 Asymptotic behavior when $L_1/L_2 \notin \mathbb{Q}$ 30
B.3 Periodic solutions for the undamped system 32
B.4 Periodic solutions for the persistently damped system 32

C Proof of Theorem 3.3 **33**

D A combinatorial estimate **35**

1 Introduction

Consider the following system of $N \geq 2$ coupled transport equations,

$$\begin{cases} \partial_t u_i(t,x) + \partial_x u_i(t,x) + \alpha_i(t)\chi_i(x)u_i(t,x) = 0, & t \geq 0,\ x \in [0,L_i],\ 1 \leq i \leq N_d, \\ \partial_t u_i(t,x) + \partial_x u_i(t,x) = 0, & t \geq 0,\ x \in [0,L_i],\ N_d+1 \leq i \leq N, \\ u_i(t,0) = \displaystyle\sum_{j=1}^{N} m_{ij} u_j(t,L_j), & t \geq 0,\ 1 \leq i \leq N, \\ u_i(0,x) = u_{i,0}(x), & x \in [0,L_i],\ 1 \leq i \leq N. \end{cases} \tag{1.1}$$

For $1 \leq i \leq N$, the corresponding transport equation is defined in the space domain $[0,L_i]$ with $L_i > 0$. The integer N_d denotes the number of equations with a damping term. For $1 \leq i \leq N_d$, the activity of the damping of the i-th equation in space is determined by the function χ_i, which is assumed to be the characteristic function of an interval $[a_i,b_i] \subset [0,L_i]$ with $a_i < b_i$, whereas its activity in time is determined by the function α_i, which is assumed to be a signal in $L^\infty(\mathbb{R},[0,1])$. The coupling between the N transport equations is determined by the coefficients $m_{ij} \in \mathbb{R}$ for $1 \leq i,j \leq N$. The goal of the paper consists in studying the stability properties of (1.1) when the signals α_i are persistently exciting, as defined below.

Definition 1.1. Let T, μ be two positive constants with $T \geq \mu > 0$. A function $\alpha \in L^\infty(\mathbb{R},[0,1])$ is said to be a (T,μ)-*persistently exciting signal* if, for every $t \in \mathbb{R}$, one has

$$\int_t^{t+T} \alpha(s)ds \geq \mu. \tag{1.2}$$

The set of all (T,μ)-persistently exciting signals is denoted by $\mathcal{G}(T,\mu)$.

System (1.1) is a system of N transport equations defined on intervals $[0,L_i]$, $1 \leq i \leq N$, which may be identified with circles $C_1,C_2,\ldots,C_N$ of respective lengths $L_1,L_2,\ldots,L_N$. Moreover, there exists a point O such that any two distinct circles only intersect at O (see Figure 1.1). The transmission condition at O can be written as

$$\begin{pmatrix} u_1(t,0) \\ u_2(t,0) \\ \vdots \\ u_N(t,0) \end{pmatrix} = M \begin{pmatrix} u_1(t,L_1) \\ u_2(t,L_2) \\ \vdots \\ u_N(t,L_N) \end{pmatrix}, \tag{1.3}$$

where $M = (m_{ij})_{1\leq i,j\leq N}$ is called the *transmission matrix* of the system. The topology of the network considered in this paper is star-shaped with respect to the point O. Note that any other network configuration falls into the present framework by a suitable choice of transition matrix M, namely, the fact that two circles C_i and C_j are not inward-outward adjacent translates to $m_{ij} = 0$. For $1 \leq i \leq N_d$, the transport equation of u_i is damped on the support $[a_i, b_i]$ of χ_i, represented in Figure 1.1. The damping is subject to the signal α_i, which can be zero on certain time intervals. When all the α_i take their values in $\{0,1\}$, (1.1) can be seen as a switched system, where the switching signal α_i controls the damping action on the interval $[a_i, b_i]$ of the circle C_i.

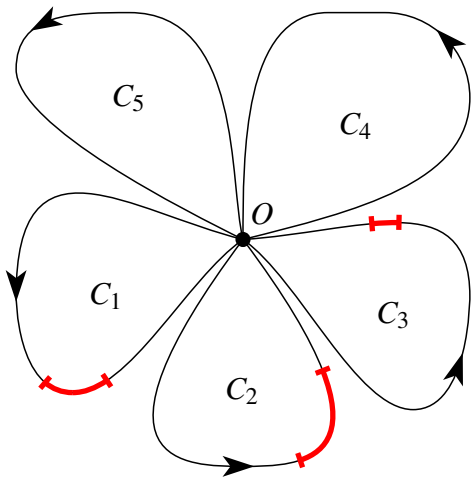

FIGURE 1.1: *Network corresponding to $N = 5$ and $N_d = 3$.*

Switching occurs in several control applications, motivating the study of systems with switched or intermittent actuators (see [15, 17, 20, 24, 27, 32] and references therein). In this context, the activity of the actuator is guaranteed by appropriate conditions, for instance existence of a positive dwell-time or average dwell-time [24]. In this paper we rely instead on the integral condition (1.2) to guarantee the damping activity. This condition finds its origin in problems of identification and adaptive control [3–5], where it is used in a more general form as a necessary and sufficient condition for the global exponential stability of some linear time-dependent systems. Linear control systems of the kind

$$\dot{x}(t) = Ax(t) + \alpha(t)Bu(t),$$

where a persistently exciting signal α determines the activity of the control u, are usually called *persistently excited systems*. They have been considered in the finite-dimensional setting in [8, 9, 11, 13, 28, 29], dealing mostly with problems concerning stabilizability by a linear feedback law (see [12] for a thorough presentation of persistently excited systems). In such systems, the persistently exciting signal α is a convenient tool to model several phenomena, such as failures in links between systems, resource allocation, or other internal or external processes that affect control efficiency. Further discussion on the practical interest of persistently excited systems can be found, e.g., in [25]. Its infinite-dimensional counterpart is much less present in the literature, due to the fact that finite-dimensional results cannot be straightforwardly generalized (see [19], where illustrative examples are provided together with sufficient conditions ensuring stability).

System (1.1) is a "toy model" to study infinite-dimensional systems under persistent excitation. It is a simple case of a *multi-body structure*. These type of problems model strings, membranes, plates, by partial differential equations defined on several coupled domains. They are an active research subject attracting much interest due to both its applications and the complexity of its analysis (see [1, 2, 6, 7, 23, 26] and references therein).

Particular multi-body systems that attracted much interest in recent years are networks of strings [14, 17, 35, 37]. Their constituent elements are one-dimensional vibrating strings distributed along a planar graph. By decomposing each one-dimensional wave equation intro traveling waves according to D'Alembert decomposition, one can replace an edge of the graph by a pair of oriented edges and consider the transport equation in each edge. Hence, when (1.1) is undamped (i.e., when $N_d = 0$), it actually represents the D'Alembert decomposition of a star-shaped network of strings. The damping term in (1.1) does not come from the above decomposition of the wave equation and thus the results of this paper cannot be directly applied to networks of strings.

This paper addresses the issue of exponential stability of (1.1), uniformly with respect to the signals α_i in a class $\mathcal{G}(T,\mu)$: given $T \geq \mu > 0$, is system (1.1) uniformly exponentially stable with respect to $\alpha_i \in \mathcal{G}(T,\mu)$, $1 \leq i \leq N_d$? The answer clearly depends on the transmission matrix M, since this matrix can amplify or reduce the solutions when they pass through O, as well as on the rationality of the ratios L_i/L_j, since periodic solutions may exist when they are rational (see Sections 2.2 and 2.3 below). The main result of this paper is the following.

Theorem 1.2. *Suppose that $N \geq 2$, $N_d \geq 1$, $|M|_{\ell^1} \leq 1$, $m_{ij} \neq 0$ for every $1 \leq i, j \leq N$, and that there exist $1 \leq i_*, j_* \leq N$ such that $L_{i_*}/L_{j_*} \notin \mathbb{Q}$. Then, for every $T \geq \mu > 0$, there exist $C, \gamma > 0$ such that, for every $p \in [1,+\infty]$, every initial condition $u_{i,0} \in L^p(0,L_i)$, $1 \leq i \leq N$, and every choice of signals $\alpha_i \in \mathcal{G}(T,\mu)$, $1 \leq i \leq N_d$, the corresponding solution of* (1.1) *satisfies*

$$\sum_{i=1}^{N} \|u_i(t)\|_{L^p(0,L_i)} \leq Ce^{-\gamma t} \sum_{i=1}^{N} \left\|u_{i,0}\right\|_{L^p(0,L_i)}, \qquad \forall t \geq 0.$$

Our argument is based on explicit formulas for the solutions of system (1.1), which allow one to efficiently track down the effects of the persistency of the damping. This approach can be worked out since system (1.1) consists of constant-speed transport equations with local damping. On the other hand, the usual techniques from PDE control, such as Carleman estimates, spectral criteria, Ingham estimates or microlocal analysis, do not seem well-adapted here, since they do not allow to handle the effects due to the time-dependency induced by the persistently exciting signals α_i. Extensions of our result to the case of state-dependent speed of transport and non-local damping would probably require more refined techniques.

The idea of relying on explicit representations for solutions of (1.1) to address control and identification issues has already been used in [16, 34]. Note, however, that, in these two references, rational dependence assumptions were necessary to derive tractable explicit formulas, which is not the case in this paper.

The paper is organized as follows. In Section 2, we give the main notations and definitions used through this paper, discuss the well-posedness of (1.1) and explain the role of the hypotheses of Theorem 1.2. Section 3 provides the explicit formula for the solutions of (1.1), first in the undamped case, where the notations are simpler and the formulas easier to write, and then in the general case. Our main result is proved in Section 4, where we study the asymptotic behavior of coefficients appearing in the explicit solution obtained in Section 3. We finally collect in a series of appendices various technical results used in the paper.

2 Notations, definitions and preliminary facts

In this paper, we denote by $\mathbb{Z}$ the set of all integers, $\mathbb{N} = \{0,1,2,3,\dots\}$ the set of nonnegative integers, $\mathbb{N}^* = \{1,2,3,\dots\}$ the set of positive integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, $\mathbb{R}_+ = [0,+\infty)$ the set of nonnegative real numbers and $\mathbb{R}_+^* = (0,+\infty)$ the set of

positive real numbers. For $a,b \in \mathbb{R}$, let $[\![a,b]\!] = [a,b] \cap \mathbb{Z}$, with the convention that $[a,b] = \emptyset$ if $a > b$.

The set of $d \times m$ matrices with real coefficients is denoted by $\mathcal{M}_{d,m}(\mathbb{R})$, or simply by $\mathcal{M}_d(\mathbb{R})$ when $d = m$. As usual, we identify column matrices in $\mathcal{M}_{d,1}(\mathbb{R})$ with vectors in $\mathbb{R}^d$. The identity matrix in $\mathcal{M}_d(\mathbb{R})$ is denoted by Id_d. For $p \in [1,+\infty]$, $|\cdot|_{\ell^p}$ indicates both the ℓ^p norm of a vector of $\mathbb{R}^d$ and the induced matrix norm of a linear map on $\mathbb{R}^d$.

All Banach and Hilbert spaces considered are supposed to be real. The scalar product between two elements u,v on a Hilbert space X is denoted by $\langle u,v\rangle_{\mathsf{X}}$, and the norm of an element u on a Banach space X is denoted by $\|u\|_{\mathsf{X}}$. The indices X will be dropped from the previous notations when the context is clear.

We shall refer to linear operators in a Banach space X simply as *operators*. The domain of an operator T on X is denoted by $D(T)$. The set of all bounded operators from a Banach space X to a Banach space Y is denoted by $\mathcal{L}(\mathsf{X},\mathsf{Y})$ and is endowed with its usual norm $\|\cdot\|_{\mathcal{L}(\mathsf{X},\mathsf{Y})}$. The notation $\mathcal{L}(\mathsf{X})$ is used for $\mathcal{L}(\mathsf{X},\mathsf{X})$.

For two topological spaces X and Y, $\mathcal{C}^0(X,Y)$ denotes the set of continuous functions from X to Y. If I is an interval of $\mathbb{R}$, X is a Banach space and $k \in \mathbb{N}$, $\mathcal{C}^k(I,X)$ denotes the set of k-times differentiable X-valued functions defined on I, and $\mathcal{C}^k_c(I,X)$ is the subset of $\mathcal{C}^k(I,X)$ of compactly supported functions. When $X = \mathbb{R}$ we omit it from the notation.

For $x \in \mathbb{R}$, $\lfloor x \rfloor \in \mathbb{Z}$ (resp. $\lceil x \rceil$) denotes the greatest (resp. smallest) integer $k \in \mathbb{Z}$ such that $k \le x$ (resp. $k \ge x$). For $y > 0$, we denote by $\{x\}_y$ the number $\{x\}_y = x - \lfloor x/y \rfloor y$. For every $n,m \in \mathbb{N}$ with $m \le n$, we use the binomial coefficient notation $\binom{n}{m} = \frac{n!}{m!(n-m)!}$. We use $\#F$ and δ_{ij} to denote, respectively, the cardinality of a finite set F and the Kronecker symbol of $i,j \in \mathbb{Z}$.

We refer to System (1.1) as being *undamped* by setting $\alpha_i \equiv 0$ for every $i \in [\![1,N_d]\!]$, in which case it is written as

$$\begin{cases} \partial_t u_i(t,x) + \partial_x u_i(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L_i],\ i \in [\![1,N]\!], \\ u_i(t,0) = \displaystyle\sum_{j=1}^N m_{ij} u_j(t,L_j), & t \in \mathbb{R}_+,\ i \in [\![1,N]\!], \\ u_i(0,x) = u_{i,0}(x), & x \in [0,L_i],\ i \in [\![1,N]\!]. \end{cases} \tag{2.1}$$

We say that System (1.1) has an *always active damping* if $\alpha_i \equiv 1$ for every $i \in [\![1,N_d]\!]$, in which case it becomes

$$\begin{cases} \partial_t u_i(t,x) + \partial_x u_i(t,x) + \chi_i(x) u_i(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L_i],\ i \in [\![1,N_d]\!], \\ \partial_t u_i(t,x) + \partial_x u_i(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L_i],\ i \in [\![N_d+1,N]\!], \\ u_i(t,0) = \displaystyle\sum_{j=1}^N m_{ij} u_j(t,L_j), & t \in \mathbb{R}_+,\ i \in [\![1,N]\!], \\ u_i(0,x) = u_{i,0}(x), & x \in [0,L_i],\ i \in [\![1,N]\!]. \end{cases} \tag{2.2}$$

The general case of (1.1) can be written as

$$\begin{cases} \partial_t u_i(t,x) + \partial_x u_i(t,x) + \alpha_i(t)\chi_i(t) u_i(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L_i],\ i \in [\![1,N]\!], \\ u_i(t,0) = \displaystyle\sum_{j=1}^N m_{ij} u_j(t,L_j), & t \in \mathbb{R}_+,\ i \in [\![1,N]\!], \\ u_i(0,x) = u_{i,0}(x), & x \in [0,L_i],\ i \in [\![1,N]\!], \end{cases} \tag{2.3}$$

with the convention that $\alpha_i \equiv 1$ and $a_i = b_i = L_i$ for $i \in [\![N_d+1,N]\!]$, implying that $\chi_i = 0$ almost everywhere in $[0,L_i]$ for $i \in [\![N_d+1,N]\!]$. In the case where $\alpha_1,\dots,\alpha_{N_d}$ belong to a class $\mathcal{G}(T,\mu)$ for the same fixed $T \ge \mu > 0$, System (2.3) is referred to as a *persistently damped system*.

Remark 2.1. Assuming that all the persistently exciting signals $\alpha_1, \dots, \alpha_{N_d}$, in (2.3) belong to the class $\mathcal{G}(T,\mu)$, with the same constants $T \geq \mu > 0$, is not actually a restriction. Indeed, if $\alpha_i \in \mathcal{G}(T_i,\mu_i)$ with $T_i \geq \mu_i > 0$ for $i \in [\![1,N_d]\!]$, then we clearly have, for every $i \in [\![1,N_d]\!]$, $\alpha_i \in \mathcal{G}(T,\mu)$ with $T = \max_{i\in[\![1,N_d]\!]} T_i$ and $\mu = \min_{i\in[\![1,N_d]\!]} \mu_i$.

2.1 Formulation and well-posedness of the Cauchy problem

The goal of this section consists in providing a rigorous definition for a solution of (2.3) and in guaranteeing that, given any initial data, the required solution exists, is unique and depends continuously on the initial data.

Definition 2.2. Let $p \in [1,+\infty)$. We set $\mathsf{X}_p = \prod_{i=1}^N L^p(0,L_i)$, endowed with the norm $\|z\|_{\mathsf{X}_p} = \left(\sum_{i=1}^N \|u_i\|^p_{L^p(0,L_i)}\right)^{1/p}$ for $z = (u_1,\dots,u_N) \in \mathsf{X}_p$.

We define the operator $A : D(A) \subset \mathsf{X}_p \to \mathsf{X}_p$ on its domain $D(A)$ by

$$\begin{gathered} D(A) = \left\{ (u_1,\dots,u_N) \in \prod_{i=1}^N W^{1,p}(0,L_i) \;\middle|\; \forall i \in [\![1,N]\!],\ u_i(0) = \sum_{j=1}^N m_{ij} u_j(L_j) \right\}, \\ A(u_1,\dots,u_N) = \left(-\frac{du_1}{dx}, \dots, -\frac{du_N}{dx} \right). \end{gathered} \tag{2.4}$$

For $i \in [\![1,N_d]\!]$, we define the operator $B_i \in \mathcal{L}(\mathsf{X}_p)$ by

$$B_i(u_1,\dots,u_N) = (0,\dots,0,-\chi_i u_i,0,\dots,0),$$

where the term $-\chi_i u_i$ is in the i-th position.

Remark 2.3. Even though Theorem 1.2 is stated for every $p \in [1,+\infty]$, we restrict ourselves in the sequel to the case $p \in [1,+\infty)$. The main reason for this is that, when $p = +\infty$, the domain $D(A)$ of the operator A defined by (2.4) is not dense in $\prod_{i=1}^N L^\infty(0,L_i)$, and thus some of our arguments given for p finite do not apply. However, once we prove Theorem 1.2 for $p \in [1,+\infty)$, we obtain the case $p = +\infty$ by suitable continuity arguments, as detailed in Remark 4.3.

With the operators A and B_i defined above, System (2.3) can be written as

$$\begin{cases} \dot{z}(t) = Az(t) + \displaystyle\sum_{i=1}^{N_d} \alpha_i(t) B_i z(t), \\ z(0) = z_0, \end{cases} \tag{2.5}$$

with $z_0 = (u_{1,0},\dots,u_{N,0})$ and $\alpha_1,\dots,\alpha_{N_d} \in L^\infty(\mathbb{R},[0,1])$. The case of the undamped system (2.1) can be written as

$$\begin{cases} \dot{z}(t) = Az(t), \\ z(0) = z_0, \end{cases} \tag{2.6}$$

and the system with an always active damping (2.2) becomes

$$\begin{cases} \dot{z}(t) = Az(t) + \displaystyle\sum_{i=1}^{N_d} B_i z(t), \\ z(0) = z_0. \end{cases} \tag{2.7}$$

The well-posedness of (2.5) is established in the sense of the following theorem, whose proof is deferred in Appendix A.

Theorem 2.4. *Let $p \in [1,+\infty)$ and $\alpha_i \in L^\infty(\mathbb{R},[0,1])$ for $i \in [\![1,N_d]\!]$. There exists a unique evolution family $\{T(t,s)\}_{t\geq s\geq 0}$ of bounded operators in X_p such that, for every $s \geq 0$ and $z_0 \in D(A)$, $t \mapsto z(t) = T(t,s)z_0$ is the unique continuous function such that $z(s) = z_0$, $z(t) \in D(A)$ for every $t \geq s$, z is differentiable for almost every $t \geq s$, $\dot{z} \in L^\infty_{\mathrm{loc}}([s,+\infty),\mathsf{X}_p)$, and $\dot{z}(t) = Az(t) + \sum_{i=1}^{N_d} \alpha_i(t)B_i z(t)$ for almost every $t \geq s$.*

The definition of an evolution family is recalled in Appendix A. The function z in Theorem 2.4 is said to be a *regular solution* of (2.5) with initial condition $z_0 \in D(A)$. When $z_0 \in \mathsf{X}_p \setminus D(A)$, the function $t \mapsto z(t) = T(t,s)z_0$ is still well-defined and continuous, and is said to be a *mild solution* of (2.5). We use the word *solution* to refer to both regular and mild solutions, according to the context.

Theorem 2.4 also provides solutions to (2.6) and (2.7) as particular cases. Since these equations are time-independent, we can actually obtain more regular solutions, thanks to the fact that A and $A + \sum_{i=1}^{N_d} B_i$ generate strongly continuous semigroups, as we detail in Appendix A.

Theorem 2.5. *Let $p \in [1,+\infty)$. The operators A and $A + \sum_{i=1}^{N_d} B_i$ generate strongly continuous semigroups $\{e^{tA}\}_{t\geq 0}$ and $\{e^{t(A+\sum_{i=1}^{N_d} B_i)}\}_{t\geq 0}$. In particular, for every $z_0 \in D(A)$, the function $t \mapsto e^{tA}z_0$ is the unique function in $\mathcal{C}^0(\mathbb{R}_+,D(A)) \cap \mathcal{C}^1(\mathbb{R}_+,\mathsf{X})$ satisfying (2.6) and the function $t \mapsto e^{t(A+\sum_{i=1}^{N_d} B_i)}z_0$ is the unique function in $\mathcal{C}^0(\mathbb{R}_+,D(A)) \cap \mathcal{C}^1(\mathbb{R}_+,\mathsf{X})$ satisfying (2.7).*

2.2 Some examples of asymptotic behavior

It is useful to have in mind some illustrative examples of the asymptotic behaviors of (1.1) under no damping, an always active damping and a persistent damping, respectively.

Example 2.6. Consider the case of a single transport equation on a circle of length L,

$$\begin{cases} \partial_t u(t,x) + \partial_x u(t,x) + \alpha(t)\chi(x)u(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L], \\ u(t,0) = u(t,L), & t \in \mathbb{R}_+, \\ u(0,x) = u_0(x), & x \in [0,L], \\ \alpha \in \mathcal{G}(T,\mu), \end{cases} \tag{2.8}$$

where χ is the characteristic function of the interval $[a,b] \subset [0,L]$. This corresponds to (2.3) with persistent damping, $N_d = N = 1$, and $m_{11} = 1$. Due to the condition $u(t,0) = u(t,L)$, it can be seen as a transport equation on a circle of length L.

When (2.8) is undamped, all its solutions are L-periodic. Indeed, for $u_0 \in \mathsf{X}_p = L^p(0,L)$, the corresponding solution of (2.8) is $u(t,x) = u_0(\{x-t\}_L)$, where we recall that $\{x\}_y = x - \lfloor x/y \rfloor y$, and this function is clearly L-periodic.

When (2.8) has an always active damping, all its solutions converge exponentially to zero. Indeed, every solution of (2.8) satisfies $u(t,x) = e^{-(b-a)}u(t-L,x)$ for every $x \in [0,L]$ and $t \geq L$, so that $\|u(t)\|_{L^p(0,L)} = e^{-(b-a)}\|u(t-L)\|_{L^p(0,L)}$. It is also clear that $\|u(t)\|_{L^p(0,L)} \leq \|u_0\|_{L^p(0,L)}$ for every $t \geq 0$, and so $\|u(t)\|_{L^p(0,L)} \leq Ce^{-\gamma t}\|u_0\|_{L^p(0,L)}$ for every $t \geq 0$, with $\gamma = \frac{b-a}{L}$ and $C = e^{\gamma L}$.

When (2.8) has a persistent damping and the damping interval $[a,b]$ is a proper subset of $[0,L]$, there exist $T > \mu > 0$, a persistently exciting signal $\alpha \in \mathcal{G}(T,\mu)$ and a nontrivial initial condition $u_0 \in L^p(0,L)$ such that the corresponding solution of (2.8) is L-periodic. Indeed, suppose that $a = 0$ and $b < L$. Take $u_0 \in \mathcal{C}^\infty([0,L]) \setminus \{0\}$ such that the support of u_0 is contained in $[b, \frac{b+L}{2}]$. Take $\alpha \in L^\infty(\mathbb{R},[0,1])$ defined by

$$\alpha(t) = \begin{cases} 1, & \text{if } 0 \leq \{t\}_L \leq \frac{L-b}{2}, \\ 0, & \text{if } \{t\}_L > \frac{L-b}{2}. \end{cases}$$

Then $\alpha \in \mathcal{G}\left(L, \frac{L-b}{2}\right)$ and one can easily verify that the corresponding solution $u(t,x)$ of (2.8) is equal to $u_0(\{x-t\}_L)$. Hence (2.8) admits a L-periodic solution. $\square$

Example 2.6 shows that the asymptotic behavior of (2.3) can be different if the damping is always active or if it is submitted to a persistently exciting signal, and this is due to the fact that the support of the solution may not be in the damping interval $[a,b]$ when the damping is active. We now consider a second example showing that, when we have more than one circle, the rationality of the ratios L_i/L_j for $i \neq j$ plays an important role in the asymptotic behavior.

Example 2.7. Consider the case of System (2.3) with persistent damping, $N = 2$, $N_d = 1$, and $m_{ij} = 1/2$ for $i, j \in \{1,2\}$, i.e.,

$$\begin{cases} \partial_t u_1(t,x) + \partial_x u_1(t,x) + \alpha(t)\chi(x)u_1(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L_1], \\ \partial_t u_2(t,x) + \partial_x u_2(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L_2], \\ u_1(t,0) = u_2(t,0) = \dfrac{u_1(t,L_1) + u_2(t,L_2)}{2}, & t \in \mathbb{R}_+, \\ u_i(0,x) = u_{i,0}(x), & x \in [0,L_i],\ i \in \{1,2\}, \\ \alpha \in \mathcal{G}(T,\mu), \end{cases} \tag{2.9}$$

where χ is the characteristic function of the interval $[a,b] \subset [0,L_1]$. In order to simplify the discussion, let us fix $p = 2$ and set $\mathsf{X}_2 = L^2(0,L_1) \times L^2(0,L_2)$.

When (2.9) is undamped its asymptotic behavior depends on the rationality of the ratio L_1/L_2, as stated in the next theorem, which is proved in Appendix B.

Theorem 2.8. *Consider* (2.9) *with* $\chi \equiv 0$.

- **i.** *If* $L_1/L_2 \notin \mathbb{Q}$, *each solution converges to a constant function* $(\lambda,\lambda) \in \mathsf{X}_2$ *with* $\lambda \in \mathbb{R}$.
- **ii.** *If* $L_1/L_2 \in \mathbb{Q}$, *there exists a non-constant periodic solution.*

When (2.9) has an always active damping all solutions converge exponentially to zero, independently of the rationality of the ratio L_1/L_2, as it follows, for instance, from Remark 4.10.

When (2.9) has a persistent damping, the rationality of the ratio L_1/L_2 plays once again a role in the asymptotic behavior of the system: if $L_1/L_2 \notin \mathbb{Q}$, all its solutions converge exponentially to zero, as it follows from our main result, Theorem 1.2. However, if $L_1/L_2 \in \mathbb{Q}$ and the damping interval $[a,b]$ is small enough, there exist $T > \mu > 0$, a persistently exciting signal $\alpha \in \mathcal{G}(T,\mu)$ and a nontrivial initial condition $u_0 \in L^p(0,L)$ such that the corresponding solution of (2.9) is periodic, as we show in Appendix B.4. $\square$

Both in Example 2.6 and in Example 2.7 in the case $L_1/L_2 \in \mathbb{Q}$, the lack of exponential stability is illustrated by the existence of a periodic solution for the persistently damped system which is actually a solution to the undamped one for which a persistently exciting signal α inactivates the damping whenever the support of the solution passes through the damping interval. The heuristic of the proof of Theorem 1.2 is that the support of every initial condition spreads with time and eventually covers the entire network. Hence every solution of the persistently excited system eventually passes through a damping interval at a time where the damping is active.

2.3 Discussion on the hypotheses of Theorem 1.2

Recall that the two main assumptions of Theorem 1.2 are the following.

Hypothesis 2.9. There exist $i_*, j_* \in [\![1,N]\!]$ such that $L_{i_*}/L_{j_*} \notin \mathbb{Q}$.

Hypothesis 2.10. The matrix M satisfies $|M|_{\ell^1} \leq 1$ and $m_{ij} \neq 0$ for every $i, j \in [\![1,N]\!]$.

At the light of Example 2.7, one cannot expect exponential stability of (2.3) with persistent damping in general if $L_i/L_j \in \mathbb{Q}$ for every $i, j \in [\![1,N]\!]$. This is why it is reasonable to make Hypothesis 2.9.

Even though the well-posedness of (2.3) discussed in Section 2.1 and the explicit formula for its solutions given later in Section 3 are obtained for every $M \in \mathcal{M}_N(\mathbb{R})$, the asymptotic behavior of System (2.3) clearly depends on the choice of the matrix M, since this matrix determines the coupling among the N transport equations.

The hypothesis $|M|_{\ell^1} \leq 1$ can be written as

$$\sum_{i=1}^{N} |m_{ij}| \leq 1, \qquad \forall j \in [\![1,N]\!]. \tag{2.10}$$

The coefficient m_{ij} can be interpreted as the proportion of mass in the circle C_j that goes to the circle C_i as it passes the contact point O. Hence, (2.10) states that, for every $j \in [\![1,N]\!]$, the total mass arriving at the circles C_i from the circle C_j is less than or equal the total mass leaving the circle C_j, which means that the mass never increases while passing through the junction.

The hypothesis $m_{ij} \neq 0$ for all i, j can be seen as a *strong mixing* of the solutions at the junction. It is designed to avoid reducibility phenomena which may be an obstruction to uniform exponential stability. Consider for instance the case $M = \mathrm{Id}_N$ with $N \geq 2$. Then (2.3) is reduced to N uncoupled transport equations on circles, each of them of the form (2.8). In that case, if there exists at least one index $i \in [\![1,N]\!]$ such that $b_i - a_i < L_i$, then there exist solutions not converging to 0 as $t \to +\infty$, even if there is damping, cf. Example 2.6.

Remark 2.11. Equation (2.10) is satisfied when M is *left stochastic*, i.e., $m_{ij} \geq 0$ for every $i, j \in [\![1,N]\!]$ and $\sum_{i=1}^{N} m_{ij} = 1$ for every $j \in [\![1,N]\!]$. Note that left stochasticity of M is equivalent for the undamped system (2.1) to the preservation of $\sum_{i=1}^{N} \int_0^{L_i} u_i(t,x)dx$ and monotonicity of the solutions with respect to the initial conditions.

3 Explicit solution

This section provides a general formula for the explicit solution of (2.3). We first prove our formula in Section 3.1 in the simpler case of the undamped system (2.1), before turning to the general case in Section 3.2. The coefficients appearing in the formula will be characterized in Section 3.3.

3.1 The undamped system

Remark that, in order to obtain an explicit formula for $u_i(t,x)$ for $i \in [\![1,N]\!]$, $t \geq 0$ and $x \in [0,L_i]$, it suffices to obtain a formula for $u_i(t,0)$ for $i \in [\![1,N]\!]$ and $t \geq 0$. Indeed, it is immediate to derive the following.

Lemma 3.1. *Let $(u_{1,0},\dots,u_{N,0}) \in D(A)$ and let $(u_1,\dots,u_N) \in \mathcal{C}^0(\mathbb{R}_+, D(A)) \cap \mathcal{C}^1(\mathbb{R}_+, \mathsf{X}_p)$ be the corresponding solution of* (2.1)*. Then, for every $i \in [\![1,N]\!]$, $t \geq 0$, and $x \in [0,L_i]$, we have*

$$u_i(t,x) = \begin{cases} u_{i,0}(x-t), & \text{if } 0 \leq t \leq x, \\ u_i(t-x,0), & \text{if } t \geq x. \end{cases} \tag{3.1}$$

In order to express $u_i(t,0)$ in terms of the initial condition $(u_{1,0},\dots,u_{N,0}) \in D(A)$, we need to introduce some notation.

Definition 3.2. **i.** We define $\mathfrak{N} = \mathbb{N}^N$ and, for $i \in [\![1,N]\!]$, $\mathfrak{N}_i = \mathbb{N}^{i-1} \times \{0\} \times \mathbb{N}^{N-i}$.

ii. We write $\mathbf{0} = (0,0,\dots,0) \in \mathfrak{N}$ and, for every $j \in [\![1,N]\!]$ and $\mathfrak{n} = (n_1,\dots,n_N) \in \mathfrak{N}$, $\mathbf{1}_j = (\delta_{ij})_{i=1,\dots,N} \in \mathfrak{N}$ and $\hat{\mathfrak{n}}_j = (n_1,n_2,\dots,n_{j-1},0,n_{j+1},\dots,n_N) = \mathfrak{n} - n_j\mathbf{1}_j \in \mathfrak{N}_j$.

iii. We define the function $L : \mathfrak{N} \to \mathbb{R}_+$ by

$$L(n_1,\dots,n_N) = \sum_{i=1}^{N} n_i L_i.$$

With these notations, the general formula for the solutions of (2.1) can be written as follows.

Theorem 3.3. *Let $(u_{1,0},\dots,u_{N,0}) \in D(A)$. The corresponding solution $(u_1,\dots,u_N)$ of (2.1) is given by*

$$u_i(t,x) = \begin{cases} u_{i,0}(x-t), & \text{if } 0 \le t \le x, \\ u_i(t-x,0), & \text{if } t \ge x, \end{cases} \tag{3.2}$$

with

$$u_i(t,0) = \sum_{j=1}^{N} \sum_{\substack{\mathfrak{n}\in\mathfrak{N}_j \\ L(\mathfrak{n})\le t}} \beta^{(i)}_{j,\mathfrak{n}+\left\lfloor \frac{t-L(\mathfrak{n})}{L_j}\right\rfloor \mathbf{1}_j} u_{j,0}\left(L_j - \{t-L(\mathfrak{n})\}_{L_j}\right), \tag{3.3}$$

and where the coefficients $\beta^{(i)}_{j,\mathfrak{n}}$ are defined by the relations

$$\beta^{(i)}_{j,\mathbf{0}} = m_{ij}, \qquad i,j \in [\![1,N]\!], \tag{3.4a}$$

and

$$\beta^{(i)}_{j,\mathfrak{n}} = \sum_{\substack{k=1 \\ n_k\ge 1}}^{N} m_{kj}\beta^{(i)}_{k,\mathfrak{n}-\mathbf{1}_k}, \qquad i,j \in [\![1,N]\!],\ \mathfrak{n} \in \mathfrak{N}\setminus\{\mathbf{0}\}. \tag{3.4b}$$

The above result follows by iterating Equations (3.1) together with Equation (1.3). Indeed, using the notations of the theorem, one has for $i \in [\![1,N]\!]$,

$$u_i(t,0) = \sum_{j=1}^{N} m_{ij} u_j(t,L_j). \tag{3.5}$$

According to Equation (3.1), each $u_j(t,L_j)$ is either equal to $u_{j,0}(L_j - t)$ or $u_j(t-L_j,0)$ whether $t \le L_j$ or not. In the latter case, we express $u_j(t-L_j,0)$ by using Equation (3.5) and we repeat the procedure a finite number of times until obtaining $u_i(t,0)$ as a linear combination involving only evaluations of the initial condition at finitely many points on the circles. This yields Equation (3.3) with an explicit expression of both the coefficients of this linear combination and the points on the circles.

The complete proof of Theorem 3.3 is provided in Appendix C and consists in verifying that the explicit formula given in the above statement is indeed the solution of (2.1).

We next provide with Figure 3.1 a geometric interpretation of (3.3) in the case $N = 2$. The point O is identified with the origin of the plane $(x_1,\ x_2)$ and the horizontal (resp. vertical) segments in the grid represented in Figure 3.1 correspond to identical copies of the circle C_1 (resp. the circle C_2). The intersection of the dashed line $l_t :\ x_1 + x_2 = t$ and the grid exactly represents the set of points of the circles where the initial condition $(u_{1,0},u_{2,0})$ is evaluated in Equation (3.3). Note that the coefficients in Equation (3.3) appearing in front of the evaluation of the initial condition at P can be expressed as a sum of products of the m_{ij}'s, each product corresponding to a path on the grid between P and O.

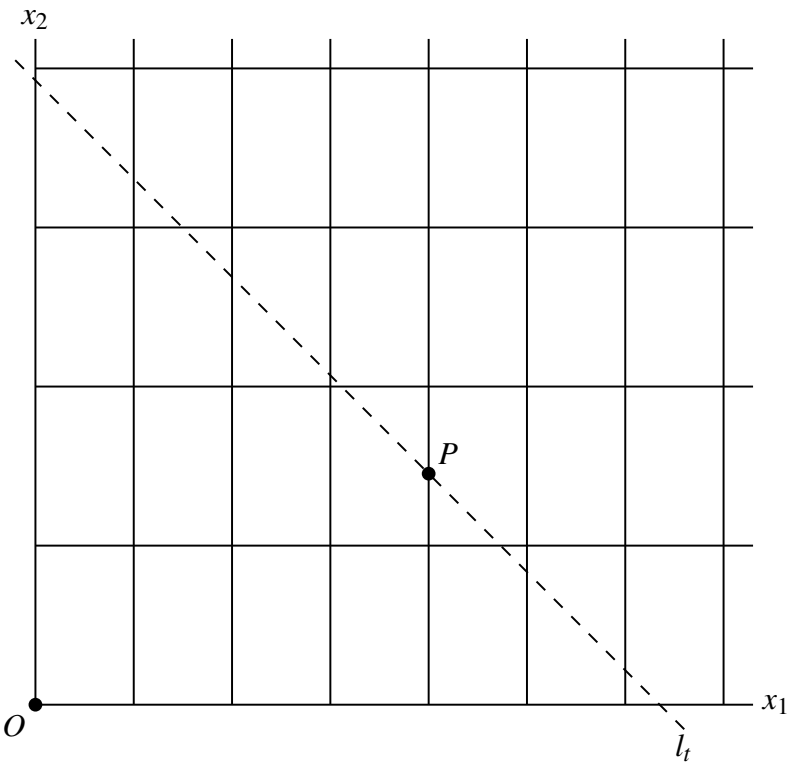

FIGURE 3.1: *Geometric construction for the explicit formula for the solution of* (2.1) *in the case* $N = 2$.

3.2 Formula for the explicit solution in the general case

We first notice that, as in Lemma 3.1, it suffices to study $u_i(t,0)$ for every $t \geq 0$ and $i \in [\![1,N]\!]$ in order to obtain the whole solution $(u_1(t),\dots,u_N(t))$. Recall that by convention we have set $\alpha_i \equiv 1$ and $a_i = b_i$ (and thus $\chi_i = 0$ almost everywhere) for $i \in [\![N_d+1,N]\!]$.

Proposition 3.4. *Let* $(u_{1,0},\dots,u_{N,0}) \in D(A)$. *Then the corresponding solution* $(u_1,\dots,u_N)$ *of* (2.3) *satisfies, for* $i \in [\![1,N]\!]$,

$$u_i(t,x) = \begin{cases} u_{i,0}(x-t)\exp\left(-\int_{[0,t]\cap[t-x+a_i,t-x+b_i]} \alpha_i(s)ds\right), & \text{if } 0 \leq t \leq x, \\ u_i(t-x,0)\exp\left(-\int_{[0,t]\cap[t-x+a_i,t-x+b_i]} \alpha_i(s)ds\right), & \text{if } t \geq x. \end{cases} \tag{3.6}$$

Proof. Let $i \in [\![1,N]\!]$. Equation (3.6) is obtained by integrating the differential equation

$$\frac{d}{ds}u_i(t+s,x+s) = -\alpha_i(t+s)\chi_i(x+s)u_i(t+s,x+s),$$

on the interval $[-t,0]$ if $t \leq x$ and on $[-x,0]$ if $t \geq x$. ■

Thanks to the fact that all the exponential decays appearing in (3.6) are upper bounded by 1, one obtains trivially the following corollary.

Corollary 3.5. *If* $(u_1,\dots,u_N)$ *is the solution of* (2.3) *with an initial condition* $(u_{1,0},\dots,u_{N,0})$, *then, for* $i \in [\![1,N_d]\!]$, u_i *satisfies the estimate*

$$|u_i(t,x)| \leq \begin{cases} |u_{i,0}(x-t)|, & \text{if } 0 \leq t \leq x, \\ |u_i(t-x,0)|, & \text{if } t \geq x. \end{cases}$$

For every $p \in [1,+\infty]$, $i \in [\![1,N]\!]$, and $t \geq L_i$, we have

$$\|u_i(t,\cdot)\|_{L^p(0,L_i)} \leq \|u_i(\cdot,0)\|_{L^p(t-L_i,t)},$$

with equality if $i \in [\![N_d+1,N]\!]$.

This corollary allows us to replace the spatial L^p-norm of u_i at a given time t by its L^p-norm in a time interval of length L_i at the fixed position $x=0$.

We can now write the explicit formula for the solutions of (2.3) using the notations from Definition 3.2. The proof follows the same steps as that of Theorem 3.3.

Theorem 3.6. *Let $(u_{1,0},\dots,u_{N,0}) \in D(A)$. The corresponding solution $(u_1,\dots,u_N)$ of (2.3) is given by (3.6), where $u_i(t,0)$ is given for $t \geq 0$ by*

$$u_i(t,0) = \sum_{j=1}^{N} \sum_{\substack{\mathfrak{n}\in\mathfrak{N}_j \\ L(\mathfrak{n})\leq t}} \vartheta^{(i)}_{j,\mathfrak{n}+\left\lfloor \frac{t-L(\mathfrak{n})}{L_j}\right\rfloor \mathbf{1}_j, L_j-\{t-L(\mathfrak{n})\}_{L_j},t} u_{j,0}\left(L_j-\{t-L(\mathfrak{n})\}_{L_j}\right) \tag{3.7}$$

and the coefficients $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ are defined for $i,j \in [\![1,N]\!]$, $\mathfrak{n}\in\mathfrak{N}$, $x\in[0,L_j]$ and $t\in\mathbb{R}$ by

$$\vartheta^{(i)}_{j,\mathfrak{n},x,t} = \varepsilon_{j,\mathfrak{n},x,t}\,\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}, \tag{3.8a}$$

with

$$\varepsilon_{j,\mathfrak{n},x,t} = \exp\left(-\int_{I_{j,\mathfrak{n},x,t}} \alpha_j(s)ds\right), \tag{3.8b}$$

where $I_{j,\mathfrak{n},x,t} = [t-L(\mathfrak{n})-L_j+\max(x,a_j), t-L(\mathfrak{n})-L_j+b_j]$, and

$$\vartheta^{(i)}_{j,\mathbf{0},L_j,t} = m_{ij}, \tag{3.8c}$$

$$\vartheta^{(i)}_{j,\mathfrak{n},L_j,t} = \sum_{\substack{k=1\\ n_k\geq 1}}^{N} m_{kj}\,\vartheta^{(i)}_{k,\mathfrak{n}-\mathbf{1}_k,0,t}. \tag{3.8d}$$

Let us provide a geometrical interpretation of the above theorem in the case $N=2$, $N_d=1$ with damping on the circle C_1. With respect to Figure 3.1, Figure 3.2 now includes segments corresponding to the intervals $[a_1,b_1]$ in C_1, on which the solution is damped. Similarly to (3.3), the new explicit formula (3.7) expresses $u_i(t,0)$ as a linear combination involving only evaluations of the initial condition at finitely many points on the circles. The coefficients, which in (3.3) were sum of products of the m_{ij}'s, each product corresponding to a path between P and O, have now an analogous expression, with the following modification: each factor of the original product is multiplied by an additional term of the type $\varepsilon_{j,\mathfrak{n},x,t}$, which takes into account the effect of the damping along the path.

Remark 3.7. The explicit formula for the solution of the undamped equation (2.1) given in Theorem 3.3 can be obtained as a particular case of Theorem 3.6 by setting $\alpha_j \equiv 0$ for every $j \in [\![1,N_d]\!]$. Similarly, we can obtain the explicit formula for the solution of (2.2) as a particular case of Theorem 3.6 by setting $\alpha_j \equiv 1$ for every $j \in [\![1,N_d]\!]$, yielding $\varepsilon_{j,\mathfrak{n},x,t} = e^{-\mathrm{meas}([a_j,b_j]\cap[x,b_j])}$.

Remark 3.8. In the general case $(u_{1,0},\dots,u_{N,0}) \in \mathsf{X}_p$, the mild solution $(u_1,\dots,u_N)$ of (2.3) can still be characterized by (3.6) and (3.7) (yielding an equality in X_p for every $t \geq 0$). This follows by a simple density argument of $D(A)$ in X_p.

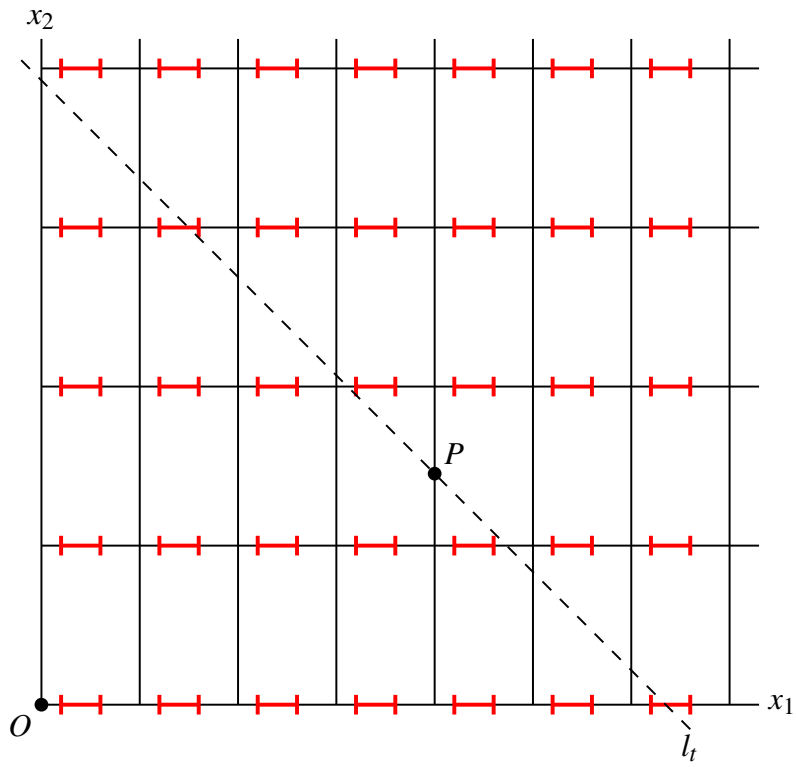

FIGURE 3.2: *Geometric interpretation of the explicit formula* (3.7) *for the solution of* (2.3) *in the case* $N = 2$.

3.3 Recursive formula for the coefficients

We now wish to determine a recursive formula with K steps for the coefficients $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ appearing in the expression of the explicit solution (3.7). For $v \in [\![1,N]\!]^K$ and $k \in [\![1,N]\!]$, we denote

$$\varphi_{k,K}(v) = \sum_{s=1}^{K} \delta_{kv_s} = \#\{s \in [\![1,K]\!] \mid v_s = k\},$$

and, for $\mathfrak{n} \in \mathfrak{N}$ with $|\mathfrak{n}|_{\ell^1} \geq K$, we introduce the set

$$\Phi_K(\mathfrak{n}) = \{v \in [\![1,N]\!]^K \mid n_j \geq \varphi_{j,K}(v) \text{ for all } j \in [\![1,N]\!]\}.$$

Then we have the following result.

Proposition 3.9. *Let $K \in \mathbb{N}^*$ and suppose that $\mathfrak{n} \in \mathfrak{N}$ is such that $|\mathfrak{n}|_{\ell^1} \geq K$. Then, for every $i, j \in [\![1,N]\!]$ and $t \in \mathbb{R}$, we have*

$$\vartheta^{(i)}_{j,\mathfrak{n},L_j,t} = \sum_{v \in \Phi_K(\mathfrak{n})} \left[\left(m_{v_1 j} \prod_{s=2}^{K} m_{v_s v_{s-1}} \right) \left(\prod_{s=1}^{K} \varepsilon_{v_s, \mathfrak{n} - \sum_{r=1}^{s} \mathbf{1}_{v_r}, 0, t} \right) \vartheta^{(i)}_{v_K, \mathfrak{n} - \sum_{s=1}^{K} \mathbf{1}_{v_s}, L_{v_K}, t} \right]. \tag{3.9}$$

Proof. The proof is done by induction on K. If $K = 1$, we have

$$\Phi_1(\mathfrak{n}) = \{v \in [\![1,N]\!] \mid n_j \geq \delta_{jv} \text{ for all } j \in [\![1,N]\!]\} = \{v \in [\![1,N]\!] \mid n_v \geq 1\},$$

and so, by (3.8a) and (3.8d),

$$\sum_{v \in \Phi_1(\mathfrak{n})} \left[m_{vj} \varepsilon_{v, \mathfrak{n} - \mathbf{1}_v, 0, t} \vartheta^{(i)}_{v, \mathfrak{n} - \mathbf{1}_v, L_v, t} \right] = \sum_{\substack{v=1 \\ n_v \geq 1}}^{N} m_{vj} \varepsilon_{v, \mathfrak{n} - \mathbf{1}_v, 0, t} \vartheta^{(i)}_{v, \mathfrak{n} - \mathbf{1}_v, L_v, t} = \vartheta^{(i)}_{j, \mathfrak{n}, L_j, t}.$$

Suppose now that $K \in \mathbb{N}^*$ with $K \leq |\mathfrak{n}|_{\ell^1}$ and (3.9) holds true for $K-1$. Then we have, by (3.8d),

$$\begin{aligned}
\vartheta^{(i)}_{j,\mathfrak{n},L_j,t} &= \sum_{v' \in \Phi_{K-1}(\mathfrak{n})} \left[\left(m_{v'_1 j} \prod_{s=2}^{K-1} m_{v'_s v'_{s-1}} \right) \left(\prod_{s=1}^{K-1} \varepsilon_{v'_s, \mathfrak{n} - \sum_{r=1}^{s} \mathbf{1}_{v'_r}, 0, t} \right) \vartheta^{(i)}_{v'_{K-1}, \mathfrak{n} - \sum_{s=1}^{K-1} \mathbf{1}_{v'_s}, L_{v'_{K-1}}, t} \right] \\
&= \sum_{v' \in \Phi_{K-1}(\mathfrak{n})} \left[\left(m_{v'_1 j} \prod_{s=2}^{K-1} m_{v'_s v'_{s-1}} \right) \left(\prod_{s=1}^{K-1} \varepsilon_{v'_s, \mathfrak{n} - \sum_{r=1}^{s} \mathbf{1}_{v'_r}, 0, t} \right) \right. \\
&\qquad \left. \left(\sum_{\substack{k=1 \\ n_k > \varphi_{k,K-1}(v')}}^{N} m_{k v'_{K-1}} \varepsilon_{k, \mathfrak{n} - \sum_{s=1}^{K-1} \mathbf{1}_{v'_s} - \mathbf{1}_k, 0, t} \vartheta^{(i)}_{k, \mathfrak{n} - \sum_{s=1}^{K-1} \mathbf{1}_{v'_s} - \mathbf{1}_k, L_k, t} \right) \right] \\
&= \sum_{v \in \Phi_K(\mathfrak{n})} \left[\left(m_{v_1 j} \prod_{s=2}^{K} m_{v_s v_{s-1}} \right) \left(\prod_{s=1}^{K} \varepsilon_{v_s, \mathfrak{n} - \sum_{r=1}^{s} \mathbf{1}_{v_r}, 0, t} \right) \vartheta^{(i)}_{v_K, \mathfrak{n} - \sum_{s=1}^{K} \mathbf{1}_{v_s}, L_{v_K}, t} \right],
\end{aligned}$$

where we take $v = (v', v_N = k) = (v'_1, \ldots, v'_{N-1}, k)$ and notice that $\varphi_{j,K}(v', k) = \varphi_{j,K-1}(v') + \delta_{jk}$ for every $j \in [\![1, N]\!]$, so that

$$\begin{aligned}
&\{(v', k) \in \Phi_{K-1}(\mathfrak{n}) \times [\![1, N]\!] \mid n_k > \varphi_{k,K-1}(v')\} \\
&\quad = \{(v', k) \in [\![1, N]\!]^{K-1} \times [\![1, N]\!] \mid n_j \geq \varphi_{j,K-1}(v') \text{ for all } j \in [\![1, N]\!] \text{ and } n_k > \varphi_{k,K-1}(v')\} \\
&\qquad = \{v \in [\![1, N]\!]^K \mid n_j \geq \varphi_{j,K}(v) \text{ for all } j \in [\![1, N]\!]\} = \Phi_K(\mathfrak{n}).
\end{aligned}$$

This proves (3.9) by induction. ■

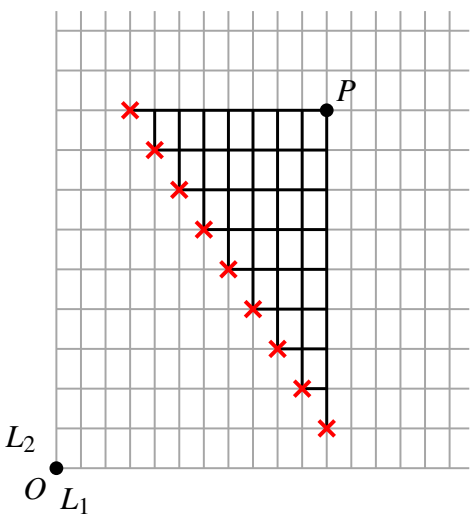

FIGURE 3.3: *Here* $N = 2$, $\mathfrak{n} = (11, 9)$ *and* $K = 8$.

Remark 3.10. In terms of the construction of Figure 3.2, each $v \in \Phi_K(\mathfrak{n})$ corresponds to a path between P and a point represented by a cross in Figure 3.3. The term

$$\prod_{s=1}^{K} \varepsilon_{v_s, \mathfrak{n} - \sum_{r=1}^{s} \mathbf{1}_{v_r}, 0, t}$$

in (3.9) represents the total decay along this path.

Remark 3.11. Under Hypothesis 2.10, the terms $m_{\nu_1 j}\prod_{s=2}^{K} m_{\nu_s\nu_{s-1}}$ in (3.9) satisfy

$$\sum_{\nu\in\Phi_K(\mathfrak{n})}\left|m_{\nu_1 j}\prod_{s=2}^{K} m_{\nu_s\nu_{s-1}}\right|\leq 1. \tag{3.10}$$

Indeed, we have

$$\sum_{\nu\in\Phi_K(\mathfrak{n})}\left|m_{\nu_1 j}\prod_{s=2}^{K} m_{\nu_s\nu_{s-1}}\right|\leq \sum_{\nu\in[\![1,N]\!]^K}\left|m_{\nu_1 j}\prod_{s=2}^{K} m_{\nu_s\nu_{s-1}}\right| = \sum_{\nu_1=1}^{N}\cdots\sum_{\nu_K=1}^{N}\left|m_{\nu_K\nu_{K-1}}\right|\cdots\left|m_{\nu_1 j}\right|\leq 1,$$

by applying iteratively (2.10). We also remark that Proposition 3.9 implies that the coefficient $\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}$ belongs to the convex hull of the points $\pm\vartheta^{(i)}_{\nu_K,\mathfrak{n}-\sum_{s=1}^{K}\mathbf{1}_{\nu_s},L_{\nu_K},t}$, for ν in $\Phi_K(\mathfrak{n})$.

4 Proof of the main result

We now study the asymptotic behavior of the solutions of (2.3) with persistent damping, through their explicit formula obtained in Section 3. We first show that, in order to obtain the exponential stability of the solutions of (2.3), it suffices to obtain the exponential convergence as $|\mathfrak{n}|_{\ell^1}\to+\infty$ of the coefficients $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ of the explicit formula (3.7).

4.1 Convergence of the coefficients implies convergence of the solution

The section is devoted to the proof of the following result.

Proposition 4.1. *Let $\mathcal{F}\subset L^\infty(\mathbb{R},[0,1])$. Suppose that there exist constants $C_0,\gamma_0>0$ such that, for every $\alpha_k\in\mathcal{F}$, $k\in[\![1,N_d]\!]$, we have*

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right|\leq C_0 e^{-\gamma_0|\mathfrak{n}|_{\ell^1}},\qquad \forall i,j\in[\![1,N]\!],\ \forall\mathfrak{n}\in\mathfrak{N},\ \forall x\in[0,L_j],\ \forall t\in\mathbb{R}.$$

Then there exist constants $C,\gamma>0$ such that, for every $p\in[1,+\infty)$ and every initial condition $z_0\in\mathsf{X}_p$, the corresponding solution $z(t)$ of (2.3) satisfies

$$\|z(t)\|_{\mathsf{X}_p}\leq Ce^{-\gamma t}\|z_0\|_{\mathsf{X}_p},\qquad \forall t\geq 0. \tag{4.1}$$

Remark 4.2. The conclusion (4.1) of Proposition 4.1 can be written, in terms of the evolution family $\{T(t,s)\}_{t\geq s\geq 0}$ associated with (2.3), as

$$\|T(t,0)\|_{\mathcal{L}(\mathsf{X}_p)}\leq Ce^{-\gamma t},\qquad t\geq 0.$$

When the class $\mathcal{F}$ is invariant by time-translation (e.g., for $\mathcal{F}=\mathcal{G}(T,\mu)$), this is actually equivalent to

$$\|T(t,s)\|_{\mathcal{L}(\mathsf{X}_p)}\leq Ce^{-\gamma(t-s)},\qquad t\geq s\geq 0.$$

Proof. Set $L_{\max}=\max_{i\in[\![1,N]\!]}L_i$ and $L_{\min}=\min_{i\in[\![1,N]\!]}L_i$. Take $z_0=(u_{1,0},\dots,u_{N,0})\in\mathsf{X}_p$ and denote by $z(t)=(u_1(t),\dots,u_N(t))$ the corresponding solution of (2.3). By Corollary 3.5, Theorem 3.6, and Remark 3.8, we have, for $t\geq L_{\max}$,

$$\|z(t)\|^p_{\mathsf{X}_p}=\sum_{i=1}^{N}\|u_i(t)\|^p_{L^p(0,L_i)}\leq\sum_{i=1}^{N}\|u_i(\cdot,0)\|^p_{L^p(t-L_i,t)}, \tag{4.2}$$

with $u_i(t,0)$ given by (3.7). Denoting

$$Y_j(t) = \#\{\mathfrak{n} \in \mathfrak{N}_j \,|\, L(\mathfrak{n}) \le t\},$$

we have

$$\begin{aligned}
\|u_i(\cdot,0)\|^p_{L^p(t-L_i,t)} &= \int_{t-L_i}^t |u_i(s,0)|^p\,ds \\
&\le N^{p-1}\sum_{j=1}^N \int_{t-L_i}^t \Big|\sum_{\substack{\mathfrak{n}\in\mathfrak{N}_j\\ L(\mathfrak{n})\le s}} \vartheta^{(i)}_{j,\mathfrak{n}+\left\lfloor\frac{s-L(\mathfrak{n})}{L_j}\right\rfloor \mathbf{1}_j, L_j-\{s-L(\mathfrak{n})\}_{L_j},s} u_{j,0}(L_j-\{s-L(\mathfrak{n})\}_{L_j})\Big|^p ds \\
&\le N^{p-1}\sum_{j=1}^N \int_{t-L_i}^t Y_j(s)^{p-1} \sum_{\substack{\mathfrak{n}\in\mathfrak{N}_j\\ L(\mathfrak{n})\le s}} \Big|\vartheta^{(i)}_{j,\mathfrak{n}+\left\lfloor\frac{s-L(\mathfrak{n})}{L_j}\right\rfloor \mathbf{1}_j, L_j-\{s-L(\mathfrak{n})\}_{L_j},s} u_{j,0}(L_j-\{s-L(\mathfrak{n})\}_{L_j})\Big|^p ds \\
&\le N^{p-1}C_0^p\sum_{j=1}^N Y_j(t)^{p-1} \sum_{\substack{\mathfrak{n}\in\mathfrak{N}_j\\ L(\mathfrak{n})\le t}} \int_{t-L_i}^t e^{-p\gamma_0\left(|\mathfrak{n}|_{\ell^1}+\left\lfloor\frac{s-L(\mathfrak{n})}{L_j}\right\rfloor\right)} \left|u_{j,0}(L_j-\{s-L(\mathfrak{n})\}_{L_j})\right|^p ds \\
&\le N^{p-1}C_0^p e^{2p\gamma_0} e^{-\frac{p\gamma_0}{L_{\max}}t}\sum_{j=1}^N Y_j(t)^{p-1} \sum_{\substack{\mathfrak{n}\in\mathfrak{N}_j\\ L(\mathfrak{n})\le t}} \int_{t-L_{\max}}^t \left|u_{j,0}(L_j-\{s-L(\mathfrak{n})\}_{L_j})\right|^p ds,
\end{aligned} \tag{4.3}$$

where we use that

$$|\mathfrak{n}|_{\ell^1} + \left\lfloor\frac{s-L(\mathfrak{n})}{L_j}\right\rfloor = \sum_{k=1}^N n_k + \left\lfloor\frac{s-L(\mathfrak{n})}{L_j}\right\rfloor \ge \frac{L(\mathfrak{n})}{L_{\max}} + \left\lfloor\frac{s-L(\mathfrak{n})}{L_{\max}}\right\rfloor \ge \frac{s}{L_{\max}} - 1 \ge \frac{t}{L_{\max}} - 2,$$

for $\mathfrak{n} \in \mathfrak{N}_j$ with $L(\mathfrak{n}) \le t$ and $s \in [t-L_i, t]$.

According to its definition, $Y_j(t)$ can be upper bounded as follows

$$\begin{aligned}
Y_j(t) &\le \#\{\mathfrak{n}\in\mathfrak{N}_j \,|\, n_iL_i \le t \text{ for all } i \in [\![1,N]\!]\setminus\{j\}\} \\
&= \#\left(\left[\!\left[0,\frac{t}{L_1}\right]\!\right] \times\cdots\times \left[\!\left[0,\frac{t}{L_{j-1}}\right]\!\right] \times \{0\} \times \left[\!\left[0,\frac{t}{L_{j+1}}\right]\!\right] \times\cdots\times \left[\!\left[0,\frac{t}{L_N}\right]\!\right]\right) \\
&\le \left(\frac{t}{L_{\min}}+1\right)^{N-1}.
\end{aligned} \tag{4.4}$$

We next estimate $\int_{t-L_{\max}}^t \left|u_{j,0}(L_j-\{s-L(\mathfrak{n})\}_{L_j})\right|^p ds$ with $j \in [\![1,N]\!]$. Notice that $[t-L_{\max}, t) \subset \cup_{k=k_{\min}}^{k_{\max}} \big[L(\mathfrak{n})+kL_j, L(\mathfrak{n})+(k+1)L_j\big)$ with

$$k_{\min} = \max\{k\in\mathbb{Z} \,|\, L(\mathfrak{n})+kL_j \le t-L_{\max}\}, \quad k_{\max} = \min\{k\in\mathbb{Z} \,|\, L(\mathfrak{n})+(k+1)L_j \ge t\}.$$

We deduce that

$$\begin{aligned}
\int_{t-L_{\max}}^t \left|u_{j,0}(L_j-\{s-L(\mathfrak{n})\}_{L_j})\right|^p ds &\le \sum_{k=k_{\min}}^{k_{\max}} \int_0^{L_j} \left|u_{j,0}(\sigma)\right|^p d\sigma \\
&= (k_{\max}-k_{\min}+1)\left\|u_{j,0}\right\|^p_{L^p(0,L_j)} \\
&\le \left(\frac{L_{\max}}{L_j}+2\right)\left\|u_{j,0}\right\|^p_{L^p(0,L_j)}.
\end{aligned} \tag{4.5}$$

Inserting (4.4) and (4.5) into (4.3) finally gives (4.1) thanks to (4.2). Notice that the coefficients γ and C can be chosen to be independent of p. ■

Remark 4.3. Even though the well-posedness of (2.5) was considered in Section 2.1 only for $p \in [1,+\infty)$, we extend it here below to the case $p = +\infty$ and we verify that Proposition 4.1 still holds true in this case.

First set $\mathsf{X}_\infty = \prod_{i=1}^N L^\infty(0,L_i)$ with its usual norm $\|z\|_{\mathsf{X}_\infty} = \max_{i\in[\![1,N]\!]} \|u_i\|_{L^\infty(0,L_i)}$ for $z = (u_1,\dots,u_N) \in \mathsf{X}_\infty$. Fix $z_0 \in \mathsf{X}_\infty$. Since $z_0 \in \mathsf{X}_p$ for every $p \in [1,+\infty)$, then (2.5) admits a unique mild solution $z(t) = T(t,0)z_0$ in $\bigcap_{p\in[1,+\infty)} \mathcal{C}^0(\mathbb{R}_+,\mathsf{X}_p)$ with initial condition $z(0) = z_0$. As noticed in Remark 3.8, $z(t)$ is characterized as an element of X_p by equations (3.6) and (3.7). Hence $z(t) \in \mathsf{X}_\infty$ for every $t \geq 0$. We can thus refer to $z(\cdot)$ as the solution of the Cauchy problem (2.5) in X_∞.

Suppose now that the hypotheses of Proposition 4.1 are satisfied and let $C,\gamma > 0$ be as in its statement. By (4.1), we have

$$\|z(t)\|_{\mathsf{X}_p} \leq Ce^{-\gamma t}\|z_0\|_{\mathsf{X}_p} \leq C\left(\sum_{i=1}^N L_i\right)^{1/p} e^{-\gamma t}\|z_0\|_{\mathsf{X}_\infty}.$$

Since $\|y\|_{\mathsf{X}_\infty} = \lim_{p\to+\infty}\|y\|_{\mathsf{X}_p}$ for every $y \in \mathsf{X}_\infty$, we conclude that

$$\|z(t)\|_{\mathsf{X}_\infty} \leq Ce^{-\gamma t}\|z_0\|_{\mathsf{X}_\infty}.$$

4.2 Preliminary estimates of $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$

In this section we establish estimates on the growth of $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ based on combinatorial arguments.

Proposition 4.4. *For every $i,j \in [\![1,N]\!]$, $\mathfrak{n} \in \mathfrak{N}$, $x \in [0,L_j]$, $t \in \mathbb{R}$, and $\alpha_k \in L^\infty(\mathbb{R},[0,1])$, $k \in [\![1,N_d]\!]$, we have*

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| \leq |M|_{\ell^1}^{|\mathfrak{n}|_{\ell^1}+1}. \tag{4.6}$$

Proof. We show (4.6) by induction on $|\mathfrak{n}|_{\ell^1}$. For every $i,j \in [\![1,N]\!]$, $x \in [0,L_j]$, $t \in \mathbb{R}$, and $\alpha_k \in L^\infty(\mathbb{R},[0,1])$, $k \in [\![1,N_d]\!]$, we have, by (3.8),

$$\left|\vartheta^{(i)}_{j,\mathbf{0},x,t}\right| \leq \left|\vartheta^{(i)}_{j,\mathbf{0},L_j,t}\right| = \left|m_{ij}\right| \leq |M|_{\ell^1}.$$

If $R \in \mathbb{N}$ is such that (4.6) holds for every $i,j \in [\![1,N]\!]$, $\mathfrak{n} \in \mathfrak{N}$ with $|\mathfrak{n}|_{\ell^1} = R$, $x \in [0,L_j]$, $t \in \mathbb{R}$, and $\alpha_k \in L^\infty(\mathbb{R},[0,1])$, $k \in [\![1,N_d]\!]$, then, for $\mathfrak{n} \in \mathfrak{N}$ with $|\mathfrak{n}|_{\ell^1} = R+1$, we have, by (3.8),

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| \leq \left|\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}\right| \leq \sum_{\substack{r=1\\ n_r\geq 1}}^N \left|m_{rj}\right| \left|\vartheta^{(i)}_{r,\mathfrak{n}-\mathbf{1}_r,L_r,t}\right| \leq \sum_{r=1}^N \left|m_{rj}\right| |M|_{\ell^1}^{R+1} \leq |M|_{\ell^1}^{R+2},$$

since $|M|_{\ell^1} = \max_{j\in[\![1,N]\!]} \sum_{r=1}^N \left|m_{rj}\right|$. The result thus follows by induction. ■

As a consequence of Propositions 4.1 and 4.4 and Remark 4.3, we deduce at once the following corollary.

Corollary 4.5. *Suppose that $|M|_{\ell^1} < 1$. Then there exist $C,\gamma > 0$ such that, for every $p \in [1,+\infty]$ and every initial condition $z_0 \in \mathsf{X}_p$, the corresponding solution $z(t)$ of the undamped equation* (2.1) *satisfies*

$$\|z(t)\|_{\mathsf{X}_p} \leq Ce^{-\gamma t}\|z_0\|_{\mathsf{X}_p}, \qquad \forall t \geq 0.$$

Another trivial but important consequence of Proposition 4.4 is that, if $|M|_{\ell^1} \leq 1$, the coefficients $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ are all bounded in absolute value by 1.

Corollary 4.6. *Suppose that $|M|_{\ell^1} \leq 1$. Then, for every $i, j \in [\![1,N]\!]$, $\mathfrak{n} \in \mathfrak{N}$, $x \in [0,L_j]$, $t \in \mathbb{R}$, and $\alpha_k \in L^\infty(\mathbb{R},[0,1])$, $k \in [\![1,N_d]\!]$, we have*

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| \leq 1. \tag{4.7}$$

Our second estimate on the coefficients $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ is the following.

Lemma 4.7. *Suppose that Hypothesis 2.10 is satisfied. Then there exists $\nu \in (0,1)$ such that, for every $i, j, k \in [\![1,N]\!]$, $\mathfrak{n} \in \mathfrak{N}$, $x \in [0,L_j]$, $t \in \mathbb{R}$, and $\alpha_r \in L^\infty(\mathbb{R},[0,1])$, $r \in [\![1,N_d]\!]$, we have*

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| \leq \binom{|\mathfrak{n}|_{\ell^1}}{n_k} \nu^{|\mathfrak{n}|_{\ell^1}}. \tag{4.8}$$

Proof. Up to a permutation in the set of indices, we can suppose, without loss of generality, that $k = N$. Let

$$\mu_N = \max_{j \in [\![1,N]\!]} \sum_{i=1}^{N-1} \left|m_{ij}\right|, \qquad \nu_N = \max_{j \in [\![1,N]\!]} \left|m_{Nj}\right|.$$

By Hypothesis 2.10, we have both $\sum_{i=1}^{N-1} \left|m_{ij}\right| < 1$ and $\left|m_{Nj}\right| < 1$. Hence $\mu_N, \nu_N \in (0,1)$.

We prove by induction on $|\mathfrak{n}|_{\ell^1}$ that

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| \leq \binom{|\mathfrak{n}|_{\ell^1}}{n_k} \mu_N^{|\mathfrak{n}|_{\ell^1}-n_N} \nu_N^{n_N}. \tag{4.9}$$

For every $i, j \in [\![1,N]\!]$, $x \in [0,L_j]$, $t \in \mathbb{R}$, and $\alpha_r \in L^\infty(\mathbb{R},[0,1])$, $r \in [\![1,N_d]\!]$, we have, by (3.8),

$$\left|\vartheta^{(i)}_{j,\mathbf{0},x,t}\right| \leq \left|\vartheta^{(i)}_{j,\mathbf{0},L_j,t}\right| = \left|m_{ij}\right| \leq 1,$$

so that (4.9) is satisfied for $\mathfrak{n} = \mathbf{0}$.

Suppose now that $R \in \mathbb{N}$ is such that (4.9) is satisfied for every $i, j \in [\![1,N]\!]$, $\mathfrak{n} \in \mathfrak{N}$ with $|\mathfrak{n}|_{\ell^1} = R$, $x \in [0,L_j]$, $t \in \mathbb{R}$, and $\alpha_r \in L^\infty(\mathbb{R},[0,1])$, $r \in [\![1,N_d]\!]$. If $\mathfrak{n} \in \mathfrak{N}$ is such that $|\mathfrak{n}|_{\ell^1} = R+1$, we have, by (3.8),

$$\begin{aligned}
\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| &\leq \left|\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}\right| \leq \sum_{\substack{r=1\\ n_r \geq 1}}^{N-1} \left|m_{rj}\right| \left|\vartheta^{(i)}_{r,\mathfrak{n}-\mathbf{1}_r,L_r,t}\right| + \left|m_{Nj}\right| \left|\vartheta^{(i)}_{N,\mathfrak{n}-\mathbf{1}_N,L_N,t}\right| \\
&\leq \sum_{\substack{r=1\\ n_r \geq 1}}^{N-1} \left|m_{rj}\right| \binom{|\mathfrak{n}|_{\ell^1}-1}{n_N} \mu_N^{|\mathfrak{n}|_{\ell^1}-n_N-1} \nu_N^{n_N} + \left|m_{Nj}\right| \binom{|\mathfrak{n}|_{\ell^1}-1}{n_N-1} \mu_N^{|\mathfrak{n}|_{\ell^1}-n_N} \nu_N^{n_N-1} \\
&\leq \binom{|\mathfrak{n}|_{\ell^1}-1}{n_N} \mu_N^{|\mathfrak{n}|_{\ell^1}-n_N} \nu_N^{n_N} + \binom{|\mathfrak{n}|_{\ell^1}-1}{n_N-1} \mu_N^{|\mathfrak{n}|_{\ell^1}-n_N} \nu_N^{n_N} \\
&= \binom{|\mathfrak{n}|_{\ell^1}}{n_N} \mu_N^{|\mathfrak{n}|_{\ell^1}-n_N} \nu_N^{n_N},
\end{aligned}$$

with the convention that $\vartheta^{(i)}_{N,\mathfrak{n}-\mathbf{1}_N,L_N,t} = 0$ if $n_N = 0$. Hence (4.9) holds for $\mathfrak{n}$, which proves the result by induction. We conclude by taking $\nu = \max\{\mu_k, \nu_k \,|\, k \in [\![1,N]\!]\}$. ■

4.3 Exponential decay of $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ in $\mathfrak{N}_b(\rho)$

The proof of the exponential decay of the coefficients $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ as $|\mathfrak{n}|_{\ell^1} \to +\infty$, uniformly with respect to $\alpha_k \in \mathcal{G}(T,\mu)$, $k \in [\![1,N_d]\!]$, is split into two cases. We first estimate $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ for $\mathfrak{n}$ in a subset $\mathfrak{N}_b(\rho)$ of $\mathfrak{N}$, namely when one of the components of $\mathfrak{n}$ is much smaller than the others. The parameter $\rho \in (0,1)$ is a measure of such a smallness and will be fixed later. For $\mathfrak{n} \in \mathfrak{N}_b(\rho)$, the exponential decay of $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ does not result from the presence of the persistent damping but solely from combinatorial considerations. We then proceed in Section 4.4 to estimate $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ in the set $\mathfrak{N}_c(\rho) = \mathfrak{N} \setminus \mathfrak{N}_b(\rho)$, where the decay comes from the persistent damping in (2.3).

Definition 4.8. For $k \in [\![1,N]\!]$ and $\rho \in (0,1)$, we define

$$\mathfrak{N}_b(\rho,k) = \{\mathfrak{n} = (n_1,\dots,n_N) \in \mathfrak{N} \mid n_k \le \rho\, |\mathfrak{n}|_{\ell^1}\},$$
$$\mathfrak{N}_b(\rho) = \bigcup_{k=1}^{N} \mathfrak{N}_b(\rho,k), \qquad \mathfrak{N}_c(\rho) = \mathfrak{N} \setminus \mathfrak{N}_b(\rho).$$

We now deduce from Lemma 4.7 the exponential decay of $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ in the set $\mathfrak{N}_b(\rho)$.

Theorem 4.9. *Suppose that Hypothesis 2.10 is satisfied. There exist $\rho \in (0,1/2)$ and constants $C,\gamma > 0$ such that, for every $i,j \in [\![1,N]\!]$, $\mathfrak{n} \in \mathfrak{N}_b(\rho)$, $x \in [0,L_j]$, $t \in \mathbb{R}$, and $\alpha_r \in L^\infty(\mathbb{R},[0,1])$, $r \in [\![1,N_d]\!]$, we have*

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| \le Ce^{-\gamma|\mathfrak{n}|_{\ell^1}}. \tag{4.10}$$

Proof. Let $\nu \in (0,1)$ be as in Lemma 4.7. According to Lemma D.1 in appendix, there exist $\rho \in (0,1/2)$, $C,\gamma > 0$ such that for every $n \in \mathbb{N}$ and $k \in [\![0,\rho n]\!]$, we have $\binom{n}{k}\nu^n \le Ce^{-\gamma n}$. Take $i,j \in [\![1,N]\!]$, $\mathfrak{n} \in \mathfrak{N}_b(\rho)$, $x \in [0,L_j]$, and $\alpha_r \in L^\infty(\mathbb{R},[0,1])$ for $r \in [\![1,N_d]\!]$. Since $\mathfrak{n} \in \mathfrak{N}_b(\rho)$, there exists $k \in [\![1,N]\!]$ such that $\mathfrak{n} \in \mathfrak{N}_b(\rho,k)$, i.e., $n_k \le \rho\, |\mathfrak{n}|_{\ell^1}$. Then, by Lemmas 4.7 and D.1, we have

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| \le \binom{|\mathfrak{n}|_{\ell^1}}{n_k} \nu^{|\mathfrak{n}|_{\ell^1}} \le Ce^{-\gamma|\mathfrak{n}|_{\ell^1}}.$$

■

Remark 4.10. The above estimate is actually sufficient to derive the conclusion of Theorem 1.2 when the damping is always active; indeed, in this case, one can easily deduce by an inductive argument using (3.8) that

$$\vartheta^{(i)}_{j,\mathfrak{n},L_j,t} = \beta^{(i)}_{j,\mathfrak{n}} e^{-n_1(b_1-a_1)} e^{-n_2(b_2-a_2)} \dots e^{-n_{N_d}(b_{N_d}-a_{N_d})}$$

and the exponential decay in $\mathfrak{N}_c(\rho)$ follows straightforwardly. Notice that in this case Hypothesis 2.9 is not necessary.

4.4 Exponential decay of $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ in $\mathfrak{N}_c(\rho)$

In this section, we establish the exponential decay of $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ in the set $\mathfrak{N}_c(\rho)$. The main difficulty in proving it lies in the fact that $\alpha_i(t)$ can be equal to zero for certain time intervals, so that the term $\varepsilon_{j,\mathfrak{n},x,t}$ defined by (3.8b) can be equal to 1. Recall that

$$\varepsilon_{j,\mathfrak{n},0,t} = e^{-\int_{t-L(\mathfrak{n})-L_j+a_j}^{t-L(\mathfrak{n})-L_j+b_j} \alpha_j(s)ds}. \tag{4.11}$$

Our goal consists in showing in Lemma 4.15 that $\varepsilon_{j,\mathfrak{n},0,t}$ is smaller than a certain value "often enough". The first step in this direction is the following lemma.

Lemma 4.11. *Let $T \geq \mu > 0$ and $j \in [\![1,N_d]\!]$. For $\rho > 0$ and $\alpha \in \mathcal{G}(T,\mu)$, define*

$$\mathcal{I}_{j,\rho,\alpha} = \left\{ \tau \in \mathbb{R} \,\middle|\, \int_{\tau+a_j}^{\tau+b_j} \alpha(s)ds \geq \rho \right\}. \tag{4.12}$$

There exist $\rho_j > 0$ and $\ell_j > 0$, depending only on μ, T and $b_j - a_j$, such that, for every $t \in \mathbb{R}$ and $\alpha \in \mathcal{G}(T,\mu)$, $\mathcal{I}_{j,\rho_j,\alpha} \cap [t,t+T]$ contains an interval of length ℓ_j.

Proof. We set $\rho_j = \frac{\mu(b_j-a_j)}{2T}$, $\ell_j = \min\{\rho_j, T\}$. Take $\alpha \in \mathcal{G}(T,\mu)$ and define the function $A : \mathbb{R} \to \mathbb{R}$ by

$$A(\tau) = \int_{\tau+a_j}^{\tau+b_j} \alpha(s)ds.$$

Since $\alpha \in L^\infty(\mathbb{R},[0,1])$, A is 1-Lipschitz continuous. We also have, for every $t \in \mathbb{R}$,

$$\int_t^{t+T} A(\tau)d\tau = \int_t^{t+T} \int_{a_j}^{b_j} \alpha(s+\tau)dsd\tau = \int_{a_j}^{b_j} \int_{s+t}^{s+t+T} \alpha(\tau)d\tau ds \geq \mu(b_j - a_j). \tag{4.13}$$

Take $t \in \mathbb{R}$. There exists $t_\star \in [t,t+T]$ such that $A(t_\star) \geq \frac{\mu(b_j-a_j)}{T} = 2\rho_j$, for otherwise (4.13) would not be satisfied. Since A is 1-Lipschitz continous, we have $A(\tau) \geq \rho_j$ for $\tau \in [t_\star - \rho_j, t_\star + \rho_j]$, and thus

$$\left[t_\star - \rho_j, t_\star + \rho_j\right] \cap [t,t+T] \subset \mathcal{I}_{j,\rho_j,\alpha} \cap [t,t+T].$$

But, since $t_\star \in [t,t+T]$, $[t_\star - \rho_j, t_\star + \rho_j] \cap [t,t+T]$ is an interval of length at least ℓ_j, which concludes the proof. ■

Lemma 4.11 translates the persistence of excitation of α into a property on the integrals appearing in (4.11).

As remarked in Section 2.3, one cannot expect to obtain a general result concerning the exponential stability of (2.3) without taking into account the rationality of the ratios L_i/L_j. The following lemma uses the irrationality of L_i/L_j for certain $i,j \in [\![1,N]\!]$ to give a further step into the understanding of $\varepsilon_{j,\mathfrak{n},0,t}$.

Lemma 4.12. *Let $T \geq \mu > 0$ and let $\rho_j > 0$, $j \in [\![1,N_d]\!]$, be as in Lemma 4.11. There exists $K \in \mathbb{N}$ such that, for every $k_1 \in [\![1,N]\!]$, $k_2 \in [\![1,N_d]\!]$ with $L_{k_1}/L_{k_2} \notin \mathbb{Q}$, $t \in \mathbb{R}$, $\mathfrak{n} \in \mathfrak{N}$, and $\alpha \in \mathcal{G}(T,\mu)$, there exists $\mathfrak{r} \in \mathfrak{N}$ with $n_j \leq r_j \leq K + n_j$, $j \in \{k_1,k_2\}$, and $r_j = n_j$ for $j \in [\![1,N]\!] \setminus \{k_1,k_2\}$, such that*

$$t - L(\mathfrak{r}) \in \mathcal{I}_{k_2,\rho_{k_2},\alpha}.$$

Proof. We shall prove the following simpler statement: for every $k_1 \in [\![1,N]\!]$ and $k_2 \in [\![1,N_d]\!]$ with $L_{k_1}/L_{k_2} \notin \mathbb{Q}$, there exist $N_1 = N_1(k_1,k_2) \in \mathbb{N}$ and $N_2 = N_2(k_1,k_2) \in \mathbb{N}$ such that, for every $t \in \mathbb{R}$, $\mathfrak{n} \in \mathfrak{N}$, and $\alpha \in \mathcal{G}(T,\mu)$, there exist $\mathfrak{r} \in \mathfrak{N}$ with $n_{k_j} \leq r_{k_j} \leq N_j + n_{k_j}$, $j \in \{1,2\}$, and $r_j = n_j$ for $j \in [\![1,N]\!] \setminus \{k_1,k_2\}$, such that

$$t - L(\mathfrak{r}) \in \mathcal{I}_{k_2,\rho_{k_2},\alpha}.$$

From this result, one can easily obtain the statement of the lemma by taking

$$K = \max\left\{ N_1(k_1,k_2),\, N_2(k_1,k_2) \,\middle|\, k_1 \in [\![1,N]\!],\, k_2 \in [\![1,N_d]\!] \text{ such that } \frac{L_{k_1}}{L_{k_2}} \notin \mathbb{Q} \right\}.$$

We decompose the argument into two steps.

Step 1. *Definition of the points x_j and y_j.*

Let $\rho_{k_2} > 0$ and $\ell_{k_2} > 0$ be obtained from μ, T and $b_{k_2} - a_{k_2}$ as in Lemma 4.11. Let $\kappa = 3\lceil T/\ell_{k_2}\rceil$ and set

$$x_j = \frac{j}{\kappa}T, \qquad j \in [\![0,\kappa]\!],$$

which satisfy $x_j - x_{j-1} = \frac{\kappa}{T} \le \frac{\ell_{k_2}}{3}$ for $j \in [\![1,\kappa]\!]$. Hence, for every interval J of length ℓ_{k_2} contained in $[0,T]$, there exists $j \in [\![1,\kappa]\!]$ such that $x_{j-1}, x_j \in J$.

We now construct intermediate points between the x_j, $j \in [\![0,\kappa]\!]$. Since $L_{k_1}/L_{k_2} \notin \mathbb{Q}$, the set

$$\{n_1 L_{k_1} + n_2 L_{k_2} \,|\, n_1, n_2 \in \mathbb{Z}\} \tag{4.14}$$

is dense in $\mathbb{R}$. Hence we can find $n_{1,j}, n_{2,j} \in \mathbb{Z}$, $j \in [\![1,\kappa]\!]$, such that the numbers $y_j = n_{1,j}L_{k_1} + n_{2,j}L_{k_2}$ satisfy

$$0 = x_0 < y_1 < x_1 < y_2 < x_2 < \cdots < y_\kappa < x_\kappa = T. \tag{4.15}$$

As a consequence, for any interval J of length ℓ_{k_2} contained in $[0,T]$, there exists $j \in [\![1,\kappa]\!]$ such that $y_j \in J$.

Step 2. *Characterization of $\mathfrak{r} \in \mathfrak{N}$ and conclusion.*

Let $N_1^\star = \max\{|n_{1,1}|, \dots, |n_{1,\kappa}|\}$, $N_2^\star = \max\{|n_{2,1}|, \dots, |n_{2,\kappa}|\}$ and $N_1 = 2N_1^\star$, $N_2 = 2N_2^\star$. Take $t \in \mathbb{R}$ and $\mathfrak{n} \in \mathfrak{N}$. For $j \in [\![1,\kappa]\!]$, define $\mathfrak{r}_j = (r_{1,j}, \dots, r_{N,j}) \in \mathfrak{N}$ by

$$r_{k_1,j} = n_{k_1} + n_{1,j} + N_1^\star, \qquad r_{k_2,j} = n_{k_2} + n_{2,j} + N_2^\star,$$

and $r_{i,j} = n_i$ for $i \in [\![1,N]\!] \setminus \{k_1,k_2\}$; it is clear, by this definition, that $n_{k_i} \le r_{k_i,j} \le N_i + n_{k_i}$ for $i \in \{1,2\}$ and $j \in [\![1,\kappa]\!]$. Set

$$z_j = t - L(\mathfrak{r}_j), \qquad j \in [\![1,\kappa]\!];$$

we thus have

$$z_j = t - n_{1,j}L_{k_1} - n_{2,j}L_{k_2} - Z^\star = t - Z^\star - y_j$$

with $Z^\star = L(\mathfrak{n}) + N_1^\star L_{k_1} + N_2^\star L_{k_2}$. Since, by construction, $y_j \in (0,T)$ for $j \in [\![1,\kappa]\!]$, we have $z_j \in [t - Z^\star - T, t - Z^\star]$.

Take $\alpha \in \mathcal{G}(T,\mu)$. By Lemma 4.11, $\mathcal{I}_{k_2,\rho_{k_2},\alpha} \cap [t - Z^\star - T, t - Z^\star]$ contains an interval J of length ℓ_{k_2}. Consider the interval $J' = -J + t - Z^\star$, which is a subinterval of $[0,T]$ of length ℓ_{k_2}. By Step 1, there exists $j \in [\![1,\kappa]\!]$ such that $y_j \in J'$, and thus $z_j \in J \subset \mathcal{I}_{k_2,\rho_{k_2},\alpha}$. Since $z_j = t - L(\mathfrak{r}_j)$, we obtain the desired result with $\mathfrak{r} = \mathfrak{r}_j$. ■

Remark 4.13. The only instance in the proof of Lemma 4.12 where we use the fact that $L_{k_1}/L_{k_2} \notin \mathbb{Q}$ is when we establish the existence of numbers y_j, $j = 1, \dots, \kappa$, of the form $y_j = n_{1,j}L_{k_1} + n_{2,j}L_{k_2}$ with $n_{1,j}, n_{2,j} \in \mathbb{Z}$ satisfying (4.15), which we do by using the density of the set (4.14). When $L_{k_1}/L_{k_2} \in \mathbb{Q}$ and we write $L_{k_1}/L_{k_2} = p/q$ for coprime $p, q \in \mathbb{N}^*$, the set given in (4.14) is, by Bézout's Lemma,

$$\left\{ L_{k_2}\frac{n_1 p + n_2 q}{q} \,\middle|\, n_1, n_2 \in \mathbb{Z} \right\} = \left\{ k\frac{L_{k_2}}{q} \,\middle|\, k \in \mathbb{Z} \right\}.$$

Hence the construction of $y_j = n_{1,j}L_{k_1} + n_{2,j}L_{k_2}$ with $n_{1,j}, n_{2,j} \in \mathbb{Z}$ satisfying (4.15) is still possible if $L_{k_2}/q < \kappa/T$, i.e., if $q > L_{k_2}\kappa/T$, and thus Lemma 4.12 still holds true if $L_{k_1}/L_{k_2} = p/q$ with coprime $p, q \in \mathbb{N}$ and q large enough.

Recalling that $\kappa = 3\lceil T/\ell_{k_2}\rceil$ with $\ell_{k_2} = \min\left\{\frac{\mu(b_{k_2}-a_{k_2})}{2T}, T\right\}$, we can even give a more explicit sufficient condition on q to still have Lemma 4.12: if

$$q \geq 3L_{k_2}\left(\max\left\{\frac{2T}{\mu(b_{k_2}-a_{k_2})}, \frac{1}{T}\right\} + \frac{1}{T}\right),$$

then one can easily check that $q > L_{k_2}\kappa/T$ and hence we are in the previous situation.

More explicitly, we can replace Hypothesis 2.9 by the following one.

Hypothesis 4.14. There exist $i \in [\![1,N]\!]$ and $j \in [\![1,N_d]\!]$ for which we have either $L_i/L_j \notin \mathbb{Q}$ or $L_i/L_j = p/q$ with coprime $p,q \in \mathbb{N}^*$ satisfying

$$q \geq 3L_j\left(\max\left\{\frac{2T}{\mu(b_j-a_j)}, \frac{1}{T}\right\} + \frac{1}{T}\right). \tag{4.16}$$

Notice that condition (4.16) only depends on the constants T, μ of the PE condition, on the length $b_j - a_j$ of the damping interval j and on the length L_j.

As a consequence of the previous lemma we deduce the following property.

Lemma 4.15. *Let $T \geq \mu > 0$. There exist $\eta \in (0,1)$ and $K \in \mathbb{N}$ such that, for every $k_1 \in [\![1,N]\!]$, $k_2 \in [\![1,N_d]\!]$ with $L_{k_1}/L_{k_2} \notin \mathbb{Q}$, $t \in \mathbb{R}$, $\mathfrak{n} \in \mathfrak{N}$, and $\alpha_{k_2} \in \mathcal{G}(T,\mu)$, there exists $\mathfrak{r} \in \mathfrak{N}$ with $n_j \leq r_j \leq K + n_j$, $j \in \{k_1,k_2\}$, and $r_j = n_j$ for $j \in [\![1,N]\!] \setminus \{k_1,k_2\}$, such that*

$$\varepsilon_{k_2,\mathfrak{r},0,t} \leq \eta.$$

Proof. Take $\rho_j > 0$, $j \in [\![1,N_d]\!]$, as in Lemma 4.11 and $K \in \mathbb{N}$ as in Lemma 4.12. Define $\eta = \max_{j\in[\![1,N_d]\!]} e^{-\rho_j} \in (0,1)$. Take $k_1 \in [\![1,N]\!]$ and $k_2 \in [\![1,N_d]\!]$ with $L_{k_1}/L_{k_2} \notin \mathbb{Q}$. Let $t \in \mathbb{R}$, $\mathfrak{n} \in \mathfrak{N}$, and $\alpha_{k_2} \in \mathcal{G}(T,\mu)$. Applying Lemma 4.12 at $t - L_{k_2}$, we deduce the existence of $\mathfrak{r} \in \mathfrak{N}$ such that

$$t - L_{k_2} - L(\mathfrak{r}) \in \mathcal{I}_{k_2,\rho_{k_2},\alpha_{k_2}}.$$

By the definition (4.12) of $\mathcal{I}_{k_2,\rho_{k_2},\alpha_{k_2}}$, this means that

$$\int_{t-L(\mathfrak{r})-L_{k_2}+a_{k_2}}^{t-L(\mathfrak{r})-L_{k_2}+b_{k_2}} \alpha_{k_2}(s)ds \geq \rho_{k_2}.$$

By (4.11), we thus obtain that $\varepsilon_{k_2,\mathfrak{r},0,t} \leq e^{-\rho_{k_2}} \leq \eta$. ■

Remark 4.16. Notice that the hypothesis that $L_{k_1}/L_{k_2} \notin \mathbb{Q}$ is only used to apply Lemma 4.12, and thus, by Remark 4.13, Lemma 4.15 still holds true if k_1 and k_2 are chosen as in Hypothesis 4.14.

Remark 4.17. Figure 4.1 helps illustrating Lemma 4.15 in the case $N = 2$. Taking $N_d = 1$, the lemma states that there exists $K \in \mathbb{N}$ ($K = 7$ in the picture) such that every rectangle of size $(K+1)L_{k_1} \times (K+1)L_{k_2}$ that we place in the grid pictured in Figure 4.1 contains at least one horizontal segment (highlighted in the picture) where $\varepsilon_{k_2,\mathfrak{r},0,t} \leq \eta$. Let us remark that K and η do not depend on $\alpha_1 \in \mathcal{G}(T,\mu)$: hence the position of the highlighted segments may change if we change the persistently exciting signal, but we can guarantee that on every rectangle there exists at least one such segment.

We now apply Lemma 4.15 to obtain the following property, which is a preliminary step towards the exponential decay of $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ in $\mathfrak{N}_c(\rho)$.

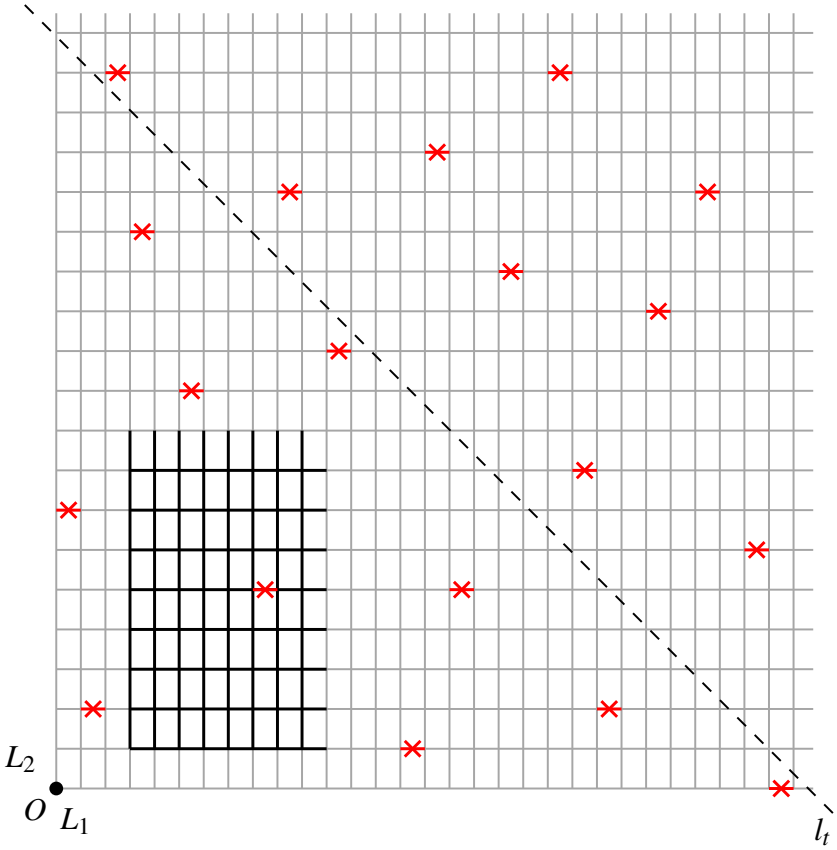

FIGURE 4.1: *Interpretation of Lemma 4.15 in the case $N = 2$ and $N_d = 1$.*

Lemma 4.18. *Suppose that Hypotheses 2.9 and 2.10 are satisfied. Let $T \geq \mu > 0$ and $\rho \in (0,1)$. Then there exist $\lambda \in (0,1)$ and $K \in \mathbb{N}^*$ such that, for every $\mathfrak{n} \in \mathfrak{N}_c(\rho)$ with $\min_{i\in[\![1,N]\!]} n_i \geq K$, $i, j \in [\![1,N]\!]$, $t \in \mathbb{R}$, and $\alpha_k \in \mathcal{G}(T,\mu)$, $k \in [\![1,N_d]\!]$, we have*

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}\right| \leq \lambda \max_{\substack{\mathfrak{p}\in\mathfrak{N} \,|\, |\mathfrak{p}|_{\ell^1}=K \\ p_r\leq n_r,\, \forall r\in[\![1,N]\!] \\ s\in[\![1,N]\!] \,|\, p_s>0}} \left|\vartheta^{(i)}_{s,\mathfrak{n}-\mathfrak{p},L_s,t}\right|.$$

Proof. Let $\eta \in (0,1)$ and $K_0 \in \mathbb{N}^*$ be as in Lemma 4.15. Let $L_{\min} = \min_{i\in[\![1,N]\!]} L_i$. We take $K = 2K_0 + 1$.

Take $\mathfrak{n} \in \mathfrak{N}_c(\rho)$ with $\min_{i\in[\![1,N]\!]} n_i \geq K$ and let $k_1 \in [\![1,N]\!]$, $k_2 \in [\![1,N_d]\!]$ be such that $L_{k_1}/L_{k_2} \notin \mathbb{Q}$. Since $\mathfrak{n} \in \mathfrak{N}_c(\rho)$, one has $n_{k_i} > \rho\, |\mathfrak{n}|_{\ell^1}$ for $i \in \{1,2\}$. Take $i, j \in [\![1,N]\!]$, $t \in \mathbb{R}$ and $\alpha_k \in \mathcal{G}(T,\mu)$ for $k \in [\![1,N_d]\!]$. Since $|\mathfrak{n}|_{\ell^1} \geq \min_{i\in[\![1,N]\!]} n_i \geq K$, we can apply Proposition 3.9 and deduce from (3.9) the estimate

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}\right| \leq \Theta \sum_{\nu\in\Phi_K(\mathfrak{n})} \left[\left(\left|m_{\nu_1 j}\right| \prod_{s=2}^{K} \left|m_{\nu_s\nu_{s-1}}\right|\right)\left(\prod_{s=1}^{K} \varepsilon_{\nu_s,\mathfrak{n}-\sum_{r=1}^{s}\mathbf{1}_{\nu_r},0,t}\right)\right], \tag{4.17}$$

where

$$\Theta = \max_{\substack{\mathfrak{p}\in\mathfrak{N} \,|\, |\mathfrak{p}|_{\ell^1}=K \\ p_r\leq n_r,\, \forall r\in[\![1,N]\!] \\ s\in[\![1,N]\!] \,|\, p_s>0}} \left|\vartheta^{(i)}_{s,\mathfrak{n}-\mathfrak{p},L_s,t}\right|.$$

Let us now apply Lemma 4.15 to the point $\mathfrak{n}' = \mathfrak{n} - K_0\mathbf{1}_{k_1} - (K_0+1)\mathbf{1}_{k_2}$. Notice that $\mathfrak{n}' \in \mathfrak{N}$ since $n_{k_1}, n_{k_2} > K_0$. Hence there exists $\mathfrak{r} \in \mathfrak{N}$ with

$$\begin{aligned} n_{k_1} - K_0 \le r_{k_1} \le n_{k_1}, \\ n_{k_2} - K_0 - 1 \le r_{k_2} \le n_{k_2} - 1, \\ r_j = n_j \text{ for } j \in [\![1,N]\!] \setminus \{k_1,k_2\}, \end{aligned}$$

such that $\varepsilon_{k_2,\mathfrak{r},0,t} \le \eta$.

We next show that there exists $v_0 = (v_{0,1},\dots,v_{0,K}) \in \Phi_K(\mathfrak{n})$ and $s_0 \in [\![1,K]\!]$ such that $v_{0,s_0} = k_2$ and $\mathfrak{n} - \sum_{r=1}^{s_0} \mathbf{1}_{v_{0,r}} = \mathfrak{r}$. For that purpose, take $v_0 \in [\![1,N]\!]^K$ with $v_{0,1} = v_{0,2} = \cdots = v_{0,n_{k_1}-r_{k_1}} = k_1$ and $v_{0,n_{k_1}-r_{k_1}+1} = v_{0,n_{k_1}-r_{k_1}+2} = \cdots = v_{0,K} = k_2$. Such a v_0 is well-defined in $\Phi_K(\mathfrak{n})$ since $0 \le n_{k_1} - r_{k_1} \le K_0 < K$. By construction, $\varphi_{k_1,K}(v_0) = n_{k_1} - r_{k_1} \le n_{k_1}$, $\varphi_{k_2,K}(v_0) = K - (n_{k_1} - r_{k_1}) \le K \le n_{k_2}$ and $\varphi_{k,K}(v_0) = 0 \le n_k$ for $k \in [\![1,N]\!] \setminus \{k_1,k_2\}$. Hence v_0 is in $\Phi_K(\mathfrak{n})$. Taking $s_0 = n_{k_1} - r_{k_1} + n_{k_2} - r_{k_2} \in [\![1,K]\!]$, we have $v_{0,s_0} = k_2$ and $\mathfrak{n} - \sum_{r=1}^{s_0} \mathbf{1}_{v_{0,r}} = \mathfrak{r}$.

Let $\delta = \min_{i,j\in[\![1,N]\!]} |m_{ij}| > 0$ and $\lambda = 1 - \delta^K(1-\eta)$. One clearly has that λ is in $(0,1)$, since $\eta, \delta \in (0,1)$. Using (3.10), we get from (4.17) that

$$\begin{aligned} \left|\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}\right| &\le \Theta \Bigg[\left(|m_{v_{0,1}j}| \prod_{s=2}^{K} |m_{v_{0,s}v_{0,s-1}}| \right) \left(\varepsilon_{v_{0,s_0},\mathfrak{n}-\sum_{r=1}^{s_0}\mathbf{1}_{v_{0,r}},0,t} \prod_{\substack{s=1\\ s\neq s_0}}^{K} \varepsilon_{v_{0,s},\mathfrak{n}-\sum_{r=1}^{s}\mathbf{1}_{v_{0,r}},0,t} \right) \\ &\quad + \sum_{v\in\Phi_K(\mathfrak{n})\setminus\{v_0\}} \left(|m_{v_1 j}| \prod_{s=2}^{K} |m_{v_s v_{s-1}}| \right) \left(\prod_{s=1}^{K} \varepsilon_{v_s,\mathfrak{n}-\sum_{r=1}^{s}\mathbf{1}_{v_r},0,t} \right) \Bigg] \\ &\le \Theta \left[\left(|m_{v_{0,1}j}| \prod_{s=2}^{K} |m_{v_{0,s}v_{0,s-1}}| \right) \eta + \sum_{v\in\Phi_K(\mathfrak{n})\setminus\{v_0\}} \left(|m_{v_1 j}| \prod_{s=2}^{K} |m_{v_s v_{s-1}}| \right) \right] \\ &= \Theta \left[\left(|m_{v_{0,1}j}| \prod_{s=2}^{K} |m_{v_{0,s}v_{0,s-1}}| \right) (\eta - 1) + \sum_{v\in\Phi_K(\mathfrak{n})} \left(|m_{v_1 j}| \prod_{s=2}^{K} |m_{v_s v_{s-1}}| \right) \right] \\ &\le \Theta \left[\delta^K(\eta-1) + \sum_{v\in\Phi_K(\mathfrak{n})} \left(|m_{v_1 j}| \prod_{s=2}^{K} |m_{v_s v_{s-1}}| \right) \right] \le \Theta\left[1 - \delta^K(1-\eta)\right] = \lambda\Theta. \end{aligned}$$

■

We now obtain the exponential decay of $\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}$ in the set $\mathfrak{N}_c(\sigma)$.

Theorem 4.19. *Suppose that Hypotheses 2.9 and 2.10 are satisfied. Let $T \ge \mu > 0$ and $\sigma \in (0,1)$. Then there exist $C \ge 1$, $\gamma > 0$ and $K \in \mathbb{N}^*$ such that, for every $\mathfrak{n} \in \mathfrak{N}_c(\sigma)$ with $\min_{i\in[\![1,N]\!]} n_i \ge K$, $i,j \in [\![1,N]\!]$, $x \in [0,L_j]$, $t \in \mathbb{R}$, and $\alpha_k \in \mathcal{G}(T,\mu)$, $k \in [\![1,N_d]\!]$, we have*

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| \le Ce^{-\gamma|\mathfrak{n}|_{\ell^1}}. \tag{4.18}$$

Proof. Take $\rho = \sigma/2$ and let $\lambda \in (0,1)$ and $K \in \mathbb{N}^*$ be as in Lemma 4.18. For $\mathfrak{n} \in \mathfrak{N}_c(\rho)$ with $\min_{i\in[\![1,N]\!]} n_i \ge K$, we set

$$q_{\max}(\mathfrak{n}) = \max\left\{ q \in \mathbb{N} \mid \mathfrak{n} - \mathfrak{r} \in \mathfrak{N}_c(\rho) \text{ for every } \mathfrak{r} \in \mathfrak{N} \text{ with } |\mathfrak{r}|_{\ell^1} = qK \right\}.$$

From Lemma 4.18, one deduces by an immediate inductive argument that for every $q \in [\![1, q_{\max}(\mathfrak{n})]\!]$, $i,j \in [\![1,N]\!]$, $t \in \mathbb{R}$, and $\alpha_k \in \mathcal{G}(T,\mu)$, $k \in [\![1,N_d]\!]$, we have

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}\right| \le \lambda^q \max_{\substack{\mathfrak{p}\in\mathfrak{N} \mid |\mathfrak{p}|_{\ell^1}=qK \\ p_r\le n_r,\ \forall r\in[\![1,N]\!] \\ s\in[\![1,N]\!] \mid p_s>0}} \left|\vartheta^{(i)}_{s,\mathfrak{n}-\mathfrak{p},L_s,t}\right|.$$

By Corollary 4.6, it holds $\left|\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}\right| \leq 1$ for every $i,j \in [\![1,N]\!]$, $\mathfrak{n} \in \mathfrak{N}$ and $t \in \mathbb{R}$. Therefore, for every $\mathfrak{n} \in \mathfrak{N}_c(\rho)$ with $\min_{i\in[\![1,N]\!]} \mathfrak{n}_i \geq K$, $i,j \in [\![1,N]\!]$, $t \in \mathbb{R}$ and $\alpha_k \in \mathcal{G}(T,\mu)$, $k \in [\![1,N_d]\!]$, we obtain

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}\right| \leq \lambda^{q_{\max}(\mathfrak{n})}. \tag{4.19}$$

Notice now that, by definition of $q_{\max}$, one also has that

$$q_{\max}(\mathfrak{n}) + 1 \geq \frac{1}{K} \min\{|\mathfrak{n} - \mathfrak{r}|_{\ell^1} \mid \mathfrak{r} \in \mathfrak{N}_b(\rho)\},$$

where, according to Definition 4.8, $\mathfrak{N}_b(\rho) = \mathfrak{N} \setminus \mathfrak{N}_c(\rho)$. One deduces at once that there exists $\xi > 0$ such that, for every $\mathfrak{n} \in \mathfrak{N}_c(\sigma)$,

$$q_{\max}(\mathfrak{n}) + 1 \geq \xi\, |\mathfrak{n}|_{\ell^1}. \tag{4.20}$$

Since $\lambda \in (0,1)$, setting $\gamma = -\xi \log \lambda > 0$ and $C = 1/\lambda$ one concludes by inserting (4.20) into (4.19) and then using (3.8a). ■

4.5 Exponential convergence of the solutions

To conclude the proof of Theorem 1.2, it suffices to combine Proposition 4.1 with the estimates of $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ for $\mathfrak{n} \in \mathfrak{N}_b(\sigma)$ and $\mathfrak{n} \in \mathfrak{N}_c(\rho)$ given by Theorems 4.9 and 4.19, respectively.

Proof of Theorem 1.2. Let $C_1, \gamma_1 > 0$ and $\rho > 0$ be as in Theorem 4.9. Take $\sigma = \rho$ in Theorem 4.19 and let $C_2, \gamma_2 > 0$ and $K \in \mathbb{N}^*$ be as in that theorem. Let

$$\gamma = \min\{\gamma_1, \gamma_2\}, \qquad C = \max\{C_1, C_2, e^{\gamma K/\rho}\}.$$

Take $i,j \in [\![1,N]\!]$, $\mathfrak{n} \in \mathfrak{N}$, $x \in [0,L_j]$, $t \in \mathbb{R}$ and $\alpha_k \in \mathcal{G}(T,\mu)$, $k \in [\![1,N_d]\!]$. If $\mathfrak{n} \in \mathfrak{N}_b(\rho)$ or $\mathfrak{n} \in \mathfrak{N}_c(\rho)$ with $\min_{i\in[\![1,N]\!]} n_i \geq K$, then one concludes directly from Theorems 4.9 and 4.19. Finally, if $\mathfrak{n} \in \mathfrak{N}_c(\rho)$ with $\min_{i\in[\![1,N]\!]} n_i < K$, note that $|\mathfrak{n}|_{\ell^1} \leq K/\rho$. Then, by Corollary 4.6, one has

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| \leq 1 \leq Ce^{-\gamma K/\rho} \leq Ce^{-\gamma|\mathfrak{n}|_{\ell^1}}.$$

Theorem 1.2 now follows from Proposition 4.1. ■

Remark 4.20. By Remark 4.13, Hypothesis 2.9 can be replaced by Hypothesis 4.14 in Lemma 4.18 and Theorem 4.19, and so the same is also true for Theorem 1.2. We recall that the case $p = +\infty$ also follows from Proposition 4.1 thanks to Remark 4.3.

A Well-posedness of the Cauchy problems (2.5), (2.6) and (2.7)

We are interested in this section in the proof of Theorem 2.4, which states the well-posedness of the Cauchy problem (2.5). This is done in two steps. First, we show that the operator A defined in (2.4) is the generator of a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$, thus establishing the well-posedness of the undamped system. We then consider the operator $B(t) = \sum_{i=1}^{N_d} \alpha_i(t) B_i$ as a bounded time-dependent perturbation of A in order to conclude the well-posedness of (2.5).

A.1 Preliminaries

Let X be a Banach space and let $A : D(A) \subset \mathsf{X} \to \mathsf{X}$ be an operator in X. The definitions of strong and weak solutions for the Cauchy problem associated with A can be found for instance in [30]. Recall that if A is densely defined with a non-empty resolvent set, then the Cauchy problem associated with A has a unique strong solution for each initial condition $z_0 \in D(A)$ if and only if A is the generator of a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$. In this case, the solution is given by $z(t) = e^{tA}z_0$ (see, for instance, [30, Chapter 4, Theorem 1.3]). Then, $t \mapsto e^{tA}z_0$ is a well-defined continuous function for every $z_0 \in \mathsf{X}$ and it is the unique weak solution of the Cauchy problem associated with A.

Definition A.1. A family of operators $\{T(t,s)\}_{t\geq s\geq 0} \subset \mathcal{L}(\mathsf{X})$ is an *evolution family* on X if

- **i.** $T(s,s) = \mathrm{Id}_{\mathsf{X}}$ for every $s \geq 0$,
- **ii.** $T(t,s) = T(t,\tau)T(\tau,s)$ for every $t \geq \tau \geq s \geq 0$,
- **iii.** for every $z \in \mathsf{X}$, $(t,s) \mapsto T(t,s)z$ is continuous for every $t \geq s \geq 0$.

An evolution family is *exponentially bounded* if it satisfies further the following property.

- **iv.** There exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t,s)\|_{\mathcal{L}(\mathsf{X})} \leq Me^{\omega(t-s)}$ for every $t \geq s \geq 0$.

For references on evolution families see, for instance, [10, 22, 31]. We are interested here in a family of the form $A(t) = A + B(t)$ where A is the generator of a strongly continuous semigroup and $B \in L^\infty(\mathbb{R}_+, \mathcal{L}(\mathsf{X}))$. We shall use here the following notions of solution.

Definition A.2. Consider the problem

$$\begin{cases} \dot{z}(t) = (A + B(t))z(t), & t \geq s \geq 0, \\ z(s) = z_0, \end{cases} \tag{A.1}$$

where A is the generator of a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ and $B \in L^\infty(\mathbb{R}_+, \mathcal{L}(\mathsf{X}))$.

- **i.** We say that $z : [s,+\infty) \to \mathsf{X}$ is a *regular solution* of (A.1) if z is continuous, $z(s) = z_0$, $z(t) \in D(A)$ for every $t \geq s$, z is differentiable for almost every $t \geq s$, $\dot{z} \in L^\infty_{\mathrm{loc}}([s,+\infty), \mathsf{X})$ and $\dot{z}(t) = (A + B(t))z(t)$ for almost every $t \geq s$.
- **ii.** We say that $z : [s,+\infty) \to \mathsf{X}$ is a *mild solution* of (A.1) if z is continuous and, for every $t \geq s$, we have
$$z(t) = e^{(t-s)A}z_0 + \int_s^t e^{(t-\tau)A}B(\tau)z(\tau)d\tau.$$

Here, the integrals of X-valued functions should be understood as Bochner integrals; see, for instance, [36]. In the following proposition, we summarize the main facts needed for the present paper.

Proposition A.3. *Let A be the generator of a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ and $B \in L^\infty(\mathbb{R}_+, \mathcal{L}(\mathsf{X}))$. Then, the following holds true:*

- **i.** *every regular solution of* (A.1) *is also a mild solution;*
- **ii.** *there exists a unique family $\{T(t,s)\}_{t\geq s\geq 0}$ of bounded operators in X such that $(t,s) \mapsto T(t,s)z$ is continuous for every $z \in \mathsf{X}$ and*
$$T(t,s)z = e^{(t-s)A}z + \int_s^t e^{(t-\tau)A}B(\tau)T(\tau,s)zd\tau, \qquad \forall z \in \mathsf{X}. \tag{A.2}$$
Furthermore, this family is an exponentially bounded evolution family;
- **iii.** *for every $z_0 \in \mathsf{X}$,* (A.1) *admits a unique mild solution z, given by $z(t) = T(t,s)z_0$.*

A.2 Proof of Theorem 2.5

Since $\sum_{i=1}^{N_d} B_i$ is a bounded operator, it suffices to show Theorem 2.5 for A.

Proposition A.4. *Let $p \in [1,+\infty)$. The operator A is closed and densely defined. Moreover, $D(A)$ endowed with the graph norm is a Banach space compactly embedded in X_p.*

Proof. The proposition follows straightforwardly by the remark that the graph norm on $D(A)$ coincides with the usual norm in $\prod_{i=1}^{N} W^{1,p}(0,L_i)$, that is,

$$\|z\|_{D(A)}^p = \sum_{i=1}^{N} \left(\|u_i\|_{L^p(0,L_i)}^p + \|u_i'\|_{L^p(0,L_i)}^p \right)$$

for $z = (u_1, \dots, u_N) \in D(A)$. ■

Proposition A.5. *Let $p \in [1,+\infty)$. The resolvent set $\rho(A)$ of A is nonempty.*

Proof. Since A is closed, we have $\lambda \in \rho(A)$ if and only if $\lambda - A$ is a bijection from $D(A)$ to X_p. A direct computation based on explicit formulas yields that $\lambda - A$ is a bijection as soon as $\lambda \in \mathbb{R}$ with $\lambda > \frac{\log|M|_{\ell^2}}{L_{\min}}$, where $L_{\min} = \min_{i \in [\![1,N]\!]} L_i$. ■

We now turn to a result of existence of solutions of (2.6) when $z_0 \in D(A)$.

Theorem A.6. *For every $z_0 = (u_{1,0}, \dots, u_{N,0}) \in D(A)$, there exists a unique strong solution $z = (u_1, \dots, u_N) \in \mathcal{C}^0(\mathbb{R}_+, D(A)) \cap \mathcal{C}^1(\mathbb{R}_+, \mathsf{X}_p)$ of* (2.6).

Proof. Let $T_0 > 0$ be such that $T_0 < L_{\min}$. Note that it suffices to show the theorem for solutions in $\mathcal{C}^0([0,T_0], D(A)) \cap \mathcal{C}^1([0,T_0], \mathsf{X}_p)$, since T_0 does not depend on $z_0 \in D(A)$. Let $z_0 = (u_{1,0}, \dots, u_{N,0}) \in D(A)$. It follows easily from the transport equation and the transmission condition (1.3) that a solution $t \mapsto z(t) = (u_1(t), \dots, u_N(t))$ of (2.6) necessarily satisfies

$$u_i(t,x) = \begin{cases} \displaystyle\sum_{j=1}^{N} m_{ij} u_{j,0}(L_j - t + x) & \text{if } 0 \le x \le t, \\ u_{i,0}(x-t) & \text{if } x > t. \end{cases} \tag{A.3}$$

Conversely, if $z = (u_1, \dots, u_N)$ is given by (A.3), then it solves (2.6) and has z_0 as initial condition. Moreover, one checks by direct computations that z fulfills the required regularity properties. ■

Proof of Theorem 2.5. From Propositions A.4 and A.5 and Theorem A.6, we obtain that A generates a strongly continuous semigroup $\{e^{tA}\}_{t \ge 0}$ (see, for instance, [30, Chapter 4, Theorem 1.3]). Since $\sum_{i=1}^{N_d} B_i \in \mathcal{L}(\mathsf{X}_p)$, $A + \sum_{i=1}^{N_d} B_i$ also generates a strongly continuous semigroup (see [30, Chapter 3, Theorem 1.1]). ■

Remark A.7. In the particular case $p = 2$ and $|M|_{\ell^2} \le 1$, one may conclude more easily that A is the generator of a strongly continuous semigroup $\{e^{tA}\}_{t \ge 0}$, without having to construct the explicit formula (A.3) for the solution as we did in Theorem A.6. Indeed, a straightforward computation shows that, for any $M \in \mathcal{M}_N(\mathbb{R})$, the adjoint operator A^* of A is given by

$$D(A^*) = \left\{ (u_1, \dots, u_N) \in \prod_{i=1}^{N} H^1(0,L_i) \;\middle|\; u_i(L_i) = \sum_{j=1}^{N} m_{ji} u_j(0) \right\},$$

$$A^*(u_1, \dots, u_N) = \left(\frac{du_1}{dx}, \dots, \frac{du_N}{dx} \right).$$

Also, for any $z=(u_1,\dots,u_N)\in D(A)$, we have

$$\langle z,Az\rangle = -\sum_{i=1}^N \int_0^{L_i} u_i u_i' = \frac{1}{2}\sum_{i=1}^N (u_i(0)^2 - u_i(L_i)^2) \le \frac{|M|_{\ell^2}^2 - 1}{2}\sum_{i=1}^N u_i(L_i)^2,$$

since, by (1.3), we have $\sum_{i=1}^N u_i(0)^2 \le |M|_{\ell^2}^2 \sum_{i=1}^N u_i(L_i)^2$. Thus, if $|M|_{\ell^2}\le 1$, we have $\langle z,Az\rangle \le 0$ for every $z\in D(A)$, so that A is dissipative. A similar computation holds for A^*, with M replaced by its transpose M^{T}, showing that A^* is also dissipative. Hence A generates a strongly continuous semigroup of contractions $\{e^{tA}\}_{t\ge 0}$ (see, for instance, [30, Chapter 1, Theorem 4.4]).

A.3 Proof of Theorem 2.4

Thanks to Theorem 2.5 and Items ii and iii of Proposition A.3, there exists a unique evolution family $\{T(t,s)\}_{t\ge s\ge 0}$ associated with (2.5) such that, for every $z_0\in \mathsf{X}_p$, $t\mapsto T(t,s)z_0$ is the unique mild solution of (2.5) with initial condition $z(s)=z_0$. In order to complete the proof of Theorem 2.4, it suffices to show that this solution is actually regular when $z_0\in D(A)$. To do so, we study the explicit formula for the solutions of (2.5) for small time, as we did with the undamped system (2.6) in Theorem A.6.

Proof of Theorem 2.4. Let $T_0>0$ be such that $T_0<L_{\min}$. As in Theorem A.6, it suffices to show the theorem for solutions in $\mathcal{C}^0([s,s+T_0],\mathsf{X}_p)$. Since the class $L^\infty(\mathbb{R},[0,1])$ is invariant by time-translation, we can also suppose without loss of generality that $s=0$. In order to simplify the notations, we define the function $\varphi_i:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}_+^*$ for $i\in[\![1,N]\!]$ by

$$\varphi_i(t,x) = e^{-\int_0^t \alpha_i(s)\chi_i(x-t+s)ds},$$

where we extend the function χ_i to $\mathbb{R}$ by 0 outside its interval of definition $[0,L_i]$. (In particular, $\varphi_i\equiv 1$ for $i\in[\![N_d+1,N]\!]$.) We have that both φ_i and $1/\varphi_i$ belong to $L^\infty(\mathbb{R}_+\times\mathbb{R})\cap\mathcal{C}^0(\mathbb{R}_+\times\mathbb{R})$ and $W^{1,\infty}(\mathbb{R}_+\times\mathbb{R})$.

Let $z_0=(u_{1,0},\dots,u_{N,0})\in D(A)$. We claim that, for $0\le t\le T_0$, the function $t\mapsto z(t)=(u_1(t),\dots,u_N(t))$ given by

$$u_i(t,x) = \begin{cases} \dfrac{\varphi_i(t,x)}{\varphi_i(t-x,0)}\displaystyle\sum_{j=1}^N m_{ij}\varphi_j(t-x,L_j)u_{j,0}(x+L_j-t), & \text{if } 0\le x\le t,\\ \varphi_i(t,x)u_{i,0}(x-t), & \text{if } x>t,\end{cases}$$

is in $\mathcal{C}^0([0,T_0],\mathsf{X}_p)$, $z(0)=z_0$, $z(t)\in D(A)$ for every $t\in[0,T_0]$, z is differentiable for almost every $t\in[0,T_0]$, $\dot z\in L^\infty([0,T_0],\mathsf{X}_p)$ and $\dot z(t)=Az(t)+\sum_{i=1}^{N_d}\alpha_i(t)B_iz(t)$ for almost every $t\in[0,T_0]$. Indeed, it is clear that z is well-defined, $z(0)=z_0$, and $z(t)\in\mathsf{X}_p$ for every $t\in[0,T_0]$. It is also clear, thanks to the regularity properties of φ_i, that, for every $t\in[0,T_0]$, $u_i(t)\in W^{1,p}(0,t)$ and $u_i(t)\in W^{1,p}(t,T_0)$, and, since $x\mapsto u_i(t,x)$ is continuous at $x=t$, we conclude that $u_i(t)\in W^{1,p}(0,L_i)$. Furthermore,

$$u_i(t,0) = \sum_{j=1}^N m_{ij}\varphi_j(t,L_j)u_{j,0}(L_j-t) = \sum_{j=1}^N m_{ij}u_j(t,L_j),$$

and thus $z(t)\in D(A)$ for every $t\in[0,T_0]$. By the same argument, we also obtain that $u_i(\cdot,x)\in W^{1,p}(0,T_0)$ for every $x\in[0,L_i]$. Computing $u_i(t+h,x)-u_i(t,x)$ for $t,t+h\in[0,T_0]$ also shows, by a straightforward estimate, that $\|u_i(t+h)-u_i(t)\|_{L^p(0,L_i)}\to 0$ as $h\to 0$, and thus $z\in\mathcal{C}^0([0,T_0],\mathsf{X}_p)$.

Since $u_i(\cdot,x) \in W^{1,p}(0,T_0)$ for every $x \in [0,L_i]$, one can also compute $\partial_t u_i(t,x)$, and it is easy to verify that $\partial_t u_i \in L^\infty([0,T_0],L^p(0,L_i))$. Hence z is differentiable almost everywhere, with $\dot z = (\partial_t u_1,\dots,\partial_t u_N) \in L^\infty([0,T_0],\mathsf{X}_p)$.

Notice now that $\dot z(t) - Az(t) = (\partial_t u_1(t) + \partial_x u_1(t),\dots,\partial_t u_N(t) + \partial_x u_N(t))$ is given by

$$\partial_t u_i(t,x) + \partial_x u_i(t,x) = \begin{cases} \dfrac{\partial_t \varphi_i(t,x) + \partial_x \varphi_i(t,x)}{\varphi_i(t-x,0)} \displaystyle\sum_{j=1}^N m_{ij}\varphi_j(t-x,L_j)u_{j,0}(x+L_j-t), & \\ & \text{if } 0 \le x \le t, \\ [\partial_t \varphi_i(t,x) + \partial_x \varphi_i(t,x)]u_{i,0}(x-t), & \text{if } x > t, \end{cases}$$

and one can compute that $\partial_t \varphi_i(t,x) + \partial_x \varphi_i(t,x) = -\alpha_i(t)\chi_i(x)\varphi_i(t,x)$ almost everywhere, so that

$$\partial_t u_i(t,x) + \partial_x u_i(t,x) = -\alpha_i(t)\chi_i(x)u_i(t,x).$$

Thus $\dot z(t) - Az(t) = -\sum_{i=1}^{N_d} \alpha_i(t)B_i z(t)$ for almost every $t \in [0,T_0]$, which concludes the proof of existence. Uniqueness results from the fact that every regular solution is in particular a mild solution, which is unique, according to Proposition A.3. ■

B Asymptotic behavior of (2.9)

We consider here System (2.9) from Example 2.7. Let X_2 be the Hilbert space $\mathsf{X}_2 = L^2(0,L_1) \times L^2(0,L_2)$. The goal of this section is to prove Theorem 2.8 concerning the asymptotic behavior of (2.9) when $\chi \equiv 0$, and also to show the existence of periodic solutions to (2.9) with a persistent damping when $L_1/L_2 \in \mathbb{Q}$ and $b-a$ is small enough. The proof of Theorem 2.8.i being based on LaSalle Principle, we recall its formulation in a Banach space in Section B.1.

B.1 LaSalle Principle in a Banach space

In this section, X denotes a Banach space and $A : D(A) \subset \mathsf{X} \to \mathsf{X}$ is a linear operator in X that generates a strongly continuous semigroup $\{e^{tA}\}_{t\ge 0}$.

Definition B.1. **i.** For $z_0 \in \mathsf{X}$, the *ω-limit set* $\omega(z_0)$ is the set of $z \in \mathsf{X}$ such that there exists a nondecreasing sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ with $t_n \to +\infty$ as $n \to \infty$ such that $e^{t_nA}z_0 \to z$ in X as $n \to \infty$. A set $M \subset \mathsf{X}$ is *positively invariant* under $\{e^{tA}\}_{t\ge 0}$ if, for every $z_0 \in M$ and $t \ge 0$, we have $e^{tA}z_0 \in M$. For $E \subset \mathsf{X}$, the *maximal positively invariant subset* M of E is the union of all positively invariant sets contained in E.

ii. A *Lyapunov function* for $\{e^{tA}\}_{t\ge 0}$ is a continuous function $V : \mathsf{X} \to \mathbb{R}_+$ such that

$$\dot V(z) = \limsup_{t\to 0+} \frac{V(e^{tA}z) - V(z)}{t} \le 0, \qquad \forall z \in \mathsf{X}.$$

The following results can be found in [18, 21, 33].

Theorem B.2. **i.** *Suppose that $\{e^{tA}z_0 \mid t \ge 0\}$ is precompact in X. Then $\omega(z_0)$ is a nonempty, compact, connected, positively invariant set.*

ii. *Let V be a Lyapunov function on X, define $E = \{z \in \mathsf{X} \mid \dot V(z) = 0\}$ and let M be the maximal positively invariant subset of E. If $\{e^{tA}z_0 \mid t \ge 0\}$ is precompact in X, then $\omega(z_0) \subset M$.*

B.2 Asymptotic behavior when $L_1/L_2 \notin \mathbb{Q}$

Let us turn to the proof of Theorem 2.8.i. We consider the undamped system

$$\begin{cases} \partial_t u_i(t,x) + \partial_x u_i(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L_i],\ i \in \{1,2\}, \\ u_1(t,0) = u_2(t,0) = \dfrac{u_1(t,L_1)+u_2(t,L_2)}{2}, & t \in \mathbb{R}_+, \\ u_i(0,x) = u_{i,0}(x), & x \in [0,L_i],\ i \in \{1,2\}. \end{cases} \tag{B.1}$$

Let $V : \mathsf{X}_2 \to \mathbb{R}$ be the function $V(z) = \|z\|^2_{\mathsf{X}_2}$ and A be the operator given in Definition 2.2 in the case $p = 2$, $N = 2$ and $m_{ij} = 1/2$ for $i, j \in \{1,2\}$, which is the operator associated with System (B.1). By Theorem 2.5, A is the generator of a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$.

Lemma B.3. *The function V is a Lyapunov function for $\{e^{tA}\}_{t\geq 0}$. If $z = (u_1,u_2) \in D(A)$, we have*

$$\dot{V}(z) = 2\langle z, Az\rangle = -\frac{(u_1(L_1)-u_2(L_2))^2}{2}.$$

Proof. Notice first that, for $z = (u_1,u_2) \in D(A)$, we have $\langle z, Az\rangle = -(u_1(L_1)-u_2(L_2))^2/4 \leq 0$. Take $z \in D(A)$, so that $t \mapsto e^{tA}z$ is continuously differentiable in $\mathbb{R}_+$; thus $t \mapsto V(e^{tA}z)$ is continuously differentiable in $\mathbb{R}_+$ with $\frac{d}{dt}V(e^{tA}z) = 2\left\langle e^{tA}z, Ae^{tA}z\right\rangle$. Hence, for every $z \in D(A)$, $\dot{V}(z) = 2\langle z, Az\rangle \leq 0$. This also shows that $\left\|e^{tA}z\right\|_{\mathsf{X}_2} \leq \|z\|_{\mathsf{X}_2}$ for every $z \in D(A)$ and $t \geq 0$, and, by the density of $D(A)$ in X_2, we obtain that $\left\|e^{tA}z\right\|_{\mathsf{X}_2} \leq \|z\|_{\mathsf{X}_2}$ for every $z \in \mathsf{X}_2$ and $t \geq 0$, i.e., $\{e^{tA}\}_{t\geq 0}$ is a contraction semigroup. Thus $\dot{V}(z) \leq 0$ for every $z \in \mathsf{X}_2$, and so V is a Lyapunov function for $\{e^{tA}\}_{t\geq 0}$. ■

It is then immediate to prove the following.

Lemma B.4. *For every $z \in D(A)$, $\{e^{tA}z \mid t \geq 0\}$ is precompact in X_2 and $\omega(z) \subset D(A)$.*

Set $E = \{z \in \mathsf{X}_2 \mid \dot{V}(z) = 0\}$ and let M be the maximal positively invariant subset of E.

Lemma B.5. *Suppose $L_1/L_2 \notin \mathbb{Q}$. Then*

$$D(A) \cap M = \{(\lambda,\lambda) \in L^2(0,L_1) \times L^2(0,L_2) \mid \lambda \in \mathbb{R}\},$$

i.e., $D(A) \cap M$ is the set of constant functions on $L^2(0,L_1) \times L^2(0,L_2)$.

Proof. Take $z_0 = (u_{1,0}, u_{2,0}) \in D(A) \cap M$. By the positive invariance of M, $e^{tA}z_0 \in D(A) \cap M$ for every $t \geq 0$.

Let us denote $z(t) = (u_1(t), u_2(t)) = e^{tA}z_0$, which is a strong solution of (B.1) with initial condition z_0. Since $z(t) \in M$, we have $\dot{V}(z(t)) = 0$ for every $t \geq 0$, which means, by Lemma B.3, that $u_1(t,L_1) = u_2(t,L_2)$ for every $t \geq 0$. Then we have that

$$u_1(t,0) = u_2(t,0) = \frac{u_1(t,L_1)+u_2(t,L_2)}{2} = u_1(t,L_1) = u_2(t,L_2), \qquad \forall t \geq 0.$$

Without any loss of generality, we can suppose that $L_1 \leq L_2$. For $t \geq L_1$ and $x \in [0,L_1]$, we have that

$$u_1(t,x) = u_1(t-x,0) = u_2(t-x,0) = u_2(t,x),$$

and so $u_1(t,x) = u_2(t,x)$. Now, for $t \geq L_1$ and $x \in [0,L_1]$, we have that

$$u_1(t+L_1,x) = u_1(t+L_1-x,0) = u_1(t+L_1-x,L_1) = u_1(t-x,0) = u_1(t,x)$$

and thus $[L_1,+\infty) \ni t \mapsto u_1(t,x)$ is a L_1-periodic function for every $x \in [0,L_1]$. Similarly, for $t \geq L_2$ and $x \in [0,L_2]$, we have that

$$u_2(t+L_2,x) = u_2(t+L_2-x,0) = u_2(t+L_2-x,L_2) = u_2(t-x,0) = u_2(t,x)$$

and thus $[L_2,+\infty) \ni t \mapsto u_2(t,x)$ is a L_2-periodic function for every $x \in [0,L_2]$. Since $u_1(t,x) = u_2(t,x)$ for $t \geq L_1$, $x \in [0,L_1]$, we obtain that $[L_2,+\infty) \ni t \mapsto u_1(t,x)$ is both L_1-periodic and L_2-periodic for every $x \in [0,L_1]$, and the fact that $L_1/L_2 \notin \mathbb{Q}$ thus implies that $[L_2,+\infty) \ni t \mapsto u_1(t,x) = u_2(t,x) =: \lambda(x)$ is constant for every $x \in [0,L_1]$. Clearly, $\lambda(x)$ does not depend on x since

$$\lambda(x) = u_1(t,x) = u_1(t-x,0) = \lambda(0), \qquad \forall t \geq L_2+L_1,\ \forall x \in [0,L_1],$$

and so we shall denote this constant value simply by λ. We thus have that

$$u_1(t,x) = u_2(t,x) = \lambda, \qquad \forall t \geq L_2,\ \forall x \in [0,L_1].$$

We deduce at once that $u_{1,0}$ and $u_{2,0}$ are both equal to the constant function λ and hence

$$D(A) \cap M \subset \{(\lambda,\lambda) \in L^2(0,L_1) \times L^2(0,L_2) \mid \lambda \in \mathbb{R}\}.$$

The converse inclusion is trivial and this concludes the proof. ■

Suppose now that $z_0 \in D(A)$. By Lemma B.4 and Theorem B.2.ii, we have that $\omega(z_0) \subset D(A) \cap M$ and thus, by Lemma B.5, if $L_1/L_2 \notin \mathbb{Q}$, we get that every function in $\omega(z_0)$ is constant. We now wish to show that $\omega(z_0)$ contains only one point in X_2, which will imply that $e^{tA}z_0$ converges to this function as $t \to +\infty$. To do so, we study a conservation law for (B.1).

We define $U : \mathsf{X}_2 \to \mathbb{R}$ by

$$U(u_1,u_2) = \frac{1}{L_1+L_2}\left(\int_0^{L_1} u_1(x)dx + \int_0^{L_2} u_2(x)dx\right).$$

Notice that U is well defined and continuous in X_2 since we have the continuous embedding $\mathsf{X}_2 \hookrightarrow L^1(0,L_1) \times L^1(0,L_2)$.

Lemma B.6. *For every $z \in \mathsf{X}_2$ and $t \geq 0$, we have $U(e^{tA}z) = U(z)$.*

Proof. By the density of $D(A)$ in X_2 and by the continuity of U, it suffices to show this for $z \in D(A)$. In this case, the function $t \mapsto U(e^{tA}z)$ is differentiable in $\mathbb{R}_+$, and, noting $e^{tA}z = (u_1(t),u_2(t))$, we have by a trivial computation that $\frac{d}{dt}U(e^{tA}z) = 0$. ■

Define the operator L on X_2 by $Lz = (U(z),U(z))$ and notice that $L \in \mathcal{L}(\mathsf{X}_2)$. The main result of this section is the following, which proves Theorem 2.8.i and gives the explicit value of the constant λ.

Theorem B.7. *Suppose $L_1/L_2 \notin \mathbb{Q}$. Then, for every $z_0 \in \mathsf{X}_2$, $\lim_{t\to+\infty} e^{tA}z_0 = Lz_0$.*

Proof. Since L is a bounded operator and the semigroup $\{e^{tA}\}_{t\geq 0}$ is uniformly bounded, it suffices by density to show this result for $z_0 \in D(A)$. By Lemma B.4 and Theorem B.2.ii, we have $\omega(z_0) \subset D(A) \cap M$ and thus, by Lemma B.5, every function in $\omega(z_0)$ is constant. Let $z = (\lambda,\lambda) \in \omega(z_0)$ with $\lambda \in \mathbb{R}$ and take $(t_n)_{n\in\mathbb{N}}$ a nondecreasing sequence in $\mathbb{R}_+$ with $t_n \to +\infty$ as $n \to \infty$ such that $e^{t_nA}z_0 \to z$ in X_2 as $n \to \infty$. By the continuity of U and by Lemma B.6, we obtain that

$$\lambda = U(z) = \lim_{n\to\infty} U(e^{t_nA}z_0) = U(z_0)$$

and thus $z = Lz_0$. Hence $\omega(z_0) = \{Lz_0\}$ and, by definition of $\omega(z_0)$, this shows that $e^{tA}z_0 \to Lz_0$ as $t \to +\infty$, which gives the desired result. ■

B.3 Periodic solutions for the undamped system

We now turn to a constructive proof of Theorem 2.8.ii.

Proof of Theorem 2.8.ii. Take $p,q \in \mathbb{N}^*$ coprime such that $L_1/L_2 = p/q$. Let $\ell = L_1/p = L_2/q$. Take $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ with support included in $(0,\ell)$. For $x \in [0,L_1]$, we define $u_{1,0}$ by

$$u_{1,0}(x) = \sum_{k=-\infty}^{+\infty} \varphi(x-k\ell). \tag{B.2}$$

Notice that, for each $x \in \mathbb{R}$, there exists at most one $k \in \mathbb{Z}$ such that $\varphi(x-k\ell) \neq 0$. In particular, $u_{1,0} \in \mathcal{C}^\infty([0,L_1])$. Similarly we define $u_{2,0} \in \mathcal{C}^\infty([0,L_2])$ by the same expression

$$u_{2,0}(x) = \sum_{k=-\infty}^{+\infty} \varphi(x-k\ell), \quad x \in [0,L_2]. \tag{B.3}$$

Define $u_j(t,x) = u_{j,0}(x-t)$ for $j = 1,2$. Since $L_1 = p\ell$, $L_2 = q\ell$, we have

$$u_1(t,L_1) = u_2(t,L_2) = u_1(t,0) = u_2(t,0).$$

Thus, (u_1,u_2) is the unique solution of (B.1) with initial data $z_0 = (u_{1,0},u_{2,0})$. It is periodic in time, and non-constant if φ is chosen to be non-constant. ■

B.4 Periodic solutions for the persistently damped system

Sections B.2 and B.3 present the asymptotic behavior of (B.1) in the cases $L_1/L_2 \notin \mathbb{Q}$ and $L_1/L_2 \in \mathbb{Q}$, showing that all solutions converge to a constant in the first case and that periodic solutions exist in the second one. When considering System (2.9) with a persistent damping, the fact that all its solutions converge exponentially to the origin when $L_1/L_2 \notin \mathbb{Q}$ is a consequence of our main result, Theorem 1.2. However, if $L_1/L_2 \in \mathbb{Q}$ and the damping interval $[a,b]$ is small enough, one may obtain periodic solutions, thus showing that Theorem 1.2 cannot hold in general for $L_1/L_2 \in \mathbb{Q}$ and any length of damping interval.

Theorem B.8. *Suppose that $L_1/L_2 \in \mathbb{Q}$. Then there exists $\ell_0 > 0$ such that, if $b-a \leq \ell_0$, there exists $\alpha \in \mathcal{G}(4\ell_0,\ell_0)$ for which* (2.9) *admits a non-zero periodic solution.*

Proof. We consider here the construction of a periodic solution for (B.1) done in the proof of Theorem 2.8.ii. Take $p,q \in \mathbb{N}^*$ coprime such that $L_1/L_2 = p/q$ and note $\ell = L_1/p = L_2/q$. Take $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ not identically zero with support included in $(0,\ell/2)$. By proceeding as in Theorem 2.8.ii we get a periodic non-zero solution (u_1,u_2) of (B.1) given by

$$u_1(t,x) = \sum_{k=-\infty}^{+\infty} \varphi(x-t-k\ell), \qquad u_2(t,x) = \sum_{k=-\infty}^{+\infty} \varphi(x-t-k\ell). \tag{B.4}$$

Take $\ell_0 = \ell/4$ and suppose that $b-a \leq \ell_0$. We construct a periodic signal $\alpha : \mathbb{R} \to \{0,1\}$ defined by

$$\alpha(t) = \begin{cases} 0, & \text{if } t \in \bigcup_{n\in\mathbb{Z}} \left[a-(n+1/2)\,\ell, b-n\ell\right], \\ 1, & \text{otherwise.} \end{cases}$$

This defines a periodic signal α with period $T = \ell = 4\ell_0$ which belongs to $\mathcal{G}(4\ell_0,\ell_0)$. One then easily checks that $\alpha(t)\chi(x)u_2(t,x) = 0$ for every $(t,x) \in \mathbb{R}_+ \times [0,L_2]$, and so (B.4) satisfies (2.9). ■

C Proof of Theorem 3.3

From now on, we use the convention $\beta^{(i)}_{j,\mathfrak{n}} = 0$ if $\mathfrak{n} = (n_1, \dots, n_N) \in \mathbb{Z}^N$ is such that $n_k < 0$ for some index $k \in [\![1, N]\!]$, so that (3.4b) can be written as

$$\beta^{(i)}_{j,\mathfrak{n}} = \sum_{k=1}^{N} m_{kj} \beta^{(i)}_{k,\mathfrak{n}-\mathbf{1}_k}.$$

One then gets by a trivial induction the following result.

Lemma C.1. *For every $i, j \in [\![1, N]\!]$ and $\mathfrak{n} = (n_1, \dots, n_N) \in \mathfrak{N} \setminus \{\mathbf{0}\}$, we have*

$$\beta^{(i)}_{j,\mathfrak{n}} = \sum_{k=1}^{N} m_{ik} \beta^{(k)}_{j,\mathfrak{n}-\mathbf{1}_k}. \tag{C.1}$$

We can now turn to the proof of Theorem 3.3.

Proof of Theorem 3.3. Let $z_0 = (u_{1,0}, \dots, u_{N,0}) \in D(A)$ and let $z = (u_1, \dots, u_N)$ be defined by (3.2), with $u_i(t, 0)$ given by (3.3). Notice that $u_i(\cdot, 0)$ is defined everywhere on $\mathbb{R}_+$ and is measurable, so that u_i is defined everywhere on $\mathbb{R}_+ \times [0, L_i]$ and is measurable. Note also that u_i is well defined, since

$$u_i(0,0) = \sum_{j=1}^{N} \beta^{(i)}_{j,\mathbf{0}} u_{j,0}(L_j) = u_{i,0}(0)$$

thanks to the fact that $\beta^{(i)}_{j,\mathbf{0}} = m_{ij}$ and $z_0 \in D(A)$.

Let $T_0 > 0$ be as in the proof of Theorem A.6. The unique solution of (2.1) with initial condition z_0 is then given by (A.3) for $0 \le t \le T_0$, and, in order to prove the theorem, it suffices to show that, for every t, τ with $0 \le \tau \le t \le \tau + T_0$, we have

$$u_i(t,x) = \begin{cases} \displaystyle\sum_{j=1}^{N} m_{ij} u_j(\tau, L_j - (t-\tau) + x), & \text{if } 0 \le x \le t - \tau, \\ u_i(\tau, x - t + \tau), & \text{if } x > t - \tau. \end{cases} \tag{C.2}$$

Indeed, if this is proved, we apply it to $\tau = 0$ to obtain that z is the solution of (2.1) with initial condition z_0 for $0 \le t \le T_0$, and also that $z(t) \in D(A)$ for every $t \in [0, T_0]$, and a simple induction allows us to conclude.

We prove (C.2) by considering three cases.

Case 1. $0 \le t - \tau < x$ and $t \le x$. By (3.2), we have $u_i(t,x) = u_{i,0}(x - t) = u_i(\tau, x - t + \tau)$.

Case 2. $0 \le t - \tau < x$ and $t > x$. Since $x - t + \tau < \tau$, we have in this case $u_i(\tau, x - t + \tau) = u_i(\tau - x + t - \tau, 0) = u_i(t - x, 0) = u_i(t, x)$.

Case 3. $t - \tau \ge x$. We notice first that it suffices to consider the case $x = 0$. Indeed, if $t - \tau \ge x$, then $t \ge x$, and it follows clearly by the definition (3.2) of u_i that $u_i(t,x) = u_i(t - x, 0)$ for $t \ge x \ge 0$. On the other hand, let us denote

$$v_{i,\tau}(t,x) = \begin{cases} \displaystyle\sum_{j=1}^{N} m_{ij} u_j(\tau, L_j - (t-\tau) + x) & \text{if } 0 \le x \le t - \tau, \\ u_i(\tau, x - t + \tau) & \text{if } x > t - \tau. \end{cases}$$

It is also clear that $v_{i,\tau}(t,x) = v_{i,\tau}(t - x, 0)$, and thus it suffices to show that $u_i(t, 0) = v_{i,\tau}(t, 0)$ for every $t \in [\tau, \tau + T_0]$ in order to conclude that $u_i(t,x) = u_i(t - x, 0) = v_{i,\tau}(t - x, 0) = v_{i,\tau}(t, x)$ for every $t \in [\tau, \tau + T_0]$ and $x \in [0, L_i]$ with $t - \tau \ge x$.

For $t \in [\tau, \tau+T_0]$, we have $v_{i,\tau}(t,0) = \sum_{j=1}^N m_{ij} u_j(\tau, L_j - t + \tau)$. Furthermore

$$u_j(\tau, L_j - t + \tau) = \begin{cases} u_{j,0}(L_j - t), & \text{if } t \le L_j, \\ u_j(t - L_j, 0), & \text{if } t \ge L_j, \end{cases}$$

and so

$$\begin{aligned}
v_{i,\tau}(t,0) &= \sum_{j=1}^N m_{ij} u_j(\tau, L_j - t + \tau) = \sum_{\substack{j=1 \\ L_j \le t}}^N m_{ij} u_j(t - L_j, 0) + \sum_{\substack{j=1 \\ L_j > t}}^N m_{ij} u_{j,0}(L_j - t) \\
&= \sum_{\substack{j=1 \\ L_j \le t}}^N m_{ij} \sum_{k=1}^N \sum_{\substack{\mathfrak{n} \in \mathfrak{N}_k \\ L(\mathfrak{n}) \le t - L_j}} \beta^{(j)}_{k, \mathfrak{n} + \left\lfloor \frac{t - L_j - L(\mathfrak{n})}{L_k} \right\rfloor \mathbf{1}_k} u_{k,0}(L_k - \{t - L_j - L(\mathfrak{n})\}_{L_k}) \\
&\quad + \sum_{\substack{j=1 \\ L_j > t}}^N m_{ij} u_{j,0}(L_j - t) \\
&= \sum_{\substack{k=1 \\ L_k \le t}}^N \sum_{\substack{j=1 \\ L_j \le t}}^N \sum_{\substack{\mathfrak{n} \in \mathfrak{N}_k \\ L(\mathfrak{n}) \le t - L_j}} m_{ij} \beta^{(j)}_{k, \mathfrak{n} + \left\lfloor \frac{t - L_j - L(\mathfrak{n})}{L_k} \right\rfloor \mathbf{1}_k} u_{k,0}(L_k - \{t - L_j - L(\mathfrak{n})\}_{L_k}) \\
&\quad + \sum_{\substack{k=1 \\ L_k > t}}^N \sum_{\substack{j=1 \\ L_j \le t}}^N \sum_{\substack{\mathfrak{n} \in \mathfrak{N}_k \\ L(\mathfrak{n}) \le t - L_j}} m_{ij} \beta^{(j)}_{k,\mathfrak{n}} u_{k,0}(L_k - (t - L_j - L(\mathfrak{n}))) + \sum_{\substack{j=1 \\ L_j > t}}^N \beta^{(i)}_{j,\mathbf{0}} u_{j,0}(L_j - t).
\end{aligned}$$

Set

$$\begin{aligned}
A_1(t) &= \{(k,j,\mathfrak{n}) \in [\![1,N]\!] \times [\![1,N]\!] \times \mathfrak{N} \mid L_k \le t,\ L_j \le t,\ \mathfrak{n} \in \mathfrak{N}_k,\ L(\mathfrak{n}) \le t - L_j\}, \\
A_2(t) &= \{(k,j,\mathfrak{n}) \in [\![1,N]\!] \times [\![1,N]\!] \times \mathfrak{N} \mid L_k > t,\ L_j \le t,\ \mathfrak{n} \in \mathfrak{N}_k,\ L(\mathfrak{n}) \le t - L_j\},
\end{aligned}$$

so that we can write

$$\begin{aligned}
v_{i,\tau}(t,0) &= \sum_{(k,j,\mathfrak{n}) \in A_1(t)} m_{ij} \beta^{(j)}_{k, \mathfrak{n} + \left\lfloor \frac{t - L_j - L(\mathfrak{n})}{L_k} \right\rfloor \mathbf{1}_k} u_{k,0}(L_k - \{t - L_j - L(\mathfrak{n})\}_{L_k}) \\
&\quad + \sum_{(k,j,\mathfrak{n}) \in A_2(t)} m_{ij} \beta^{(j)}_{k,\mathfrak{n}} u_{k,0}(L_k - (t - L_j - L(\mathfrak{n}))) + \sum_{\substack{j=1 \\ L_j > t}}^N \beta^{(i)}_{j,\mathbf{0}} u_{j,0}(L_j - t).
\end{aligned} \tag{C.3}$$

Set

$$\begin{aligned}
B_1(t) = \{(k,j,\mathfrak{m}) \in [\![1,N]\!] \times [\![1,N]\!] \times \mathfrak{N} \mid L_k \le t,\ \mathfrak{m} \in \mathfrak{N}_k, \\
L(\mathfrak{m}) \le t,\ \mathfrak{m} + \left\lfloor \frac{t - L(\mathfrak{m})}{L_k} \right\rfloor \mathbf{1}_k = (r_1, \dots, r_N) \text{ with } r_j \ge 1 \Big\},
\end{aligned}$$

$$\begin{aligned}
B_2(t) = \{(k,j,\mathfrak{m}) \in [\![1,N]\!] \times [\![1,N]\!] \times \mathfrak{N} \mid L_k > t, \\
\mathfrak{m} = (m_1, \dots, m_N) \in \mathfrak{N}_k \setminus \{\mathbf{0}\},\ m_j \ge 1,\ L(\mathfrak{m}) \le t\},
\end{aligned}$$

and define the functions $\varphi_\lambda : [\![1,N]\!] \times [\![1,N]\!] \times \mathfrak{N} \to [\![1,N]\!] \times [\![1,N]\!] \times \mathfrak{N}$, $\lambda = 1,2$, by

$$\varphi_1(k,j,\mathfrak{n}) = \begin{cases} (k,j,\mathfrak{n} + \mathbf{1}_j) & \text{if } k \ne j, \\ (k,j,\mathfrak{n}) & \text{if } k = j, \end{cases} \qquad \varphi_2(k,j,\mathfrak{n}) = (k,j,\mathfrak{n} + \mathbf{1}_j).$$

We claim that φ_λ is a bijection from $A_\lambda(t)$ to $B_\lambda(t)$, $\lambda = 1,2$. Indeed, it is easy to verify that the image of $A_\lambda(t)$ by φ_λ is included in $B_\lambda(t)$ and that $\varphi_\lambda : A_\lambda(t) \to B_\lambda(t)$ is injective for $\lambda = 1,2$. Let us check that these functions are surjective.

If $(k,j,\mathfrak{m}) \in B_1(t)$, we note $(r_1,\dots,r_N) = \mathfrak{m} + \lfloor (t-L(\mathfrak{m}))/L_k \rfloor \mathbf{1}_k$ and we set $\mathfrak{n} = \mathfrak{m}$ if $k = j$ and $\mathfrak{n} = \mathfrak{m} - \mathbf{1}_j$ if $k \neq j$. Notice first that, if $k \neq j$, then $m_j = r_j \geq 1$, so that $\mathfrak{m} - \mathbf{1}_j \in \mathfrak{N}$, and thus, in both cases $k = j$ and $k \neq j$, we have $(k,j,\mathfrak{n}) \in [\![1,N]\!] \times [\![1,N]\!] \times \mathfrak{N}$, and clearly $\varphi_1(k,j,\mathfrak{n}) = (k,j,\mathfrak{m})$, so that, in order to conclude that $\varphi_1 : A_1(t) \to B_1(t)$ is surjective, it suffices to show that $(k,j,\mathfrak{n}) \in A_1(t)$. We clearly have $L_k \leq t$ and $\mathfrak{n} \in \mathfrak{N}_k$. If $j = k$, we have $L_j = L_k \leq t$ and, since $r_j = r_k = \lfloor (t-L(\mathfrak{m}))/L_k \rfloor$ and $r_j \geq 1$, we have $(t-L(\mathfrak{m}))/L_k \geq 1$, i.e., $L(\mathfrak{n}) = L(\mathfrak{m}) \leq t - L_k = t - L_j$, so that $(k,j,\mathfrak{n}) \in A_1(t)$. If $j \neq k$, we have $m_j = r_j \geq 1$, so that $L_j \leq m_j L_j \leq L(\mathfrak{m}) \leq t$; also, $L(\mathfrak{n}) = L(\mathfrak{m}) - L_j \leq t - L_j$, so that $(k,j,\mathfrak{n}) \in A_1(t)$. Hence $(k,j,\mathfrak{n}) \in A_1(t)$ in both cases, and thus $\varphi_1 : A_1(t) \to B_1(t)$ is surjective.

If $(k,j,\mathfrak{m}) \in B_2(t)$, we set $\mathfrak{n} = \mathfrak{m} - \mathbf{1}_j$, so that $\mathfrak{n} \in \mathfrak{N}$ and $\varphi_2(k,j,\mathfrak{n}) = (k,j,\mathfrak{m})$. Now, it is clear that $L_k > t$ and $\mathfrak{n} \in \mathfrak{N}_k$, and we have $L_j \leq m_j L_j \leq L(\mathfrak{m}) \leq t$ and $L(\mathfrak{n}) = L(\mathfrak{m}) - L_j \leq t - L_j$. Hence $(k,j,\mathfrak{n}) \in A_2(t)$, and thus $\varphi_2 : A_2(t) \to B_2(t)$ is surjective.

Thanks to the bijections $\varphi_\lambda : A_\lambda(t) \to B_\lambda(t)$, $\lambda = 1,2$, we can rewrite (C.3) as

$$\begin{aligned} v_{i,\tau}(t,0) = &\sum_{(k,j,\mathfrak{m}) \in B_1(t)} m_{ij} \beta^{(j)}_{k,\mathfrak{m}-\mathbf{1}_j + \left\lfloor \frac{t-L(\mathfrak{m})}{L_k} \right\rfloor \mathbf{1}_k} u_{k,0}(L_k - \{t - L(\mathfrak{m})\}_{L_k}) \\ &+ \sum_{(k,j,\mathfrak{m}) \in B_2(t)} m_{ij} \beta^{(j)}_{k,\mathfrak{m}-\mathbf{1}_j} u_{k,0}(L_k - (t - L(\mathfrak{m}))) + \sum_{\substack{j=1 \\ L_j > t}}^{N} \beta^{(i)}_{j,\mathbf{0}} u_{j,0}(L_j - t), \end{aligned}$$

and so, by applying Lemma C.1, we obtain

$$\begin{aligned} v_{i,\tau}(t,0) = &\sum_{\substack{k=1 \\ L_k \leq t}}^{N} \sum_{\substack{\mathfrak{m} \in \mathfrak{N}_k \\ L(\mathfrak{m}) \leq t}} \sum_{\substack{j=1 \\ \left(\mathfrak{m} + \left\lfloor \frac{t-L(\mathfrak{m})}{L_k} \right\rfloor \mathbf{1}_k\right)_j \geq 1}}^{N} m_{ij} \beta^{(j)}_{k,\mathfrak{m} + \left\lfloor \frac{t-L(\mathfrak{m})}{L_k} \right\rfloor \mathbf{1}_k - \mathbf{1}_j} u_{k,0}(L_k - \{t - L(\mathfrak{m})\}_{L_k}) \\ &+ \sum_{\substack{k=1 \\ L_k > t}}^{N} \sum_{\substack{\mathfrak{m} \in \mathfrak{N}_k \setminus \{\mathbf{0}\} \\ L(\mathfrak{m}) \leq t}} \sum_{\substack{j=1 \\ m_j \geq 1}}^{N} m_{ij} \beta^{(j)}_{k,\mathfrak{m}-\mathbf{1}_j} u_{k,0}(L_k - (t - L(\mathfrak{m}))) + \sum_{\substack{k=1 \\ L_k > t}}^{N} \beta^{(i)}_{k,\mathbf{0}} u_{k,0}(L_k - t) \\ = &\sum_{\substack{k=1 \\ L_k \leq t}}^{N} \sum_{\substack{\mathfrak{m} \in \mathfrak{N}_k \\ L(\mathfrak{m}) \leq t}} \beta^{(i)}_{k,\mathfrak{m} + \left\lfloor \frac{t-L(\mathfrak{m})}{L_k} \right\rfloor \mathbf{1}_k} u_{k,0}(L_k - \{t - L(\mathfrak{m})\}_{L_k}) \\ &+ \sum_{\substack{k=1 \\ L_k > t}}^{N} \sum_{\substack{\mathfrak{m} \in \mathfrak{N}_k \setminus \{\mathbf{0}\} \\ L(\mathfrak{m}) \leq t}} \beta^{(i)}_{k,\mathfrak{m}} u_{k,0}(L_k - (t - L(\mathfrak{m}))) + \sum_{\substack{k=1 \\ L_k > t}}^{N} \beta^{(i)}_{k,\mathbf{0}} u_{k,0}(L_k - t) \\ = &\sum_{k=1}^{N} \sum_{\substack{\mathfrak{m} \in \mathfrak{N}_k \\ L(\mathfrak{m}) \leq t}} \beta^{(i)}_{k,\mathfrak{m} + \left\lfloor \frac{t-L(\mathfrak{m})}{L_k} \right\rfloor \mathbf{1}_k} u_{k,0}(L_k - \{t - L(\mathfrak{m})\}_{L_k}) = u_i(t,0). \end{aligned}$$

Hence $v_{i,\tau}(t,0) = u_i(t,0)$ for every $t \in [\tau, \tau + T_0]$, which thus concludes the proof of (C.2). ■

D A combinatorial estimate

In order to estimate the right-hand side of (4.8), one needs the following lemma.

Lemma D.1. *Let $\nu \in (0,1)$. There exist $\rho \in (0, 1/2)$, $C, \gamma > 0$ such that, for every $n \in \mathbb{N}$ and $k \in [\![0, \rho n]\!]$, we have*

$$\binom{n}{k} \nu^n \leq C e^{-\gamma n}. \tag{D.1}$$

Proof. For $n \in \mathbb{N}$, consider the function $f_n(k) = \binom{n}{k} \nu^n$ defined for $k \in [\![0, n]\!]$. Since $k \mapsto \binom{n}{k}$ is increasing for $k \in [\![0, n/2]\!]$, if $k \in [\![0, \rho n]\!]$ for a certain $\rho \in (0, 1/2)$, then

$$f_n(k) \leq f_n(\lfloor \rho n \rfloor). \tag{D.2}$$

Let us estimate $f_n(\lfloor \rho n \rfloor)$ for n large. As $n \to +\infty$, using Stirling's approximation $\log n! = n \log n - n + O(\log n)$, we get

$$\begin{aligned}
\log f_n(\lfloor \rho n \rfloor) &= \log \binom{n}{\lfloor \rho n \rfloor} + n \log \nu \\
&= n \log n - \rho n \log(\rho n) - (1-\rho) n \log[(1-\rho) n] + n \log \nu + O(\log n) \\
&= n \left[g(\rho) + O\left(\frac{\log n}{n} \right) \right],
\end{aligned}$$

where the function $g : (0,1) \to \mathbb{R}$ is defined by

$$g(\rho) = \rho \log\left(\frac{1}{\rho}\right) + (1-\rho) \log\left(\frac{1}{1-\rho}\right) + \log \nu.$$

It is a continuous function of $\rho \in (0,1)$ and $\lim_{\rho \to 0} g(\rho) = \log \nu < 0$, hence there exists $\rho \in (0, 1/2)$ depending only on ν such that $g(\rho) \leq \frac{1}{2} \log \nu < 0$. For this value of ρ, we have

$$\log f_n(\lfloor \rho n \rfloor) \leq \frac{n}{2} \log \nu + O(\log n) \leq \frac{n}{4} \log \nu + O(1).$$

The result follows by combining the above with (D.2). ■

References

[1] F. Alabau-Boussouira and M. Léautaud. Indirect controllability of locally coupled wave-type systems and applications. *J. Math. Pures Appl. (9)*, 99(5):544–576, 2013.

[2] F. Ali Mehmeti, J. von Below, and S. Nicaise, editors. *Partial differential equations on multi-structures*, volume 219 of *Lecture Notes in Pure and Applied Mathematics*, New York, 2001. Marcel Dekker Inc.

[3] B. Anderson. Exponential stability of linear equations arising in adaptive identification. *IEEE Trans. Automat. Control*, 22(1):83–88, 1977.

[4] B. Anderson, R. Bitmead, C. Johnson, P. Kokotovic, R. Kosut, I. Mareels, L. Praly, and B. Riedle. *Stability of adaptive systems: Passivity and averaging analysis*. MIT Press Series in Signal Processing, Optimization, and Control, 8. MIT Press, Cambridge, MA, 1986.

[5] S. Andersson and P. Krishnaprasad. Degenerate gradient flows: a comparison study of convergence rate estimates. In *Decision and Control, 2002, Proceedings of the 41st IEEE Conference on*, volume 4, pages 4712–4717. IEEE, 2002.

[6] G. Bastin, B. Haut, J.-M. Coron, and B. D'Andréa-Novel. Lyapunov stability analysis of networks of scalar conservation laws. *Netw. Heterog. Media*, 2(4):751–759, 2007.

[7] A. Bressan, S. Čanić, M. Garavello, M. Herty, and B. Piccoli. Flows on networks: recent results and perspectives. *EMS Surv. Math. Sci.*, 1(1):47–111, 2014.

[8] A. Chaillet, Y. Chitour, A. Loría, and M. Sigalotti. Towards uniform linear time-invariant stabilization of systems with persistency of excitation. In *Decision and Control, 2007 46th IEEE Conference on*, pages 6394–6399. IEEE, 2007.

[9] A. Chaillet, Y. Chitour, A. Loría, and M. Sigalotti. Uniform stabilization for linear systems with persistency of excitation: the neutrally stable and the double integrator cases. *Math. Control Signals Systems*, 20(2):135–156, 2008.

[10] C. Chicone and Y. Latushkin. *Evolution semigroups in dynamical systems and differential equations*, volume 70 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[11] Y. Chitour, F. Colonius, and M. Sigalotti. Growth rates for persistently excited linear systems. *Math. Control Signals Systems*, 26(4):589–616, 2014.

[12] Y. Chitour, G. Mazanti, and M. Sigalotti. Stabilization of persistently excited linear systems. In J. Daafouz, S. Tarbouriech, and M. Sigalotti, editors, *Hybrid Systems with Constraints*, chapter 4. Wiley-ISTE, London, UK, 2013.

[13] Y. Chitour and M. Sigalotti. On the stabilization of persistently excited linear systems. *SIAM J. Control Optim.*, 48(6):4032–4055, 2010.

[14] R. Dáger and E. Zuazua. *Wave propagation, observation and control in* 1*-d flexible multi-structures*, volume 50 of *Mathématiques & Applications*. Springer-Verlag, Berlin, 2006.

[15] M. Gugat. Optimal switching boundary control of a string to rest in finite time. *ZAMM Z. Angew. Math. Mech.*, 88(4):283–305, 2008.

[16] M. Gugat. Contamination source determination in water distribution networks. *SIAM J. Appl. Math.*, 72(6):1772–1791, 2012.

[17] M. Gugat and M. Sigalotti. Stars of vibrating strings: switching boundary feedback stabilization. *Netw. Heterog. Media*, 5(2):299–314, 2010.

[18] J. K. Hale. Dynamical systems and stability. *J. Math. Anal. Appl.*, 26:39–59, 1969.

[19] F. Hante, M. Sigalotti, and M. Tucsnak. On conditions for asymptotic stability of dissipative infinite-dimensional systems with intermittent damping. *Journal of Differential Equations*, 252(10):5569–5593, 2012.

[20] F. M. Hante, G. Leugering, and T. I. Seidman. Modeling and analysis of modal switching in networked transport systems. *Appl. Math. Optim.*, 59(2), 2009.

[21] D. Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1981.

[22] T. Kato. Linear evolution equations of "hyperbolic" type. II. *J. Math. Soc. Japan*, 25:648–666, 1973.

[23] J. E. Lagnese, G. Leugering, and E. J. P. G. Schmidt. *Modeling, analysis and control of dynamic elastic multi-link structures*. Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 1994.

[24] D. Liberzon. *Switching in Systems and Control*. Birkhäuser Boston, 1 edition, 2003.

[25] A. Loría, A. Chaillet, G. Besançon, and Y. Chitour. On the PE stabilization of time-varying systems: open questions and preliminary answers. In *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05. 44th IEEE Conference on*, pages 6847–6852. IEEE, 2005.

[26] G. Lumer. Connecting of local operators and evolution equations on networks. In *Potential theory, Copenhagen 1979 (Proc. Colloq., Copenhagen, 1979)*, volume 787 of *Lecture Notes in Math.*, pages 219–234. Springer, Berlin, 1980.

[27] M. Margaliot. Stability analysis of switched systems using variational principles: an introduction. *Automatica*, 42(12):2059–2077, 2006.

[28] G. Mazanti. Stabilization of persistently excited linear systems by delayed feedback laws. *Systems Control Lett.*, 68:57–67, 2014.

[29] G. Mazanti, Y. Chitour, and M. Sigalotti. Stabilization of two-dimensional persistently excited linear control systems with arbitrary rate of convergence. *SIAM J. Control Optim.*, 51(2):801–823, 2013.

[30] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[31] R. S. Phillips. Perturbation theory for semi-groups of linear operators. *Trans. Amer. Math. Soc.*, 74:199–221, 1953.

[32] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King. Stability criteria for switched and hybrid systems. *SIAM Rev.*, 49(4):545–592, 2007.

[33] M. Slemrod. The LaSalle invariance principle in infinite-dimensional Hilbert space. In *Dynamical systems approaches to nonlinear problems in systems and circuits (Henniker, NH, 1986)*, pages 53–59. SIAM, Philadelphia, PA, 1988.

[34] M. Suzuki, J.-i. Imura, and K. Aihara. Analysis and stabilization for networked linear hyperbolic systems of rationally dependent conservation laws. *Automatica J. IFAC*, 49(11):3210–3221, 2013.

[35] J. Valein and E. Zuazua. Stabilization of the wave equation on 1-D networks. *SIAM J. Control Optim.*, 48(4):2771–2797, 2009.

[36] K. Yosida. *Functional analysis*, volume 123 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 6 edition, 1980.

[37] E. Zuazua. Control and stabilization of waves on 1-d networks. In *Modelling and Optimisation of Flows on Networks*, pages 463–493. Springer, 2013.