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Généralisation et changement de base dans un graphe en analyse non-linéaire des fluides dans un milieu poreux.

Willy Dantin-Carrère

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RECHERCHES

Généralisation et changement de base
dans un graphe en analyse non-linéaire des fluides
dans un milieu poreux.

Suite d'un article de recherche
De Eugenio AULISA, Akif IBRAGIMOV and Magdalena TODA
« [Geometric methods in the analysis of non-linear flows
in porous media.](#) »

Suite d'un article de recherche écrit par
[M. Dantin-Carrère Willy](#)

L'article de Eugenio AULISA, Akif IBRAGIMOV and Magdalena TODA

« [Geometric methods in the analysis of non-linear flows in porous media.](#) »

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PARTIE 1

GÉNÉRALISATION D'UN GRAPHE

- Changement de base dans un graphe

$$\check{r}(x,y)=(f(x) ;g(y) ;\check{u}(x,y))$$

$$\check{u}(x,y)=\check{u}(x,y)$$

- au lieu du graphe $(\chi.x ;\chi.y ;\check{u}(x,y))$

$$\text{avec : } \check{u}_x(x,y)=\mu(x,y).u_x(x,y)$$

$$\check{u}_y(x,y)=\mu(x,y).u_y(x,y)$$

CALCUL DE $d\check{r}^2(x,y)$ ET DE $\Delta\check{g}=\text{trace}(\check{g}^{ij}.\check{h}_{ij})$

PARTIE 1

* AUTRE GRAPHE GÉNÉRALISATION :

$$\check{r}(x, y) := (f(x), g(y), \check{m}(x, y))$$

$$\check{m}_x(x, y) = \mu(x, y) \mu_x(x, y)$$

$$\check{m}_y(x, y) = \mu(x, y) \mu_y(x, y)$$

$$\check{r}_{xx}(x, y) = (f'(x), 0, \mu(x, y) \mu_x(x, y))$$

$$\check{r}_{yy}(x, y) = (0, g'(y), \mu(x, y) \mu_y(x, y))$$

$$\left(\check{r}_{xx}(x, y) \right)_y = \left(\check{r}_{yy}(x, y) \right)_x$$

$$\left(\check{r}_{xx}(x, y) \right)_y = (0, 0, \mu_y(x, y) \mu_x(x, y) + \mu(x, y) \mu_{xxy}(x, y))$$

$$\left(\check{r}_{yy}(x, y) \right)_x = (0, 0, \mu_x(x, y) \mu_y(x, y) + \mu(x, y) \mu_{xyy}(x, y))$$

$$\mu_y \mu_{xc} + \mu \mu_{xxy} = \mu_{xc} \mu_y + \mu \mu_{xyy}$$

$$\mu_y(x, y) \mu_x(x, y) = \mu_x(x, y) \mu_y(x, y)$$

$$\check{r}_{xxx} = (f''(x), 0, \mu_x(x, y) \mu_x(x, y) + \mu(x, y) \mu_{xxx}(x, y))$$

$$\check{r}_{yyy} = (0, g''(y), \mu_y(x, y) \mu_y(x, y) + \mu(x, y) \mu_{yyy}(x, y))$$

$$\check{N} := \frac{\check{r}_{xx} \times \check{r}_{yy}}{\|\check{r}_{xx} \times \check{r}_{yy}\|}$$

\check{N} : vecteur normal unitaire

$$\vec{n}_{zx} = (f'(x), 0, \mu \mu_{zx}) \text{ avec entendeur } \mu = \mu(x, y) \\ \text{et } \mu_{zx} = \mu_{zx}(x, y)$$

$$\vec{n}_{yz} = (0, g'(y), \mu \mu_{yz})$$

$$\vec{n}_{zx} \times \vec{n}_{yz} = (f'(x) \cdot g'(y), f'(x) \mu \mu_{yz}, g'(y) \mu \mu_{zx})$$

$$\|\vec{n}_{zx} \times \vec{n}_{yz}\| = \left\{ [f'(x) \cdot g'(y)]^2 + (f'(x) \mu \mu_{yz})^2 + (g'(y) \mu \mu_{zx})^2 \right\}^{1/2}$$

$$\vec{h}_{11} = \left\langle \left(\frac{-\mu \mu_{yz} f'}{\sqrt{\dots}}, \frac{-\mu \mu_{zx} g'}{\sqrt{\dots}}, \frac{f' \cdot g'}{\sqrt{\dots}} \right) (f''(x); 0; \mu_{zx} \mu_{zx} + \mu \mu_{zxzx}) \right\rangle$$

$$\vec{h}_{22} = \left\langle \left(\frac{-\mu \mu_{yz} f'}{\sqrt{\dots}}, \frac{-\mu \mu_{zx} g'}{\sqrt{\dots}}, \frac{f' \cdot g'}{\sqrt{\dots}} \right) (0; g''(y); \mu_{yz} \mu_{yz} + \mu \mu_{yzyz}) \right\rangle$$

$$\vec{h}_{12} = \left\langle \left(\frac{-\mu \mu_{yz} f'}{\sqrt{\dots}}, \frac{-\mu \mu_{zx} g'}{\sqrt{\dots}}, \frac{f' \cdot g'}{\sqrt{\dots}} \right) (0; 0; \mu_{zx} \mu_{yz} + \mu \mu_{zxyz}) \right\rangle$$

$$\vec{N} = \frac{-f' \cdot \mu \mu_{yz} \cdot i - g' \cdot \mu \cdot \mu_{zx} \cdot j + k \cdot f' \cdot g'}{\left\{ (f' \cdot g')^2 + (f' \cdot \mu \mu_{yz})^2 + (g' \cdot \mu \cdot \mu_{zx})^2 \right\}^{1/2}}$$

$$\sqrt{\dots} = \left\{ (f' \cdot g')^2 + (f' \cdot \mu \mu_{yz})^2 + (g' \cdot \mu \cdot \mu_{zx})^2 \right\}^{1/2}$$

$$f'' = f''(x) \quad f'(x) = f' \quad \mu = \mu(x, y) \quad \mu_{yz} = \mu_{yz}(x, y)$$

$$\mu_{zx} = \mu_{zx}(x, y) \quad \mu_{zxzx} = \mu_{zxzx}(x, y) \quad \mu_{yzyz} = \mu_{yzyz}(x, y) \dots$$

$$\vec{h}_{11} = \frac{-f'' \cdot f' \cdot \mu \cdot \mu_{yz} + f' \cdot g' \cdot (\mu_{zx} \mu_{zx} + \mu \cdot \mu_{zxzx})}{\left\{ (f' \cdot g')^2 + (f' \cdot \mu \cdot \mu_{yz})^2 + (g' \cdot \mu \cdot \mu_{zx})^2 \right\}^{1/2}}$$

$$\vec{h}_{22} = \frac{-g'' \cdot g' \cdot \mu \cdot \mu_{zx} + f' \cdot g' \cdot (\mu_{yz} \mu_{yz} + \mu \cdot \mu_{yzyz})}{\left\{ (f' \cdot g')^2 + (f' \cdot \mu \cdot \mu_{yz})^2 + (g' \cdot \mu \cdot \mu_{zx})^2 \right\}^{1/2}}$$

$$\check{h}_{12} = \frac{f' \cdot g' \cdot (\mu_x \cdot \mu_y + \mu \mu_{xy})}{\{(f' \cdot g')^2 + (f' \cdot \mu \mu_y)^2 + (g' \cdot \mu \cdot \mu_x)^2\}^{1/2}}$$

⊗ ON RETOURNE SUR L'EXEMPLE: $f'(x) = g'(y) = X$ / $f(x) = X \cdot x$
 $f''(x) = g''(y) = 0$ $g(y) = X \cdot y$

$$\check{h}_{11} = \frac{X^2 (\mu_{xx} \mu_{xx} + \mu \mu_{xxx})}{X \sqrt{X^2 + \mu^2 (\mu_{xx}^2 + \mu_y^2)}} = \frac{X (\mu_{xx} \mu_{xx} + \mu \mu_{xxx})}{\sqrt{X^2 + \mu^2 (\mu_{xx}^2 + \mu_y^2)}}$$

$$\check{h}_{12} = \frac{X (\mu_{xx} \mu_y + \mu \mu_{xy})}{\sqrt{X^2 + \mu^2 (\mu_{xx}^2 + \mu_y^2)}} = \frac{X (\mu_y \mu_{xx} + \mu \mu_{xy})}{\sqrt{X^2 + \mu^2 (\mu_{xx}^2 + \mu_y^2)}}$$

$$\check{h}_{22} = \frac{X (\mu_y \mu_y + \mu \mu_{yy})}{\sqrt{X^2 + \mu^2 (\mu_{xx}^2 + \mu_y^2)}}$$

$$d\check{r}^2 = |\check{r}_x|^2 dx^2 + 2 \langle \check{r}_x, \check{r}_y \rangle dx dy + |\check{r}_y|^2 dy^2$$

$$\check{r}_x = (f'(x), 0, \mu \mu_{xx}) \quad \check{r}_y = (0, g'(y), \mu \mu_y)$$

$$|\check{r}_x|^2 = [(f'(x))^2 + \mu^2 \mu_{xx}^2] \quad |\check{r}_y|^2 = [(g'(y))^2 + \mu^2 \mu_y^2]$$

$$\langle \check{r}_x, \check{r}_y \rangle = f'(x) \cdot 0 + g'(y) \cdot 0 + \mu^2 \mu_x \mu_y$$

$$\langle \check{r}_x, \check{r}_y \rangle = \mu^2 \cdot \mu_{xx} \cdot \mu_y$$

$$d\check{r}^2 = [(f'(x))^2 + \mu^2(x, y) \mu_{xx}^2(x, y)] dx^2 + 2 \mu^2(x, y) \mu_{xx}(x, y) \mu_y(x, y) dx dy + [(g'(y))^2 + \mu^2(x, y) \mu_y^2(x, y)] dy^2$$

$$\check{g}_{ij} = \begin{pmatrix} [(f'(x))^2 + \mu^2(x, y) \mu_{xx}^2(x, y)] & \mu^2(x, y) \mu_{xx}(x, y) \mu_y(x, y) \\ \mu^2(x, y) \mu_{xx}(x, y) \mu_y(x, y) & [(g'(y))^2 + \mu^2(x, y) \mu_y^2(x, y)] \end{pmatrix}$$

$$\det \check{g}_{ij} = [(f'(x))^2 + \mu^2 u_x^2] \times [(g'(y))^2 + \mu^2 u_y^2] - \mu^4 u_x^2 u_y^2$$

$$\det \check{g}_{ij} = (f'(x) \cdot g'(y))^2 + (f'(x))^2 \mu^2(x, y) u_y^2(x, y) + (g'(y))^2 \mu^2(x, y) u_x^2(x, y) + \mu^4 u_x^2 u_y^2 - \mu^4 u_x^2 u_y^2$$

$$\det \check{g}_{ij} = (f'(x) \cdot g'(y))^2 + (f'(x))^2 \mu^2(x, y) u_y^2(x, y) + (g'(y))^2 \mu^2(x, y) u_x^2(x, y)$$

$$\check{g}^{ij} = \frac{1}{\det(\check{g}_{ij})} \begin{pmatrix} (g')^2 + \mu^2 u_y^2 & -\mu^2 u_x u_y \\ -\mu^2 u_x u_y & (f')^2 + \mu^2 u_x^2 \end{pmatrix}$$

$$\alpha = \frac{1}{[\det(\check{g}_{ij})]^{\frac{1}{2}}}$$

$$\check{h}_{ij} = \alpha \cdot \begin{pmatrix} -f'' f' \mu u_y + f' g' (\mu_x u_x + \mu u_{xx}) & f' g' (\mu_x u_y + \mu u_{xy}) \\ f' g' (\mu_y u_x + \mu u_{xy}) & -g'' g' \mu u_x + f' g' (\mu_y u_y + \mu u_{yy}) \end{pmatrix}$$

$$\Delta \check{g} = \text{trace}(\check{g}^{ij} \cdot \check{h}_{ij})$$

$$\Delta \check{g} = \left\{ \begin{aligned} & [-f'' f' \mu u_y + f' g' (\mu_x u_x + \mu u_{xx})] \cdot [(g')^2 + \mu^2 u_y^2] \\ & + [f' g' (\mu_x u_y + \mu u_{xy})] \cdot [-\mu^2 u_x u_y] \\ & + [f' g' (\mu_y u_x + \mu u_{xy})] \cdot [-\mu^2 u_x u_y] \\ & + [-g'' g' \mu u_x + f' g' (\mu_y u_y + \mu u_{yy})] \cdot [(f')^2 + \mu^2 u_x^2] \end{aligned} \right\} \cdot \alpha^3$$

* ON SAIT QUE $\mu_{xc} \mu_y = \mu_y \mu_{xc}$

$$\Delta \ddot{g} = \left\{ \begin{aligned} & \left[-f'' \cdot f' \cdot \mu \cdot \mu_y \right] \cdot \left[(g')^2 + \mu^2 \mu_y^2 \right] + f' \cdot g' \cdot \mu_{xc} \mu_{xc} \cdot (g')^2 \\ & + f' \cdot g' \cdot \mu \cdot \mu_{xcxc} \left[\mu^2 \mu_y^2 + (g')^2 \right] + (f' \cdot g' \cdot \mu \mu_{xcy}) \cdot (-\mu^2 \mu_{xc} \mu_y) \\ & + (f' \cdot g' \cdot \mu \cdot \mu_{xcy}) \cdot (-\mu^2 \mu_{xc} \mu_y) + \left[-g'' \cdot g' \cdot \mu \cdot \mu_{xc} \right] \cdot \left[(f')^2 + \mu^2 \mu_{xc}^2 \right] \\ & + f' \cdot g' \cdot \mu_y \cdot \mu_y \cdot (f')^2 + f' \cdot g' \cdot \mu \mu_{yy} \cdot \left[(f')^2 + \mu^2 \mu_{xc}^2 \right] \end{aligned} \right\} \cdot \alpha^3$$

$$\Delta \ddot{g} = \left\{ \begin{aligned} & f'(x) \cdot g'(y) \cdot \mu(x, y) \cdot \mu_{xcxc}(x, y) \cdot \left[\mu^2(x, y) \cdot \mu_y^2(x, y) + (g'(y))^2 \right] \\ & + f'(x) \cdot g'(y) \cdot \mu(x, y) \cdot \mu_{yy}(x, y) \cdot \left[\mu^2(x, y) \cdot \mu_{xc}^2(x, y) + (f'(x))^2 \right] \\ & - 2 \mu^3(x, y) \cdot f'(x) \cdot g'(y) \cdot \mu_{xc}(x, y) \cdot \mu_y(x, y) \cdot \mu_{xcy}(x, y) \\ & + f'(x) \cdot g'(y) \left[(f'(x))^2 \mu_y(x, y) \mu_y(x, y) + (g'(y))^2 \mu_{xc}(x, y) \mu_{xc}(x, y) \right] \\ & + \left[-f''(x) \cdot f'(x) \cdot \mu(x, y) \mu_y(x, y) \right] \cdot \left[(g'(y))^2 + \mu^2(x, y) \mu_y^2(x, y) \right] \\ & + \left[-g''(y) \cdot g'(y) \cdot \mu(x, y) \mu_{xc}(x, y) \right] \cdot \left[(f'(x))^2 + \mu^2(x, y) \mu_{xc}^2(x, y) \right] \end{aligned} \right\} \cdot \alpha^3$$

* SOUS-ENTENDU $f = f(x) \quad \mu_{xc} = \mu_{xc}(x, y) \dots$

$$\Delta \ddot{g} = \left\{ \begin{aligned} & f' \cdot g' \cdot \mu \left[\mu^2 \mu_y^2 + (g')^2 \right] \cdot \mu_{xcxc} \\ & + f' \cdot g' \cdot \mu \left[\mu^2 \mu_{xc}^2 + (f')^2 \right] \cdot \mu_{yy} \\ & - 2 \mu^3 \cdot f' \cdot g' \cdot \mu_{xc} \mu_y \cdot \mu_{xcy} + f' \cdot g' \cdot \left[(f')^2 \mu_y \mu_y + (g')^2 \mu_{xc} \mu_{xc} \right] \\ & + \left[f'' \cdot f' \cdot \mu \mu_y \right] \cdot \left[(g')^2 + \mu^2 \mu_y^2 \right] + \left[-g'' \cdot g' \cdot \mu \cdot \mu_{xc} \right] \cdot \left[(f')^2 + \mu^2 \mu_{xc}^2 \right] \end{aligned} \right\} \cdot \alpha^3$$

* ON RETOURNE SUR L'ÉQUATION DE BASE $f(x) = \lambda \cdot x \quad f'(x) = \lambda \quad f''(x) = 0$
 $g(y) = \lambda \cdot y \quad g'(y) = \lambda \quad g''(y) = 0$

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$$\Delta \tilde{g} = \frac{\mu [\mu^2 u_y^2 + \chi^2] \cdot u_{xc} u_c + \mu [\mu^2 u_{xc}^2 + \chi^2] \cdot u_{yy}}{\chi \cdot [\chi^2 + \mu^2 (u_y^2 + u_{xc}^2)]^{3/2}}$$

$$- \frac{2 \mu^3 \cdot u_{xc} u_y u_{xcy}}{\chi [\chi^2 + \mu^2 (u_y^2 + u_{xc}^2)]^{3/2}} + \frac{\chi [\mu_y u_y + \mu_{xc} u_{xc}]}{[\chi^2 + \mu^2 (u_y^2 + u_{xc}^2)]^{3/2}}$$

PARTIE 2

EGALITE ENTRE CHANGEMENT DE BASE ET UN GRAPHE MODIFIÉ PAR $\mu(x,y)$

- $(h(x) ; g(y) ; \hat{u}(h(x),g(y)))$ changement de base dans un graphe
- Graphe modifié par $\mu(x,y)$

C'est-à-dire $(\chi.x ; \chi.y ; \tilde{u}(x,y))$

Où $\tilde{u}_x(x,y) = \mu(x,y) \cdot u_x(x,y)$

Et $\tilde{u}_y(x,y) = \mu(x,y) u_y(x,y)$

CONDITION D'UNICITE DE $\mu(x,y)$

(si il y a égalité entre les dérivées premières de $\hat{u}(h(x),g(y))$ et $\tilde{u}(x,y)$)

CALCUL $d\hat{r}^2(x,y)$

CONDITION D'UNICITE DE $\mu(x,y)$

(si il y a égalité entre les dérivées secondes de $\hat{u}(h(x),g(y))$ et $\tilde{u}(x,y)$)

FORME DE $\mu(x,y)$

PARTIE 2

SUITE : MÉTHODES GÉOMÉTRIQUES EN ANALYSE NON-LINÉAIRE
DES FLUIDES DANS UN MILIEU POREUX.

(I) AU DÉPART $\mathcal{R}(x, y) := (x; y; \mu(x, y))$

(II) LA SUITE $\tilde{\mathcal{R}}(x, y) := (\chi \cdot x; \chi \cdot y; \hat{\mu}(x, y))$

$\hat{x} := \chi \cdot x$ $\hat{y} := \chi \cdot y$ ON NE CONNAÎT PAS LA DÉFINITION

EXACTE DE $\hat{\mu}(x, y)$ MAIS ON A :

$$\hat{\mu}_x(x, y) := \mu(x, y) \cdot \mu_x(x, y)$$

$$\hat{\mu}_y(x, y) := \mu(x, y) \cdot \mu_y(x, y)$$

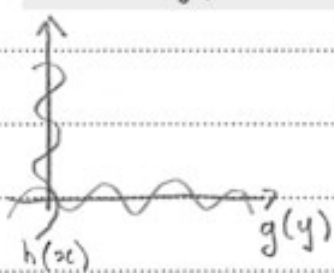
(II BIS) NOUS ON PREND $\hat{\mathcal{R}}(x, y) := (h(x); g(y); \hat{\mu}(h(x), g(y)))$

ON A DONC $\hat{x} = h(x)$ et $\hat{y} = g(y)$

$$\text{où } \hat{\mu}_x(h(x), g(y)) = h'(x) \mu_x(h(x), g(y))$$

$$\hat{\mu}_y(h(x), g(y)) = g'(y) \mu_y(h(x), g(y))$$

$$\hat{\mu}(x, y) = \mu(h(x), g(y))$$



$$\hat{\mathcal{R}}_x(x, y) = (h'(x); 0; \mu_x(h(x), g(y)) \cdot h'(x))$$

$$\hat{\mathcal{R}}_y(x, y) = (0; g'(y); \mu_y(h(x), g(y)) \cdot g'(y))$$

$$\text{CONDITION } (\hat{\mathcal{R}}_x)_y = (\hat{\mathcal{R}}_y)_x$$

$$\mu_{xy}(h(x), g(y)) \cdot h'(x) \cdot g'(y) = \mu_{yx}(h(x), g(y)) \cdot g'(y) \cdot h'(x)$$

$$\mu_{xy}(h(x), g(y)) = \mu_{yx}(h(x), g(y))$$

$$(\hat{\mathcal{R}}_x)_y = (\hat{\mathcal{R}}_y)_x \quad \square$$

(*) POUR RETROUVER $\tilde{\mathcal{R}}_x := (\chi, 0, \mu(x, y) \cdot \mu_x(x, y))$

$\tilde{\mathcal{R}}_y := (0, \chi, \mu(x, y) \cdot \mu_y(x, y))$

ON A $h'(x) = g'(y) = \chi$

$h(x) = \chi \cdot x$ et $g(y) = \chi \cdot y$

PAR AILLEURS IL FAUT AUSSI $\hat{\mu}_x(x, y) = \tilde{\mu}_x(x, y)$

$$\hat{\mu}_y(x, y) = \tilde{\mu}_y(x, y)$$

$$\hat{\mu}_x(x, y) = \mu_x(h(x), g(y)) \cdot h'(x) = \mu(x, y) \cdot \mu_{0x}(x, y) = \tilde{\mu}_x(x, y)$$

$$\hat{\mu}_y(x, y) = \mu_y(h(x), g(y)) \cdot g'(y) = \mu(x, y) \cdot \mu_{0y}(x, y) = \tilde{\mu}_y(x, y)$$

$$\mu_x(\lambda \cdot x, \lambda \cdot y) \cdot \lambda = \mu(x, y) \cdot \mu_{0x}(x, y)$$

$$\mu_y(\lambda \cdot x, \lambda \cdot y) \cdot \lambda = \mu(x, y) \cdot \mu_{0y}(x, y)$$

$$\mu(x, y) = \frac{\lambda \cdot \mu_x(\lambda \cdot x, \lambda \cdot y)}{\mu_{0x}(x, y)} \quad \text{et} \quad \mu(x, y) = \frac{\lambda \cdot \mu_y(\lambda \cdot x, \lambda \cdot y)}{\mu_{0y}(x, y)}$$

ON OBTIENT LA CONDITION POUR L'EXISTENCE DE $\mu(x, y)$

$$\frac{\mu_x(\lambda \cdot x, \lambda \cdot y)}{\mu_y(\lambda \cdot x, \lambda \cdot y)} = \frac{\mu_{0x}(x, y)}{\mu_{0y}(x, y)} \quad \square$$

SI ON A $h(x)$ ET $g(y)$ QUELCONQUE :

$$\mu_x(h(x), g(y)) \cdot h'(x) = \mu(x, y) \cdot \mu_{0x}(x, y)$$

$$\mu_y(h(x), g(y)) \cdot g'(y) = \mu(x, y) \cdot \mu_{0y}(x, y)$$

ON OBTIENT UNE AUTRE CONDITION POUR L'EXISTENCE DE $\mu(x, y)$

$$\frac{h'(x) \cdot \mu_x(h(x), g(y))}{g'(y) \cdot \mu_y(h(x), g(y))} = \frac{\mu_{0x}(x, y)}{\mu_{0y}(x, y)} \quad \square$$

$$\textcircled{*} \hat{\mu}(x, y) = (h(x); g(y); \mu(h(x), g(y)))$$

$$\hat{\mu}_x(x, y) = (h'(x), 0, \mu_x(h(x), g(y)) \cdot h'(x))$$

$$\hat{\mu}_y(x, y) = (0, g'(y), \mu_y(h(x), g(y)) \cdot g'(y))$$

$$\hat{\mu}_{xy}(x, y) = \hat{\mu}_{yx}(x, y) = (0, 0, \mu_{xy}(h(x), g(y)) \cdot h'(x) \cdot g'(y))$$

$$d\hat{r}^2(x, y) = |\hat{r}_x|^2 dx^2 + 2 \langle \hat{r}_x, \hat{r}_y \rangle dx dy + |\hat{r}_y|^2 dy^2$$

on pose $h = h(x)$ $g = g(y)$ $h' = h'(x)$ $g' = g'(y)$

$$d\hat{r}^2(x, y) = \left\{ [h']^2 + [\mu_x(h, g) \cdot h']^2 \right\} dx^2$$

$$+ \left\{ [g']^2 + [\mu_y(h, g) \cdot g']^2 \right\} dy^2$$

$$+ 2 \cdot \left\{ \mu_x(h, g) \cdot h' \cdot \mu_y(h, g) \cdot g' \right\} dx dy$$

$$d\hat{r}^2(x, y) = [h'(x)]^2 \cdot (1 + \mu_x^2(h(x), g(y))) dx^2$$

$$+ [g'(y)]^2 \cdot (1 + \mu_y^2(h(x), g(y))) dy^2$$

$$+ 2 \cdot h'(x) \cdot g'(y) \mu_x(h(x), g(y)) \mu_y(h(x), g(y)) dx dy$$

au lieu de $d\hat{r}^2(x, y) = (\chi^2 + \mu^2 \mu_x^2) dx^2 + 2 \mu^2 \mu_x \mu_y dx dy$
 avec $\mu = \mu(x, y)$ } $+ (\chi^2 + \mu^2 \mu_y^2) dy^2$
 $\mu = \mu(x, y)$ }

RAPPORT ENTRE $d\hat{r}^2(x, y)$ ET $d\tilde{r}^2(x, y)$

$$\hat{\mu}_x(x, y) = \tilde{\mu}_x(x, y) \Rightarrow \mu_x(h(x), g(y)) \cdot h'(x)$$

$$= \mu(x, y) \cdot \mu_x(x, y)$$

avec $h'(x) = \chi$ et $g'(y) = \chi$

$$\hat{\mu}_y(x, y) = \tilde{\mu}_y(x, y) \Rightarrow \mu_y(h(x), g(y)) \cdot g'(y)$$

$$= \mu(x, y) \cdot \mu_y(x, y)$$

en remplaçant dans $d\hat{r}^2(x, y)$ on RETOURNE SUR $d\tilde{r}^2(x, y)$

$$d\hat{r}^2(x, y) = \left[(h')^2 + (\mu_x(h, g) \cdot h')^2 \right] dx^2 + \left[(g')^2 + (\mu_y(h, g) \cdot g')^2 \right] dy^2$$

$$+ 2 \cdot h' \cdot g' \cdot \mu_x(h, g) \cdot \mu_y(h, g) dx dy$$

$$d\tilde{r}^2(x, y) = \left[\chi^2 + (\mu(x, y) \mu_x(x, y))^2 \right] dx^2 + \left[\chi^2 + (\mu(x, y) \mu_y(x, y))^2 \right] dy^2$$

$$+ 2 \mu(x, y)^2 \mu_x(x, y) \mu_y(x, y) dx dy$$

* VECTEUR NORMAL UNITAIRE: $\hat{N} := \frac{\hat{r}_x \times \hat{r}_y}{\|\hat{r}_x \times \hat{r}_y\|}$

$$\|\hat{r}_x \times \hat{r}_y\| = \left[(h' \cdot g')^2 + (h' \cdot g' \cdot u_y)^2 + (h' \cdot g' \cdot u_x)^2 \right]^{1/2}$$

$$\hat{r}_x \times \hat{r}_y = (h' \cdot g', h' \cdot g' \cdot u_y, h' \cdot g' \cdot u_x)$$

$$\hat{N} = \frac{-u_y(h, g) \cdot h' \cdot g' \cdot j - u_x(h, g) h' \cdot g' \cdot i + h' \cdot g' \cdot k}{\left[(h' \cdot g')^2 + (h' \cdot g' \cdot u_y(h, g))^2 + (h' \cdot g' \cdot u_x(h, g))^2 \right]^{1/2}}$$

$$h = h(x) \quad g = g(y) \quad h' = h'(x) \quad g' = g'(y) \quad h'' = h''(x) \quad g'' = g''(y)$$

$$u_x(h, g) = u_x(h(x), g(y)) \quad u_y(h, g) = u_y(h(x), g(y))$$

$$u_{xx}(h, g) = u_{xx}(h(x), g(y)) \quad u_{yy}(h, g) = u_{yy}(h(x), g(y))$$

$$u_{xy}(h, g) = u_{xy}(h(x), g(y))$$

$$\hat{r}_x(x, y) = (h', 0, u_x \cdot h') \quad \hat{r}_y(x, y) = (0, g', u_y \cdot g')$$

$$\hat{r}_{xx}(x, y) = (h'', 0, u_{xx} \cdot (h')^2 + u_x \cdot h'')$$

$$\hat{r}_{yy}(x, y) = (0, g'', u_{yy} \cdot (g')^2 + u_y \cdot g'')$$

$$\hat{r}_{xy}(x, y) = (0, 0, u_{xy} \cdot h' \cdot g')$$

$$\sqrt{\dots} = \left[(h' \cdot g')^2 + (h' \cdot g' \cdot u_y(h, g))^2 + (h' \cdot g' \cdot u_x(h, g))^2 \right]^{1/2}$$

$$\hat{h}_{xx} = \left\langle \left(\frac{-u_x \cdot h' \cdot g'}{\sqrt{\dots}}, \frac{-u_y \cdot h' \cdot g'}{\sqrt{\dots}}, \frac{h' \cdot g'}{\sqrt{\dots}} \right), \right.$$

$$\left. \left(h'', 0, u_{xx} \cdot (h')^2 + u_x \cdot h'' \right) \right\rangle$$

$$\hat{h}_{1z} = \left\langle \left(\frac{-u_x \cdot h' \cdot g'}{\sqrt{\dots}}, \frac{-u_y \cdot h' \cdot g'}{\sqrt{\dots}}, \frac{h' \cdot g'}{\sqrt{\dots}} \right), \right. \\ \left. (0, 0, u_{xy} \cdot h' \cdot g') \right\rangle$$

$$\hat{h}_{2z} = \left\langle \left(\frac{-u_x \cdot h' \cdot g'}{\sqrt{\dots}}, \frac{-u_y \cdot h' \cdot g'}{\sqrt{\dots}}, \frac{h' \cdot g'}{\sqrt{\dots}} \right), \right. \\ \left. (0, g'', u_{yy} \cdot (g')^2 + u_y \cdot g'') \right\rangle$$

$$\hat{h}_{11} = \frac{-u_x \cdot h'' \cdot h' \cdot g' + u_{xx} (h')^3 \cdot g' + u_{xx} \cdot h'' \cdot h' \cdot g'}{\sqrt{\dots}}$$

$$\hat{h}_{11} = \frac{u_{xx}(h(x), g(y)) \cdot (h'(x))^3 \cdot g'(y)}{[(h' \cdot g')^2 + (h' \cdot g' \cdot u_y(h, g))^2 + (h' \cdot g' \cdot u_{xx}(h, g))^2]^{1/2}}$$

$$\hat{h}_{12} = \frac{u_{xy}(h(x), g(y)) \cdot (h'(x) \cdot g'(y))^2}{[(h' \cdot g')^2 + (h' \cdot g' \cdot u_y(h, g))^2 + (h' \cdot g' \cdot u_{xx}(h, g))^2]^{1/2}}$$

$$\hat{h}_{22} = \frac{-u_y \cdot g'' \cdot h' \cdot g' + u_{yy} \cdot (g')^3 \cdot h' + u_y \cdot g'' \cdot h' \cdot g'}{\sqrt{\dots}}$$

$$\hat{h}_{22} = \frac{u_{yy}(h(x), g(y)) \cdot (g'(y))^3 \cdot h'(x)}{[(h' \cdot g')^2 + (h' \cdot g' \cdot u_y(h, g))^2 + (h' \cdot g' \cdot u_{xx}(h, g))^2]^{1/2}}$$

⊗ GÉNÉRALISATION $h(x)$ et $g(y)$ quelconque

$$\hat{u}_{yy} = (u_y(h(x), g(y)) \cdot g'(y))_y$$

$$\hat{u}_{yy} = u_{yy}(h(x), g(y)) \cdot (g'(y))^2 + u_y(h(x), g(y)) \cdot g''(y)$$

pareil pour \hat{u}_{xx} ($h(x), g(y)$)

$$\hat{u}_{xx} = u_{xx}(h(x), g(y)) \cdot (h'(x))^2 + u_x(h(x), g(y)) \cdot h''(x)$$

généralisation

Condition d'existence et d'unicité (dérivées secondes) de $\mu(x, y)$

$$\frac{u_{xx}(h(x), g(y)) \cdot (h'(x))^2 + u_x(h(x), g(y)) \cdot h''(x)}{u_{yy}(h(x), g(y)) \cdot (g'(y))^2 + u_y(h(x), g(y)) \cdot g''(y)}$$

$$= \frac{\mu_x(x, y) \cdot \mu_x(x, y) + \mu(x, y) \cdot \mu_{xx}(x, y)}{\mu_y(x, y) \cdot \mu_y(x, y) + \mu(x, y) \cdot \mu_{yy}(x, y)}$$

$$= \frac{\mu_x(x, y) \cdot \mu_x(x, y) + \mu(x, y) \cdot \mu_{xx}(x, y)}{\mu_y(x, y) \cdot \mu_y(x, y) + \mu(x, y) \cdot \mu_{yy}(x, y)}$$

$$= \frac{\mu_x(x, y) \cdot \mu_x(x, y) + \mu(x, y) \cdot \mu_{xx}(x, y)}{\mu_y(x, y) \cdot \mu_y(x, y) + \mu(x, y) \cdot \mu_{yy}(x, y)}$$

⊗ GÉNÉRALISATION $h''(x) \neq g''(y) \neq 0$ et $h'(x) \neq g'(y) \neq 0$

$h(x)$ et $g(y)$ quelconque

$$\left[(h' \cdot g')^2 + (h' \cdot g' \cdot u_y(h, g))^2 + (h' \cdot g' \cdot u_x(h, g))^2 \right]^{1/2}$$

$$= \left[\chi^2 + \mu^2 (u_{xx}^2(x, y) + u_{yy}^2(x, y)) \right]^{1/2}$$

$$\mu^2(x, y) = \frac{(h' \cdot g')^2 - \chi^2 + (h' \cdot g' \cdot u_y(h, g))^2 + (h' \cdot g' \cdot u_x(h, g))^2}{u_x^2(x, y) + u_y^2(x, y)}$$

$$\mu^2(x, y) = \frac{(h' \cdot g')^2 [1 + u_y^2(h, g) + u_x^2(h, g)] - \chi^2}{u_x^2(x, y) + u_y^2(x, y)}$$

PARTIE 3

GÉNÉRALISATION DE CHANGEMENT DE BASE

- Changement de base dans un graphe

$$\tilde{r}(x,y)=(f(x);g(y);\tilde{u}(x,y))$$

$$\text{avec : } \tilde{u}_x(x,y)=\mu(x,y).u_x(x,y)$$

$$\tilde{u}_y(x,y)=\mu(x,y).u_y(x,y)$$

- au lieu du graphe $(h(x);p(y);\hat{u}(h(x),p(y)))$

$$\text{avec : } \hat{u}_x(h(x),p(y))=h'(x).u_x(h(x),p(y))$$

$$\hat{u}_y(h(x),p(y))=p'(y).u_y(h(x),p(y))$$

CONDITION D'EXISTENCE ET D'UNICITE DE

$\mu(x,y)$

CALCUL DE $d\hat{r}^2(x,y)$

FORME DE $\mu(x,y)$

$$\mu^2(x, y) = \frac{(h'(x) \cdot g'(y))^2 \cdot [1 + \mu_y^2(h(x), g(y)) + \mu_x^2(h(x), g(y))] - \chi^2}{\mu_x^2(x, y) + \mu_y^2(x, y)}$$

$$\mu(x, y) = \pm \left\{ \frac{(h'(x) \cdot g'(y))^2 \cdot [1 + \mu_y^2(h(x), g(y)) + \mu_x^2(h(x), g(y))] - \chi^2}{\mu_x^2(x, y) + \mu_y^2(x, y)} \right\}^{1/2}$$

PARTIE 3

* GÉNÉRALISATION : (comparaison)

$$\hat{\mu}(x, y) := (f(x); g(y); \hat{\mu}(x, y))$$

$$\text{avec } \hat{\mu}_x(x, y) = \mu(x, y) \cdot \mu_x(x, y)$$

$$\hat{\mu}_y(x, y) = \mu(x, y) \cdot \mu_y(x, y)$$

$$\hat{\mu}(x, y) := (h(x); p(y); \hat{\mu}(h(x), p(y)))$$

$$\text{avec } \hat{\mu}_x(h(x), p(y)) = h'(x) \cdot \mu_x(h(x), p(y))$$

$$\hat{\mu}_y(h(x), p(y)) = p'(y) \cdot \mu_y(h(x), p(y))$$

$$\hat{\mu}_x(x, y) = (h'(x); 0; h'(x) \cdot \mu_x(h(x), p(y)))$$

$$\hat{\mu}_y(x, y) = (0; p'(y); p'(y) \cdot \mu_y(h(x), p(y)))$$

$$(\hat{\mu}_x(x, y))_y = (\hat{\mu}_y(x, y))_x$$

$$\hat{\mu}_{xy}(x, y) = (0; 0; \mu_{xy}(h(x), p(y)) \cdot h'(x) \cdot p'(y))$$

$$\text{Il faut que } \hat{\mu}_x(x, y) = \hat{\mu}_{yx}(x, y)$$

$$\hat{\mu}_y(x, y) = \hat{\mu}_{xy}(x, y)$$

Condition d'existence et d'unicité (dérivées premières) de $\mu(x, y)$

$$\frac{h'(x) \cdot \mu_x(h(x), p(y))}{p'(y) \cdot \mu_y(h(x), p(y))} = \frac{\mu_x(x, y)}{\mu_y(x, y)}$$

$$d\hat{r}^2(x, y) = [h'(x)]^2 \cdot (1 + u_x^2(h(x), p(y))) dx^2 + [p'(y)]^2 \cdot (1 + u_y^2(h(x), p(y))) dy^2 + 2 \cdot h'(x) \cdot p'(y) \cdot u_x(h(x), p(y)) u_y(h(x), p(y)) dx dy$$

$$\hat{N} = \frac{\hat{r}_x \times \hat{r}_y}{\|\hat{r}_x \times \hat{r}_y\|}$$

$$\|\hat{r}_x \times \hat{r}_y\| = \left[(h'(x) \cdot p'(y))^2 + (h'(x) \cdot p'(y) \cdot u_y(h(x), p(y)))^2 + (h'(x) \cdot p'(y) \cdot u_x(h(x), p(y)))^2 \right]^{1/2}$$

$$\hat{r}_x \times \hat{r}_y = (h' \cdot p' ; h' \cdot p' \cdot u_y(h, p) ; h' \cdot p' \cdot u_x(h, p))$$

$$\hat{N} = \frac{-u_y(h, p) \cdot h' \cdot p' \cdot j - u_x(h, p) \cdot h' \cdot p' \cdot i + h' \cdot p' \cdot k}{\left[(h' \cdot p')^2 + (h' \cdot p' \cdot u_y(h, p))^2 + (h' \cdot p' \cdot u_x(h, p))^2 \right]^{1/2}}$$

$$\hat{r}_x(x, y) = (h' ; 0 ; u_{xx}(h, p) \cdot h')$$

$$\hat{r}_y(x, y) = (0 ; p' ; u_{yy}(h, p) \cdot p')$$

$$\hat{r}_{xx}(x, y) = (h'' ; 0 ; u_{x^2x}(h, p) \cdot (h')^2 + u_{xx}(h, p) \cdot h'')$$

$$\hat{r}_{yy}(x, y) = (0 ; p'' ; u_{y^2y}(h, p) \cdot (p')^2 + u_{yy}(h, p) \cdot p'')$$

$$\hat{r}_{xy}(x, y) = (0 ; 0 ; u_{x^2y}(h, p) \cdot h' \cdot p')$$

$$\sqrt{\dots} = \left[(h' \cdot p')^2 + (h' \cdot p' \cdot u_y(h, p))^2 + (h' \cdot p' \cdot u_x(h, p))^2 \right]^{1/2}$$

$$\hat{h}_{xy} = \left\langle \left(\frac{-u_x(h, p) \cdot h' \cdot p'}{\sqrt{\dots}} ; \frac{-u_y(h, p) \cdot h' \cdot p'}{\sqrt{\dots}} ; \frac{h' \cdot p'}{\sqrt{\dots}} \right), \right.$$

$$\left. \left(h'' ; 0 ; u_{x^2x}(h, p) \cdot (h')^2 + u_{xx}(h, p) \cdot h'' \right) \right\rangle$$

$$\hat{h}_{12} = \dots$$

$$\hat{h}_{22} = \dots$$

ON OBTIENT :

$$\hat{h}_{11} = \frac{\mu_{xx}(h(x), p(y)) \cdot (h'(x))^3 \cdot p'(y)}{\sqrt{\dots}}$$

$$\hat{h}_{12} = \frac{\mu_{xy}(h(x), p(y)) (h'(x) \cdot p'(y))^2}{\sqrt{\dots}}$$

$$\hat{h}_{22} = \frac{\mu_{yy}(h(x), p(y)) \cdot (p'(y))^3 \cdot h'(x)}{\sqrt{\dots}}$$

$$\textcircled{*} \tilde{\alpha}(x, y) = (f(x); g(y); \tilde{\mu}(x, y))$$

$$\tilde{\alpha}_x(x, y) = (f'(x); 0; \mu(x, y) \cdot \mu_x(x, y))$$

$$\tilde{\alpha}_y(x, y) = (0; g'(y); \mu(x, y) \cdot \mu_y(x, y))$$

$$(\tilde{\alpha}_x \times \tilde{\alpha}_y) = (f'(x) \cdot g'(y); f'(x) \mu(x, y) \mu_y(x, y); \dots; g'(y) \mu(x, y) \mu_x(x, y))$$

$$\|\tilde{\alpha}_x \times \tilde{\alpha}_y\| = \left[(f'(x) \cdot g'(y))^2 + (f'(x) \cdot \mu(x, y) \mu_y(x, y))^2 + \dots + (g'(y) \cdot \mu(x, y) \mu_x(x, y))^2 \right]^{1/2}$$

$$\left[(h'(x) \cdot p'(y))^2 + (h'(x) \cdot p'(y) \cdot \mu_y(h(x), p(y)))^2 + \dots + (h'(x) \cdot p'(y) \cdot \mu_{xx}(h(x), p(y)))^2 \right]^{1/2}$$

$$= \left[(f'(x) \cdot g'(y))^2 + \mu^2(x, y) \cdot \left[(f'(x) \cdot \mu_y(x, y))^2 + (g'(y) \cdot \mu_x(x, y))^2 \right] \right]^{1/2}$$

$$\mu^2(x, y) = \pm \left\{ \frac{(h' \cdot p')^2 \cdot [1 + u_x^2(h, p) + u_y^2(h, p)] - (f' \cdot g')^2}{(f' \cdot u_y(x, y))^2 + (g' \cdot u_x(x, y))^2} \right\}^{1/2}$$

FORME DE $\mu(x, y)$

$$\mu(x, y) = \pm \left\{ \frac{(h'(x) \cdot p'(y))^2 \cdot [1 + u_x^2(h(x), p(y)) + u_y^2(h(x), p(y))] - (f'(x) \cdot g'(y))^2}{(f'(x) \cdot u_y(x, y))^2 + (g'(y) \cdot u_x(x, y))^2} \right\}^{1/2}$$

PARTIE 4

CONDITION D'EGALITE DES DERIVEES SECONDES

- $\hat{u}_{xx}(h(x),g(y))=\tilde{u}_{xx}(x,y)$
- $\hat{u}_{yy}(h(x),g(y))=\tilde{u}_{yy}(x,y)$
- $\hat{u}_{xy}(h(x),g(y))=\tilde{u}_{xy}(x,y)$

CONDITION D'EXISTENCE ET D'UNICITE DE $\mu(x,y)$

CALCUL DE $\Delta\hat{g}=\text{trace}(\hat{g}^{ij}.\hat{h}_{ij})$

PARTIE 4

$$\textcircled{*} \text{ ON POSE } \mu_x = \mu_x(x, y) \quad \mu_y = \mu_y(x, y) \quad \mu_{xx} = \mu_{xx}(x, y)$$

$$\mu_{xy} = \mu_{xy}(x, y) \quad \mu_{yy} = \mu_{yy}(x, y)$$

$$\mu_x(h, g) = \mu_x(h(x), g(y)) \quad \mu_y(h, g) = \mu_y(h(x), g(y))$$

$$h' = h'(x) \quad g' = g'(y) \quad h''(x) = h'' \quad g''(y) = g''$$

$$\textcircled{A} \mu_x(x, y) \cdot \mu_{xx}(x, y) + \mu(x, y) \cdot \mu_{xxx}(x, y)$$

$$= \mu_{xx}(h(x), g(y)) \cdot (h'(x))^2 + \mu_x(h(x), g(y)) \cdot h''(x)$$

$$\textcircled{*} \left(\mu(x, y) \cdot \mu_{xx}(x, y) \right)_y = \mu(x, y) \cdot \mu_{xxy}(x, y) + \mu_y(x, y) \cdot \mu_{xx}(x, y)$$

$$= \mu_{xy}(h(x), g(y)) \cdot h'(x) \cdot g'(y) \quad \textcircled{I}$$

$$\hat{\mu}_{xx}(x, y) = \mu_x(h(x), g(y)) \cdot h'(x)$$

$$\hat{\mu}_{xy}(x, y) = \mu_{xy}(h(x), g(y)) \cdot h'(x) \cdot g'(y)$$

$$\textcircled{II} \mu(x, y) \cdot \mu_{xy}(x, y) + \mu_x(x, y) \cdot \mu_y(x, y)$$

$$= \mu_{xy}(h(x), g(y)) \cdot h'(x) \cdot g'(y)$$

$$\textcircled{I} + \textcircled{II} \Rightarrow \mu_x(x, y) \cdot \mu_y(x, y) = \mu_y(x, y) \cdot \mu_{xx}(x, y)$$

$$\textcircled{B} \mu_y(x, y) \cdot \mu_{yy}(x, y) + \mu(x, y) \cdot \mu_{yyy}(x, y)$$

$$= \mu_{yy}(h(x), g(y)) \cdot (g'(y))^2 + \mu_y(h(x), g(y)) \cdot g''(y)$$

$$\textcircled{A} \mu(x, y) = \frac{\mu_{xx}(h, g) \cdot (h')^2 + \mu_{xx}(h, g) \cdot h'' - \mu_x \cdot \mu_{xx}}{\mu_{xx}(x, y)}$$

$$\textcircled{B} \mu(x, y) = \frac{\mu_{yy}(h, g) \cdot (g')^2 + \mu_{yy}(h, g) \cdot g'' - \mu_y \cdot \mu_{yy}}{\mu_{yy}(x, y)}$$

CONDITION D'EXISTENCE ET D'UNICITE DE $\mu(x, y)$

$$\frac{\mu_{yy}(x, y)}{\mu_{xx}(x, y)} = \frac{\mu_{yy}(h(x), g(y)) \cdot (g'(y))^2 + \mu_{yy}(h(x), g(y)) \cdot g''(y) - \mu_y(x, y) \cdot \mu_{yy}(x, y)}{\mu_{xx}(h(x), g(y)) \cdot (h'(x))^2 + \mu_{xx}(h(x), g(y)) \cdot h''(x) - \mu_x(x, y) \cdot \mu_{xx}(x, y)}$$

$$\hat{g}_{ij} = \begin{pmatrix} (h'(x))^2 + (u_x(h(x), g(y)) \cdot h'(x))^2 & u_x(h(x), g(y)) \cdot u_y(h(x), g(y)) \cdot h'(x) \cdot g'(y) \\ u_x(h(x), g(y)) \cdot u_y(h(x), g(y)) \cdot h'(x) \cdot g'(y) & (g'(y))^2 + (u_y(h(x), g(y)) \cdot g'(y))^2 \end{pmatrix}$$

$$\det \hat{g}_{ij} = (h'(x) \cdot g'(y))^2 + (h'(x) \cdot g'(y))^2 \cdot [(u_x(h(x), g(y)))^2 + (u_y(h(x), g(y)))^2]$$

$$(\hat{g}_{ij})^{-1} = \hat{g}^{ij} = \frac{1}{\det(\hat{g}_{ij})} \cdot \begin{pmatrix} (g')^2 + (u_y \cdot g')^2 & -u_x \cdot u_y \cdot h' \cdot g' \\ -u_x \cdot u_y \cdot h' \cdot g' & (h')^2 + (u_x \cdot h')^2 \end{pmatrix}$$

$$\alpha = \left(\frac{1}{\det \hat{g}_{ij}} \right)^{\frac{1}{2}}$$

$$\hat{h}_{ij} = \alpha \cdot \begin{pmatrix} u_{xx}(h, g) \cdot (h')^3 \cdot g' & u_{xy}(h, g) \cdot (h' \cdot g')^2 \\ u_{xy}(h, g) \cdot (h' \cdot g')^2 & u_{yy}(h, g) \cdot (g')^3 \cdot h' \end{pmatrix}$$

$$\Delta \hat{g} = \text{trace}(\hat{g}^{ij} \cdot \hat{h}_{ij})$$

$$\Delta \hat{g} = \alpha^3 \cdot \left\{ [(g')^2 + (u_y \cdot g')^2] \cdot u_{xx}(h, g) \cdot (h')^3 \cdot g' \right. \\ \left. - [u_x(h, g) \cdot u_y(h, g) \cdot h' \cdot g'] \cdot u_{xy}(h, g) \cdot (h' \cdot g')^2 \right. \\ \left. - [u_x(h, g) \cdot u_y(h, g) \cdot h' \cdot g'] \cdot u_{xy}(h, g) \cdot (h' \cdot g')^2 \right. \\ \left. + [(h')^2 + (u_x \cdot h')^2] \cdot u_{yy}(h, g) \cdot (g')^3 \cdot h' \right\}$$

$$\Delta \hat{g} = \frac{\left\{ \begin{aligned} & \left[(g'(y))^2 + (u_y(h(x), g(y)) \cdot g'(y))^2 \right] \cdot u_{xx}(h(x), g(y)) \cdot (h'(x))^3 \cdot g'(y) \\ & + \left[(h'(x))^2 + (u_x(h(x), g(y)) \cdot h'(x))^2 \right] \cdot u_{yy}(h(x), g(y)) \cdot (g'(y))^3 \cdot h'(x) \\ & - 2 u_x(h(x), g(y)) \cdot u_y(h(x), g(y)) \cdot u_{xy}(h(x), g(y)) \cdot (h'(x) \cdot g'(y))^3 \end{aligned} \right\}}{\left[(h'(x) \cdot g'(y))^2 + (h'(x) \cdot g'(y) \cdot u_y(h(x), g(y)))^2 + (h'(x) \cdot g'(y) \cdot u_x(h(x), g(y)))^2 \right]^{3/2}}$$