



HAL
open science

List edge and list total colorings of planar graphs without non-induced 7-cycles

Aijun Dong, Guizhen Liu, Guojun Li

► **To cite this version:**

Aijun Dong, Guizhen Liu, Guojun Li. List edge and list total colorings of planar graphs without non-induced 7-cycles. *Discrete Mathematics and Theoretical Computer Science*, 2013, Vol. 15 no. 1 (1), pp.101–106. 10.46298/dmtcs.629 . hal-00990609

HAL Id: hal-00990609

<https://inria.hal.science/hal-00990609>

Submitted on 13 May 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

List Edge and List Total Colorings of Planar Graphs without non-induced 7-cycles[†]

Aijun Dong^{1‡}Guizhen Liu^{2§}Guojun Li^{2¶}¹School of science, Shandong Jiaotong University, Jinan 250023, P. R. China²School of Mathematics, Shandong University, Jinan 250100, P. R. Chinareceived 22nd November 2010, revised 11th October 2011, 10th July 2012, accepted 19th February 2013.

Giving a planar graph G , let $\chi'_i(G)$ and $\chi''_i(G)$ denote the list edge chromatic number and list total chromatic number of G respectively. It is proved that if G is a planar graph without non-induced 7-cycles, then $\chi'_i(G) \leq \Delta(G) + 1$ and $\chi''_i(G) \leq \Delta(G) + 2$ where $\Delta(G) \geq 7$.

Keywords: List coloring; Planar graph; Choosability.

1 Introduction

The terminology and notation used but undefined in this paper can be found in [1]. Let G be a graph and we use $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, face set, maximum degree, and minimum degree of G , respectively. Let $d_G(x)$ or simply $d(x)$, denote the degree of a vertex (resp. face) x in G . A vertex (resp. face) x is called a k -vertex (resp. k -face), k^+ -vertex (resp. k^+ -face), or k^- -vertex, if $d(x) = k$, $d(x) \geq k$, or $d(x) \leq k$. We use (d_1, d_2, \dots, d_n) to denote a face f if d_1, d_2, \dots, d_n are the degrees of vertices which are incident with the face f . If u_1, u_2, \dots, u_n are the vertices on the boundary walk of a face f , then we write $f = u_1u_2 \dots u_n$. Let $\delta(f)$ denote the minimal degree of vertices which are incident with f . We use $f_i(v)$ to denote the number of i -faces which are incident with v for each $v \in V(G)$. Let $n_i(f)$ denote the number of i -vertices which are incident with f for each $f \in F(G)$. A cycle C of length k is called k -cycle, and if there is at least one edge $xy \in E(G) \setminus E(C)$ and $x, y \in V(C)$, the cycle C is called *non-induced k -cycle*.

The mapping L is said to be a *total assignment* for a graph G if it assigns a list $L(x)$ of possible colors to each element $x \in V(G) \cup E(G)$. If G has a proper total coloring $\phi(x) \in L(x)$ for all $x \in V(G) \cup E(G)$, then we say that G is *total- L -colorable*. Let $f : V(G) \cup E(G) \rightarrow N$ where f is a function into the positive integers. We say that G is *total- f -choosability* if it is total- L -colorable for every total assignment L

[†]Supported by NSFC grants 11101243, 11161035.

[‡]E-mail: dongaijun@mail.sdu.edu.cn

[§]E-mail: gzliu@sdu.edu.cn.

[¶]E-mail: guojunsdu@gmail.com.

satisfying $|L(x)| = f(x)$ for all $x \in V(G) \cup E(G)$. The *list total coloring number* $\chi''_l(G)$ of G is the smallest integer k such that G is total- f -choosability when $f(x) = k$ for each $x \in V(G) \cup E(G)$. The *list edge coloring number* $\chi'_l(G)$ of G is defined similarly in terms of coloring edges alone; and so is the concept of *edge- f -choosability*. On the list coloring number of a graph G , there is a famous conjecture known as the List Coloring Conjecture.

Conjecture 1 For a multigraph G ,

$$(a) \chi'_l(G) = \chi'(G); \quad (b) \chi''_l(G) = \chi''(G).$$

Part (a) of Conjecture 1 was formulated independently by Vizing, by Gupta, by Albersson and Collins, and by Bollobás and Harris [6, 11]. It is well known as the *List Coloring Conjecture*. Part (b) was formulated by Borodin, Kostochka and Woodall [2]. Part (a) has been proved for bipartite multigraphs [5]. Part (a) and Part (b) have been proved for outerplanar graphs [15], and graphs with $\Delta \geq 12$ which can be embedded in a surface of nonnegative characteristic [2]. There are several related results for planar graphs, such as planar graphs without 4-cycles by Hou et al.[9], planar graphs without 4- and 5-cycles or planar graphs without intersecting 4-cycles by Liu et al.[13], planar graphs without triangles adjacent 4-cycles by Li et al.[14], planar graphs without intersecting triangles by Sheng et al.[18].

To confirm Conjecture 1 is a challenging work. From the Vizing Theorem and the Total Coloring Conjecture, the following weak conjecture is presented.

Conjecture 2 For a multigraph G ,

$$(a) \chi'_l(G) \leq \Delta(G) + 1; \quad (b) \chi''_l(G) \leq \Delta(G) + 2.$$

Part (a) of Conjecture 2 has been proved for complete graphs of odd order [7]. Wang et al. confirmed part (a) of Conjecture 2 for planar graphs without 6-cycles or without 5-cycles [17, 16]. Zhang et al. proved part (a) of Conjecture 2 for planar graphs without triangles [19]. Hou et al. proved part (a) of Conjecture 2 for planar graphs without adjacent triangles or 7-cycles [8]. Cai et al. confirmed part (a) of Conjecture 2 for planar graphs without chordal 5-cycles [3]. Part (b) of Conjecture 2 was proved by Hou et al. for planar graphs G with $\Delta(G) \geq 9$ [10]. Dong et al. confirmed Conjecture 2 for planar graphs without 6-cycles with chord [4].

In this paper, we shall show the following result.

Theorem Let G be a planar graph without non-induced 7-cycles, if $\Delta(G) \geq 7$, then $\chi'_l(G) \leq \Delta(G) + 1$ and $\chi''_l(G) \leq \Delta(G) + 2$.

2 Planar graphs without non-induced 7-cycles

First let us introduce some important lemmas.

Lemma 3 Let G be a planar graph without non-induced 7-cycles. Then there is an edge $uv \in E(G)$ such that $\min\{d(u), d(v)\} \leq \lfloor \frac{\Delta(G)+1}{2} \rfloor$ and $d(u) + d(v) \leq \max\{9, \Delta(G) + 2\}$.

Proof: Suppose to the contrary that G is a minimal counterexample to Lemma 3 in terms of the number of vertices and edges. Then we have $\delta(G) \geq 3$.

By Euler's formula $|V| - |E| + |F| = 2$ and $\sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E|$, we have

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = -6(|V| - |E| + |F|) = -12.$$

Define an initial charge function w on $V(G) \cup F(G)$ by setting $w(v) = 2d(v) - 6$ if $v \in V(G)$ and $w(f) = d(f) - 6$ if $f \in F(G)$, so that $\sum_{x \in V(G) \cup F(G)} w(x) = -12$. Now redistribute the charge according to the following discharging rules.

For convenience, let $\bar{w}(v)$ denote the total charge transferred from a vertex v to all its incident 4- and 5-faces where $d(v) = 5$.

D1 Let f be a 3-face incident with a vertex v . Then v gives f charge $\frac{4-\bar{w}(v)}{f_3(v)}$ if $d(v) = 5$, $\frac{3}{2}$ if $d(v) \geq 6$.

D2 Let f be a 4-face incident with a vertex v . Then v gives f charge $\frac{1}{2}$ if $d(v) = 4, 5$ and 6 , 1 if $d(v) \geq 7$.

D3 Let f be a 5-face incident with a vertex v . Then v gives f charge $\frac{1}{5}$ if $d(v) = 4, 5$ and 6 , $\frac{1}{3}$ if $d(v) \geq 7$.

Let the new charge of each element x be $w'(x)$ for each $x \in V(G) \cup F(G)$.

In the following, let us check the new charge $w'(x)$ of each element $x \in V(G) \cup F(G)$.

Suppose $d(v) = 3$. Then $w'(v) = w(v) = 0$.

Suppose $d(v) = 4$. Then $w(v) = 2$, $f_4(v) \leq 4$. If $2 \leq f_4(v) \leq 4$, then $f_5(v) = 0$ for G contains no non-induced 7-cycles. We have $w'(v) \geq 2 - \frac{1}{2} \times 4 = 0$ by *D2*. Otherwise, i.e. $f_4(v) \leq 1$, then $f_5(v) \leq 4$. Thus we have $w'(v) > 2 - \frac{1}{2} - \frac{1}{5} \times 4 = \frac{7}{10} > 0$ by *D2* and *D3*.

Suppose $d(v) = 5$. Then $w(v) = 4$, $f_3(v) \leq 5$. If $1 \leq f_3(v)$, then $w'(v) \geq 4 - \frac{4-\bar{w}(v)}{f_3(v)} f_3(v) - \bar{w}(v) = 0$ by *D1*. Otherwise, i.e. $f_3(v) = 0$, then $f_4(v) + f_5(v) \leq 5$. It is clear that $w'(v) > 4 - \frac{1}{2} \times 5 = \frac{3}{2} > 0$ by *D2* and *D3*.

Suppose $d(v) = 6$. Then $w(v) = 6$, $f_3(v) \leq 4$ for G contains no non-induced 7-cycles. If $f_3(v) = 4$, then $f_4(v) = 0$ and $f_5(v) = 0$ for G contains no non-induced 7-cycles. We have $w'(v) \geq 6 - \frac{3}{2} \times 4 = 0$ by *D1*. If $f_3(v) \leq 3$, then it is clear that $w'(v) > 6 - \frac{3}{2} \times 3 - \frac{1}{2} \times 3 = 0$ by *D1*, *D2* and *D3*.

Suppose $d(v) = 7$. Then $w(v) = 8$, $f_3(v) \leq 5$ for G contains no non-induced 7-cycles.

Suppose $f_3(v) = 5$. Then $f_4(v) = 0$ and $f_5(v) = 0$ for G contains no non-induced 7-cycles. We can get $w'(v) \geq 8 - \frac{3}{2} \times 5 = \frac{1}{2} > 0$ by *D1*.

Suppose $f_3(v) = 4$. Then $f_4(v) \leq 2$. If $f_4(v) = 2$, then $f_5(v) = 0$ for G contains no non-induced 7-cycles. We have $w'(v) \geq 8 - \frac{3}{2} \times 4 - 1 \times 2 = 0$ by *D1* and *D2*. If $f_4(v) \leq 1$, then $f_5(v) \leq 1$ for G contains no non-induced 7-cycles. We have $w'(v) \geq 8 - \frac{3}{2} \times 4 - 1 - \frac{1}{3} = \frac{2}{3} > 0$ by *D1*, *D2* and *D3*.

Suppose $f_3(v) = 3$. Then $f_4(v) \leq 2$ and $f_5(v) \leq 2$ for G contains no non-induced 7-cycles. It is clear that $w'(v) > 8 - \frac{3}{2} \times 3 - 1 \times 2 - \frac{1}{3} \times 2 = \frac{5}{6} > 0$ by *D1*, *D2* and *D3*.

Suppose $f_3(v) \leq 2$. Then it is clear that $w'(v) > 8 - \frac{3}{2} \times 2 - 1 \times 5 = 0$ by *D1*, *D2* and *D3*.

Suppose $d(v) = 8$. Then $w(v) = 10$, $f_3(v) \leq 6$ for G contains no non-induced 7-cycles. If $f_3(v) = 6$, then $f_4(v) = 0$ and $f_5(v) = 0$ for G contains no non-induced 7-cycles. We have $w'(v) \geq 10 - \frac{3}{2} \times 6 = 1 > 0$ by *D1*. If $f_3(v) = 5$, then $f_4(v) \leq 1$ and $f_5(v) \leq 1$ for G contains no non-induced 7-cycles. We can get $w'(v) \geq 10 - \frac{3}{2} \times 5 - 1 - \frac{1}{3} = \frac{7}{6} > 0$ by *D1*, *D2* and *D3*. If $f_3(v) \leq 4$, then it is clear that $w'(v) \geq 10 - \frac{3}{2} \times 4 - 1 \times 4 = 0$ by *D1*, *D2* and *D3*.

Suppose $d(v) = 9$. Then $w(v) = 12$, $f_3(v) \leq 7$ for G contains no non-induced 7-cycles. If $f_3(v) = 7$, then $f_4(v) = 0$ and $f_5(v) = 0$ for G contains no non-induced 7-cycles. We can get $w'(v) \geq 12 - \frac{3}{2} \times 7 = \frac{3}{2} > 0$ by *D1*. If $f_3(v) \leq 6$, then it is clear that $w'(v) > 12 - \frac{3}{2} \times 6 - 1 \times 3 = 0$ by *D1*, *D2* and *D3*.

Suppose $d(v) \geq 10$. Then $w(v) = 2d(v) - 6$, $f_4(v) + f_5(v) \leq d(v) - f_3(v)$. Thus we have $w'(v) \geq 2d(v) - 6 - \frac{3}{2}f_3(v) - f_4(v) - \frac{1}{3}f_5(v) \geq d(v) - 6 - \frac{1}{2}f_3(v)$ by *D1*, *D2* and *D3*. Since $f_3(v) \leq \frac{4}{5}d(v)$, we have $w'(v) \geq \frac{3}{5}d(v) - 6 \geq 0$.

Suppose $d(f) = 3$. Then $w(f) = -3$.

Suppose $\delta(f) = 3$. Then f is a $(3, 7^+, 7^+)$ -face by assumption. We have $w'(f) = -3 + \frac{3}{2} \times 2 = 0$ by *D1*.

Suppose $\delta(f) = 4$. Then f is a $(4, 6^+, 6^+)$ -face by assumption. We have $w'(f) = -3 + \frac{3}{2} \times 2 = 0$ by *D1*.

Suppose $\delta(f) = 5$. Then f is a $(5, 5^+, 5^+)$ -face.

Suppose f is a $(5, 5, 5)$ -face. For convenience, let $f = uvw$. Of the three vertices u, v and w , there is at most one vertex which is incident with at least four 3-faces for the reason that G contains no non-induced 7-cycles. Without loss of generality, let $f_3(u) \geq 4$. Then $f_3(u) + f_4(u) + f_5(u) \leq 5$, $f_3(v) + f_4(v) + f_5(v) \leq 3$ and $f_3(w) + f_4(w) + f_5(w) \leq 3$ for G contains no non-induced 7-cycles. We have $\frac{4-\bar{w}(v)}{f_3(u)} \geq \frac{4}{5}$, $\frac{4-\bar{w}(v)}{f_3(v)} \geq \frac{4}{3}$, $\frac{4-\bar{w}(v)}{f_3(w)} \geq \frac{4}{3}$ by *D2* and *D3*. Thus $w'(f) \geq -3 + \frac{4}{5} + \frac{4}{3} \times 2 = \frac{7}{15} > 0$ by *D1*. Now we assume that $f_3(u) \leq 3$, $f_3(v) \leq 3$, $f_3(w) \leq 3$. Then we have $\frac{4-\bar{w}(v)}{f_3(u)} \geq 1$ for G contains no non-induced 7-cycles and by *D1*, *D2* and *D3*. Thus $w'(f) \geq -3 + 1 \times 3 = 0$ by *D1*.

Suppose f is a $(5, 5, 6^+)$ -face. For convenience, let $f = uvw$ where $d(u) = d(v) = 5$. Since $f_3(u) + f_4(u) + f_5(u) \leq 5$, $f_3(v) + f_4(v) + f_5(v) \leq 5$, we have $\frac{4-\bar{w}(v)}{f_3(u)} \geq \frac{4}{5}$, $\frac{4-\bar{w}(v)}{f_3(v)} \geq \frac{4}{5}$ by *D2* and *D3*. Thus $w'(f) \geq -3 + \frac{4}{5} \times 2 + \frac{3}{2} = \frac{1}{10} > 0$ by *D1*.

Suppose f is a $(5, 6^+, 6^+)$ -face. Then we have $w'(f) > -3 + \frac{3}{2} \times 2 = 0$ by *D1*.

Suppose $\delta(f) \geq 6$. Then we have $w'(f) = -3 + \frac{3}{2} \times 3 = \frac{3}{2} > 0$ by *D1*.

Suppose $d(f) = 4$. Then $w(f) = -2$. If $\delta(f) = 3$, then f is a $(3, 3^+, 7^+, 7^+)$ -face by assumption. We have $w'(f) \geq -2 + 1 \times 2 = 0$ by *D2*. If $\delta(f) \geq 4$, then f is a $(4^+, 4^+, 4^+, 4^+)$ -face. We have $w'(f) \geq -2 + \frac{1}{2} \times 4 = 0$ by *D2*.

Suppose $d(f) = 5$. Then $w(f) = -1$.

Suppose $\delta(f) = 3$. Then $n_3(f) \leq 2$ by assumption. If $n_3(f) = 2$, then f is a $(3, 3, 7^+, 7^+, 7^+)$ -face by assumption. We have $w'(f) \geq -1 + \frac{1}{3} \times 3 = 0$ by *D3*. If $n_3(f) = 1$, then f is a $(3, 4^+, 4^+, 7^+, 7^+)$ -face by assumption. We have $w'(f) \geq -1 + \frac{1}{3} \times 2 + \frac{1}{5} \times 2 = \frac{1}{15} > 0$ by *D3*.

Suppose $\delta(f) \geq 4$. Then we have $w'(f) \geq -1 + \frac{1}{5} \times 5 = 0$ by *D3*.

Suppose $d(f) \geq 6$. Then $w'(f) = w(f) \geq 0$.

From the above discussion, we obtain $-12 = \sum_{x \in V(G) \cup F(G)} w'(x) \geq 0$, a contradiction. \square

Lemma 4 *Let G be a planar graph without non-induced 7-cycles. Then $\chi'_l(G) \leq k + 1$ and $\chi''_l(G) \leq k + 2$ where $k = \max\{\Delta(G), 7\}$.*

Proof: Suppose to the contrary that G' and G'' are minimal counterexamples to the conclusions for χ'_l and χ''_l respectively. Let L' and L'' be list assignments such that $|L'(e)| = k + 1$ for each $e \in E(G)$, G' is not edge- L' -colorable, and $|L''(x)| = k + 2$ for each $x \in V(G) \cup E(G)$, G'' is not total- L'' -colorable. By Lemma 3, G' and G'' contain an edge $uv \in E(G)$ such that $\min\{d(u), d(v)\} \leq \lfloor \frac{\Delta(G)+1}{2} \rfloor$ and $d(u) + d(v) \leq \max\{9, \Delta(G) + 2\} = k + 2$.

Let $\bar{G}' = G' - uv$. Then \bar{G}' is edge- L' -colorable by assumption. For $d(u) + d(v) \leq k + 2$, there are at most k edges which are adjacent to uv in G' . Thus there is at least one color in $L'(uv)$ which we can use to color uv . Then G' is edge- L' -colorable, a contradiction.

Let $\bar{G}'' = G'' - uv$. Then \bar{G}'' is total- L'' -colorable by assumption. With loss of generality, let $d(u) = \min\{d(u), d(v)\}$. Erase the color on u , then there is at least one color in $L''(uv)$ which we can use to color uv for $d(u) + d(v) \leq k + 2$. For $d(u) \leq \lfloor \frac{\Delta(G)+1}{2} \rfloor \leq \lfloor \frac{k+1}{2} \rfloor$, then u is adjacent to at most $\lfloor \frac{k+1}{2} \rfloor$ vertices and is incident with at most $\lfloor \frac{k+1}{2} \rfloor$ edges. Thus there is at least one color in $L''(u)$ which we can use to color u . Then G'' is total- L'' -colorable, a contradiction. From the above discussion, we have $\chi'_l(G) \leq k + 1$ and $\chi''_l(G) \leq k + 2$ where $k = \max\{\Delta(G), 7\}$. \square

By Lemma 4, it is easy to obtain the main theorem.

Theorem Let G be a planar graph without non-induced 7-cycles, if $\Delta(G) \geq 7$, then $\chi'_l(G) \leq \Delta(G) + 1$ and $\chi''_l(G) \leq \Delta(G) + 2$.

References

- [1] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, North-Holland, New York, 1976.
- [2] O. Borodin, A. Kostochka, D. Woodall, List edge and list total colourings of multigraphs, J. Combin. Theory Ser. B 71 (1997) 184–204.
- [3] J. Cai, J. Hou, X. Zhang, G. Liu, Edge-choosability of planar graphs without non-induced 5-cycles, Information Processing Letters, 109 (7) (2009) 343–346.
- [4] A. DONG, G. Liu, G. Li, List edge and list total colorings of planar graphs without 6-cycles with chord, Bull. Korean Math. Soc. 49 (2012) 359–365.
- [5] F. Galvin, The list chromatic index of a bipartite multigraph, J. Combin. Theory Ser. B 63 (1995) 153–158.
- [6] R. Hägkvist, A. Chetwynd, Some upper bounds on the total and list chromatic numbers of multigraphs, J. Graph Theory 16 (1992) 503–516.
- [7] R. Hägkvist, J. Janssen, New bounds on the list-chromatic index of the complete graph and other simple graphs, Combin. Probab. Comput. 6 (1997) 295–313.
- [8] J. Hou, G. Liu, J. Cai, Edge-choosability of planar graphs without adjacent triangles or without 7-cycle, Discrete Mathematics, 309 (2009) 77–84.
- [9] J. Hou, G. Liu, J. Cai, List Edge and List Total Colorings of Planar Graphs without 4-cycles, Theoretical Computer Science, 369 (2006) 250–255.
- [10] J. Hou, G. Liu, J. Wu, Some results on list total colorings of planar graphs, Lecture Note in Computer Science, 4489 (2007) 320–328.
- [11] T. Jensen, B. Toft, Graph Coloring Problem, Wiley-Interscience, New York, 1995.

- [12] A. Kostochka, List edge chromatic number of graphs with large girth, *Discrete Math.* 101 (1992) 189–201.
- [13] B. Liu, J. Hou, G. Liu, List edge and list total colorings of planar graphs without short cycles, *Information Processing Letters*, 108 (2008) 347–351.
- [14] R. Li, B. Xu, Edge choosability and total choosability of planar graphs without no 3-cycles adjacent 4-cycles, *Discrete Mathematics*, 311 (2011) 2158–2163.
- [15] W. Wang, K. Lih, Choosability, edge-choosability and total choosability of outerplane graphs, *European J. Combin.* 22 (2001) 71–78.
- [16] W. Wang, K. Lih, Choosability and edge choosability of planar graphs without five cycles, *Appl. Math. Lett.* (2002) 15 561–565.
- [17] W. Wang, K. Lih, Structure properties and edge choosability of planar graphs without 6-cycles, *Combin. Probab. Comput.* 10 (2001) 267–276.
- [18] H. Sheng, Y. Wang, A structural theorem for planar graphs with some applications, *Discrete Applied Mathematics*, 159 (2011) 1183–1187.
- [19] L. Zhang, B. Wu, Edge choosability of planar graphs without small cycles, *Discrete Math.* 283 (2004) 289–293.