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► **To cite this version:**

Hengzhe Li, Xueliang Li, Yuefang Sun. The generalized 3-connectivity of Cartesian product graphs. Discrete Mathematics and Theoretical Computer Science, 2012, Vol. 14 no. 1 (1), pp.43-54. 10.46298/dmtcs.572 . hal-00990569

**HAL Id: hal-00990569**

**<https://inria.hal.science/hal-00990569>**

Submitted on 13 May 2014

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# The generalized 3-connectivity of Cartesian product graphs<sup>†</sup>

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received 4<sup>th</sup> May 2011, revised 20<sup>th</sup> November 2011, 11<sup>th</sup> January 2012, , accepted 19<sup>th</sup> February 2012.

The generalized connectivity of a graph, which was introduced by Chartrand et al. in 1984, is a generalization of the concept of vertex connectivity. Let  $S$  be a nonempty set of vertices of  $G$ , a collection  $\{T_1, T_2, \dots, T_r\}$  of trees in  $G$  is said to be internally disjoint trees connecting  $S$  if  $E(T_i) \cap E(T_j) = \emptyset$  and  $V(T_i) \cap V(T_j) = S$  for any pair of distinct integers  $i, j$ , where  $1 \leq i, j \leq r$ . For an integer  $k$  with  $2 \leq k \leq n$ , the  $k$ -connectivity  $\kappa_k(G)$  of  $G$  is the greatest positive integer  $r$  for which  $G$  contains at least  $r$  internally disjoint trees connecting  $S$  for any set  $S$  of  $k$  vertices of  $G$ . Obviously,  $\kappa_2(G) = \kappa(G)$  is the connectivity of  $G$ . Sabidussi's Theorem showed that  $\kappa(G \square H) \geq \kappa(G) + \kappa(H)$  for any two connected graphs  $G$  and  $H$ . In this paper, we prove that for any two connected graphs  $G$  and  $H$  with  $\kappa_3(G) \geq \kappa_3(H)$ , if  $\kappa(G) > \kappa_3(G)$ , then  $\kappa_3(G \square H) \geq \kappa_3(G) + \kappa_3(H)$ ; if  $\kappa(G) = \kappa_3(G)$ , then  $\kappa_3(G \square H) \geq \kappa_3(G) + \kappa_3(H) - 1$ . Our result could be seen as an extension of Sabidussi's Theorem. Moreover, all the bounds are sharp.

**Keywords:** Connectivity, Generalized connectivity, Internally disjoint path, Internally disjoint trees.

## 1 Introduction

All graphs in this paper are undirected, finite and simple. We refer to the book [1] for graph theoretic notations and terminologies not described here. For any graph  $G$ , the connectivity  $\kappa(G)$  of a graph  $G$  is defined as  $\min\{|S| : S \subseteq V(G) \text{ and } G - S \text{ is disconnected or trivial}\}$ . Whitney [14] showed an equivalent definition of the connectivity of a graph. For each pair of vertices  $x, y$  of  $G$ , let  $\kappa(x, y)$  denote the maximum number of internally disjoint paths connecting  $x$  and  $y$  in  $G$ . Then the connectivity  $\kappa(G)$  of  $G$  is  $\min\{\kappa(x, y) : x, y \text{ are distinct vertices of } G\}$ .

The Cartesian product of graphs is an important method to construct a bigger graph, and plays a key role in design and analysis of networks. In the past several decades, many authors have studied the (edge) connectivity of the Cartesian product graphs. Specially, Sabidussi in [11] derived the following perfect and well-known theorem on the connectivity of Cartesian product graphs.

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**Theorem 1.1 (Sabidussi's Theorem [11])** *Let  $G$  and  $H$  be two connected graphs. Then*

$$\kappa(G \square H) \geq \kappa(G) + \kappa(H).$$

More information about the (edge) connectivity of the Cartesian product graphs can be found in [4, 5, 6, 11, 12, 15, 16].

The generalized connectivity of a graph  $G$ , which was introduced by Chartrand et al. in [2], is a natural and nice generalization of the concept of vertex connectivity. A tree  $T$  is called an  $S$ -tree ( $\{u_1, u_2, \dots, u_k\}$ -tree) if  $S \subseteq V(T)$ , where  $S = \{u_1, u_2, \dots, u_k\} \in V(G)$ . A family of trees  $T_1, T_2, \dots, T_r$  are *internally disjoint  $S$ -trees* if  $E(T_i) \cap E(T_j) = \emptyset$  and  $V(T_i) \cap V(T_j) = S$  for any pair of integers  $i$  and  $j$ , where  $1 \leq i < j \leq r$ . We use  $\kappa(S)$  to denote the greatest number of internally disjoint  $S$ -trees. For an integer  $k$  with  $2 \leq k \leq n$ , the  $k$ -connectivity  $\kappa_k(G)$  of  $G$  is defined as  $\min\{\kappa(S) \mid S \in V(G) \text{ and } |S| = k\}$ . Clearly, when  $|S| = 2$ ,  $\kappa_2(G)$  is nothing new but the connectivity  $\kappa(G)$  of  $G$ , that is,  $\kappa_2(G) = \kappa(G)$ , which is the reason why one addresses  $\kappa_k(G)$  as the generalized connectivity of  $G$ . By convention, for a connected graph  $G$  with less than  $k$  vertices, we set  $\kappa_k(G) = 1$ . For any graph  $G$ , clearly,  $\kappa(G) \geq 1$  if and only if  $\kappa_k(G) \geq 1$ .

In addition to being a natural combinatorial measure, the generalized connectivity can be motivated by its interesting interpretation in practice. For example, suppose that  $G$  represents a network. If one considers to connect a pair of vertices of  $G$ , then a path is used to connect them. However, if one wants to connect a set  $S$  of vertices of  $G$  with  $|S| \geq 3$ , then a tree has to be used to connect them. This kind of tree with minimum order for connecting a set of vertices is usually called a Steiner tree, and popularly used in the physical design of VLSI, see [13]. Usually, one wants to consider how tough a network can be, for the connection of a set of vertices. Then, the number of totally independent ways to connect them is a measure for this purpose. The generalized  $k$ -connectivity can serve for measuring the capability of a network  $G$  to connect any  $k$  vertices in  $G$ .

In [8], Li and Li investigated the complexity of determining the generalized connectivity and derived that for any fixed integer  $k \geq 2$ , given a graph  $G$  and a subset  $S$  of  $V(G)$ , deciding whether there are  $k$  internally disjoint trees connecting  $S$ , namely deciding whether  $\kappa(S) \geq k$ , is NP-complete. The generalized connectivity of complete bipartite graphs was studied by Okamoto and Zhang in [10], and Li and Li in [7].

Chartrand et al. [3] got the following result for complete graphs.

**Theorem 1.2** [3] *For every two integers  $n$  and  $k$  with  $2 \leq k \leq n$ ,  $\kappa_k(K_n) = n - \lceil k/2 \rceil$ .*

**Theorem 1.3** [9] *Let  $G$  be a connected graph with at least three vertices. If  $G$  has two adjacent vertices with minimum degree  $\delta$ , then  $\kappa_3(G) \leq \delta - 1$ .*

**Theorem 1.4** [9] *For any connected graph  $G$ ,  $\kappa_3(G) \leq \kappa(G)$ . Moreover, the upper bound is sharp.*

In this paper, we study the 3-connectivity of Cartesian product graphs and get the following result.

**Theorem 1.5** *Let  $G$  and  $H$  be connected graphs such that  $\kappa_3(G) \geq \kappa_3(H)$ . The following assertions hold:*

- (i) *If  $\kappa(G) = \kappa_3(G)$ , then  $\kappa_3(G \square H) \geq \kappa_3(G) + \kappa_3(H) - 1$ . Moreover, the bound is sharp;*
- (ii) *If  $\kappa(G) > \kappa_3(G)$ , then  $\kappa_3(G \square H) \geq \kappa_3(G) + \kappa_3(H)$ . Moreover, the bound is sharp.*

The paper is organized as follows. In Section 2, we recall the definition and properties of Cartesian product graphs, and give some basic results about the internally disjoint  $S$ -trees. As usual, in order to get a general result, we begin with a special case. In Section 3, we study the 3-connectivity of the Cartesian product of a graph  $G$  and a tree  $T$ . This section is a preparation of Section 4. In Section 4, we study the 3-connectivity of the Cartesian product of two connected graphs  $G$  and  $H$ . Moreover, all the bounds are sharp. Our result could be seen as an extension of Theorem 1.1.

## 2 Some basic results

We use  $P_n$  to denote a path with  $n$  vertices. A path  $P$  is called a  $u$ - $v$  path, denoted by  $P_{u,v}$ , if  $u$  and  $v$  are the endpoints of  $P$ .

Recall that the *Cartesian product* (also called the *square product*) of two graphs  $G$  and  $H$ , written as  $G \square H$ , is the graph with vertex set  $V(G) \times V(H)$ , in which two vertices  $(u, v)$  and  $(u', v')$  are adjacent if and only if  $u = u'$  and  $(v, v') \in E(H)$ , or  $v = v'$  and  $(u, u') \in E(G)$ . Clearly, the Cartesian product is commutative, that is,  $G \square H \cong H \square G$ . The edge  $(u, v)(u', v')$  is called *one-type edge* if  $(u, u') \in E(G)$  and  $v = v'$ ; similarly, the  $(u, v)(u', v')$  is called *two-type edge* if  $u = u'$  and  $(v, v') \in E(H)$ .

Let  $G$  and  $H$  be two graphs with  $V(G) = \{u_1, u_2, \dots, u_n\}$  and  $V(H) = \{v_1, v_2, \dots, v_m\}$ , respectively. We use  $G(u_j, v_i)$  to denote the subgraph of  $G \square H$  induced by the set  $\{(u_j, v_i) \mid 1 \leq j \leq n\}$ . Similarly, we use  $H(u_j, v_i)$  to denote the subgraph of  $G \square H$  induced by the set  $\{(u_j, v_i) \mid 1 \leq i \leq m\}$ . It is easy to see  $G(u_{j_1}, v_i) = G(u_{j_2}, v_i)$  for different  $u_{j_1}$  and  $u_{j_2}$  of  $G$ . Thus, we can replace  $G(u_j, v_i)$  by  $G(v_i)$  for simplicity. Similarly, we can replace  $H(u_j, v_i)$  by  $H(u_j)$ . For any  $u, u' \in V(G)$  and  $v, v' \in V(H)$ ,  $(u, v), (u, v') \in V(H(u))$ ,  $(u', v), (u', v') \in V(H(u'))$ ,  $(u, v), (u', v) \in V(G(v))$ , and  $(u, v'), (u', v') \in V(G(v))$ . We refer to  $(u, v')$  and  $(u', v)$  as *the vertices corresponding to  $(u, v)$  in  $G(v')$  ( $= G(u, v')$ ) and  $H(u')$  ( $= H(u', v)$ )*, respectively. Similarly, we can define the path and tree corresponding to some path and tree, respectively.

In order to show our main results, we need the following well-known theorem.

**Theorem 2.1 (Menger's Theorem [1])** *Let  $G$  be a  $k$ -connected graph, and let  $x$  and  $y$  be a pair of distinct vertices in  $G$ . Then there exist  $k$  internally disjoint paths  $P_1, P_2, \dots, P_k$  in  $G$  connecting  $x$  and  $y$ .*

Let  $G$  be a connected graph, and  $S = \{x_1, x_2, x_3\} \subseteq V(G)$ . We first have the following observation about internally disjoint  $S$ -trees.

**Observation 2.1** *Let  $G$  be a connected graph,  $S = \{x_1, x_2, x_3\} \subseteq V(G)$ , and  $T$  be an  $S$ -tree. Then there exists a subtree  $T'$  of  $T$  such that  $T'$  is also an  $S$ -tree such that  $1 \leq d_{T'}(x_i) \leq 2$ ,  $|\{x_i \mid d_{T'}(x_i) = 1\}| \geq 2$  and  $\{x \mid d_{T'}(x) = 1\} \subseteq S$ . Moreover, if  $|\{x_i \mid d_{T'}(x_i) = 1\}| = 3$ , then all the vertices of  $V(T') \setminus \{x_1, x_2, x_3\}$  have degree 2 except for one vertex, say  $x$  with  $d_{T'}(x) = 3$ ; if there exists one vertex of  $S$ , say  $x_1$ , of degree 2 in  $T'$ , then  $T'$  is an  $x_2$ - $x_3$  path.*

**Proof:** It is easy to check that this observation holds by deleting vertices and edges of  $T$ . □

**Remark 2.1** (i) *Since the path between any two distinct vertices is unique in  $T$ , the tree  $T'$  obtained from  $T$  in Observation 2.1 is unique. Such a tree is called a minimal  $S$ -tree (or minimal  $\{x_1, x_2, x_3\}$ -tree).*

(ii) *Let  $S = \{x, y, z\} \subseteq V(G)$ . Throughout this paper, we can assume that each  $S$ -tree is a minimal  $S$ -tree.*

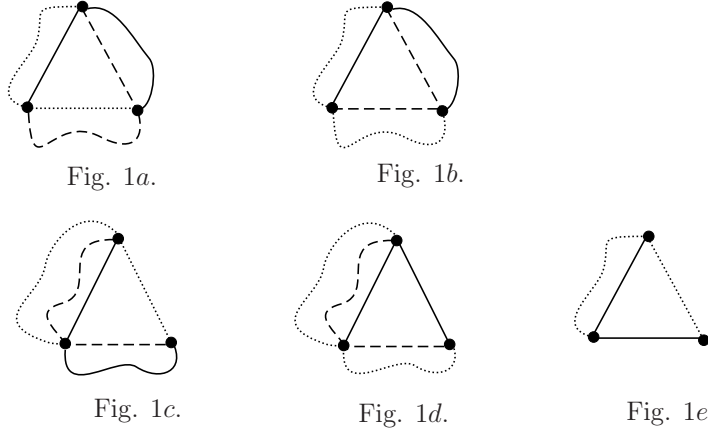
**Lemma 2.1** Let  $G$  be a graph with  $\kappa_3(G) = k \geq 2$ ,  $S = \{x, y, z\} \subseteq V(G)$ . Then, the following assertions hold:

(i) If  $G[S]$  is a clique, then there exist  $k$  internally disjoint  $S$ -trees  $T_1, T_2, \dots, T_k$ , such that  $E(T_i) \cap E(G[S]) = \emptyset$  for  $1 \leq i \leq k-2$ .

(ii) If  $G[S]$  is not a clique, then there exist  $k$  internally disjoint  $S$ -trees  $T_1, T_2, \dots, T_k$ , such that  $E(T_i) \cap E(G[S]) = \emptyset$  for  $1 \leq i \leq k-1$ .

**Proof:** We first prove (i). Clearly, by the definition of  $S$ -trees, we know  $|\{T_i \mid E(T_i) \cap E(G[S]) \neq \emptyset\}| \leq 3$ . Let  $\{T_1, T_2, \dots, T_k\}$  be  $k$  internally disjoint  $S$ -trees. If  $|\{T_i \mid E(T_i) \cap E(G[S]) \neq \emptyset\}| \leq 2$ , we are done by exchanging subscript. Thus, suppose  $|\{T_i \mid E(T_i) \cap E(G[S]) \neq \emptyset\}| = 3$ . Without loss of generality, we assume  $E(T_i) \cap E(G[S]) \neq \emptyset$ , where  $i = k-2, k-1, k$ . It is easy to check that  $T_{k-2}, T_{k-1}, T_k$  must have the structures as shown in Figures 1 a and c. But, for these two cases, we can obtain  $T'_{k-2}, T'_{k-1}, T'_k$  from  $T_{k-2}, T_{k-1}, T_k$ , such that  $E(T'_{k-2}) \cap \{xy, xz, yz\} = \emptyset$ . See Figs. 1b. and 1d, where the tree  $T'_{k-2}$  is shown by dotted lines. Thus  $T_1, T_2, \dots, T_{k-3}, T'_{k-2}, T'_{k-1}, T'_k$  are our desired  $S$ -trees.

The proof of (ii) is similar to that of (i), and thus is omitted.  $\square$



**Fig. 1:**  $T'_{k-2}, T'_{k-1}, T'_k$ . An edge is shown by a straight line. The edges (or paths) of a tree are shown by the same type of lines.

**Remark 2.2** Let  $G$  be a graph with  $\kappa_3(G) = k \geq 2$ ,  $S = \{x, y, z\} \subseteq V(G)$ . If  $|\{T_i \mid E(T_i) \cap E(G[S]) \neq \emptyset\}| \geq 2$  for any collection  $\mathcal{T}$  of  $k$  internally disjoint  $S$ -trees, then  $G[S]$  is a clique. Moreover,  $T_{k-1} \cup T_k$  must have the structure as shown in Figure 1e.

### 3 The Cartesian product of a connected graph and a path

In this section, we show the following proposition.

**Proposition 3.1** *Let  $G$  be a graph and  $P_m$  be a path with  $m$  vertices. The following assertions hold:*

- (i) *If  $\kappa_3(G) = \kappa(G) \geq 1$ , then  $\kappa_3(G \square P_m) \geq \kappa_3(G)$ . Moreover, the bound is sharp;*
- (ii) *If  $1 \leq \kappa_3(G) < \kappa(G)$ , then  $\kappa_3(G \square P_m) \geq \kappa_3(G) + 1$ . Moreover, the bound is sharp.*

We shall prove Proposition 3.1 by a series of lemmas. Since the proofs of (i) and (ii) are similar, we only show (ii). Let  $G$  be a graph with  $V(G) = \{u_1, u_2, \dots, u_n\}$  such that  $1 \leq \kappa_3(G) < \kappa(G)$ ,  $V(P_m) = \{v_1, v_2, \dots, v_m\}$  such that  $v_i$  and  $v_j$  are adjacent if and only if  $|i - j| = 1$ .

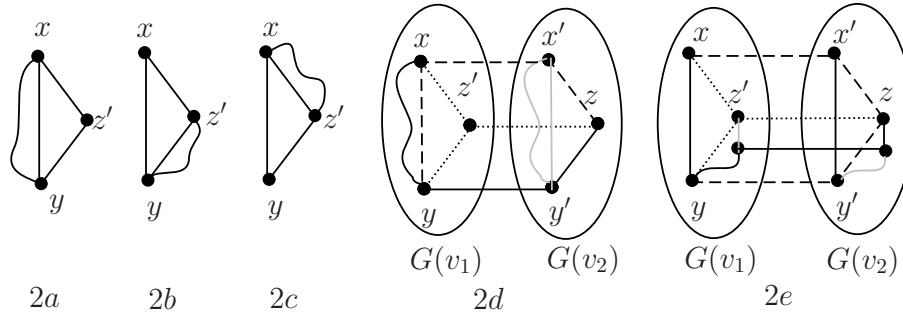
Set  $\kappa_3(G) = k$  for simplicity. To prove (ii), we need to show that for any  $S = \{x, y, z\} \subseteq V(G \square H)$ , there exist  $k + 1$  internally disjoint  $S$ -trees. We proceed our proof by the following three lemmas.

**Lemma 3.1** *If  $x, y, z$  belongs to the same  $V(G(v_i))$ ,  $1 \leq i \leq m$ , then there exist  $k + 1$  internally disjoint  $S$ -trees.*

**Proof:** Without loss of generality, we assume  $x, y, z \in V(G(v_1))$ . Since  $\kappa_3(G) = k$ , there exist  $k$  internally disjoint  $S$ -trees  $T_1, T_2, \dots, T_k$  in  $G(v_1)$ . We need another  $S$ -tree  $T_{k+1}$  such that  $T_{k+1}$  and  $T_i$  are internally disjoint, for  $i = 1, 2, \dots, k$ . Let  $x', y', z'$  be the vertices corresponding to  $x, y, z$  in  $G(v_2)$ , and  $T'_1$  be the tree corresponding to  $T_1$  in  $G(v_2)$ . Therefore, tree  $T_{k+1}$  obtained from  $T'_1$  by adding three edges  $xx', yy', zz'$  is a desired tree.  $\square$

**Lemma 3.2** *If only two vertices of  $\{x, y, z\}$  belong to some copy  $G(v_i)$ , then there exist  $k + 1$  internally disjoint  $S$ -trees.*

**Proof:** We may assume  $x, y \in V(G(v_1)), z \in V(G(v_2))$ . In the following argument, we can see that this assumption has no influence on the correctness of our proof. Let  $x', y'$  be the vertices corresponding to  $x, y$  in  $G(v_2)$ ,  $z'$  be the vertex corresponding to  $z$  in  $G(v_1)$ . Consider the following two cases.



**Fig. 2:** The edges (or paths) of a tree are shown by the same type of lines. The lightest lines stand for edges (or paths) not contained in  $T_i^*$ .

**Case 1:**  $z' \notin \{x, y\}$ .

Let  $S' = \{x, y, z'\}$ , and  $T_1, T_2, \dots, T_k$  be  $k$  internally disjoint  $S'$ -trees in  $G(v_1)$  such that  $|\{T_i \mid E(T_i) \cap E(G(v_1)[S']) \neq \emptyset\}|$  is as small as possible. We can assume that  $E(T_i) \cap E(G(v_1)[S']) = \emptyset$  for each  $i$ , where  $1 \leq i \leq k-2$  by Lemma 2.1.

For a tree  $T_i$  with  $E(T_i) \cap E(G(v_1)[S']) = \emptyset$ , let  $T_i^*$  be the tree obtained from  $T_i$  by adding  $z_i z'_i$  and  $z'_i z$ , and deleting  $z'$ , where  $z_i$  is any one neighbor of  $z'$  in  $T_i$ , and  $z'_i$  is the vertex corresponding to  $z_i$  in  $G(v_2)$ .

If  $E(T_k) \cap E(G(v_1)[S']) \neq \emptyset$ , say  $yz' \in E(T_k) \cap E(G(v_1)[S'])$ . Let  $T_k^* = T_k + zz'$  and  $T_{k+1}^* = T_k + xx' + yy'$ , where  $T_k^*$  is the tree corresponding to  $T_k$  in  $G(v_2)$ .

If  $E(T_{k-1}) \cap E(G(v_1)[S']) \neq \emptyset$  and  $E(T_k) \cap E(G(v_1)[S']) \neq \emptyset$ . Then  $T_{k-1} \cup T_k$  must have one of the structures as shown in Figures 2 a, b and c by Remark 2.2. If  $T_{k-1}$  and  $T_k$  have the structures as shown in Figure 2a, then we can obtain trees  $T_{k-1}^*, T_k^*$  and  $T_{k+1}^*$  as shown in Figure 2d. If  $T_{k-1}$  and  $T_k$  have the structures as shown in Figure 2b, then we can obtain trees  $T_{k-1}^*, T_k^*$  and  $T_{k+1}^*$  as shown in Figure 2e. If  $T_{k-1}$  and  $T_k$  have the structures as shown in Figure 2c, then we can obtain trees  $T_{k-1}^*, T_k^*$  and  $T_{k+1}^*$  similar to those in Figure 2d.

**Case 2:**  $z' \in \{x, y\}$ .

Without loss of generality, assume  $z' = y$ . Since  $\kappa(G) > \kappa_3(G) = k$ , by Menger's Theorem, there exist at least  $k+1$  internally disjoint  $x$ - $y$  paths  $P^1, P^2, \dots, P^{k+1}$  in  $G(v_1)$ . Assume that  $y_i$  is the only neighbor of  $y$  in  $P^i$ , and that  $y'_i$  is the vertex corresponding to  $y_i$  in  $G(v_2)$ . If  $x$  and  $y$  are nonadjacent in  $P^i$ , let  $T_i$  be the tree obtained from  $P^i$  by adding  $y_i y'_i$  and  $y'_i z$ . If  $x$  and  $y$  are adjacent in  $P^i$ , let  $T_i$  be the tree obtained from  $P^i$  by adding  $yz$ . Since  $G$  is a simple graph, there exists at most one path  $P^i$  such that  $x$  and  $y$  are adjacent on  $P^i$ . Thus  $T_i, 1 \leq i \leq k+1$ , are  $k+1$  internally disjoint  $S$ -trees.

□

**Lemma 3.3** *If  $x, y, z$  are contained in distinct  $G(v_i)$ s, then there exist  $k+1$  internally disjoint  $S$ -trees.*

**Proof:** We may assume that  $x \in V(G(v_1)), y \in V(G(v_2)), z \in V(G(v_3))$ . In the following argument, we can see that this assumption has no influence on the correctness of our proof. Let  $y', z'$  be the vertices corresponding to  $y, z$  in  $G(v_1)$ ,  $x', z''$  be the vertices corresponding to  $x, z$  in  $G(v_2)$  and  $x'', y''$  be the vertices corresponding to  $x, y$  in  $G(v_3)$ . We consider the following three cases.

**Case 1:**  $x, y', z'$  are distinct vertices in  $G(v_1)$

Let  $S' = \{x, y', z'\}$ , and  $T_1, T_2, \dots, T_k$  be  $k$  internally disjoint  $S'$ -trees in  $G(v_1)$  such that  $|\{T_i \mid E(T_i) \cap E(G(v_1)[S']) \neq \emptyset\}|$  is as small as possible. We can assume that  $E(T_i) \cap E(G(v_1)[S']) = \emptyset$  for each  $i$ , where  $1 \leq i \leq k-2$  by Lemma 2.1. For each  $T_i$  such that  $E(T_i) \cap E(G(v_1)[S']) = \emptyset$ , we can obtain an  $S$ -tree  $T_i^*$  from  $T_i$  similar to that in Subcase 1.1 of Lemma 3.2.

If  $E(T_{k-1}) \cap E(G(v_1)[S']) = \emptyset$  or  $E(T_{k-1}) \cap E(G(v_1)[S']) = \emptyset$ . Without loss of generality, we assume  $E(T_{k-1}) \cap E(G(v_1)[S']) = \emptyset$ . Let  $T_k^*$  be the tree obtained from  $T_k$  by adding edges

$y'y, z'z''$  and  $z''z, T_{k+1}^*$  be the tree obtained from  $T_k''$  by adding  $x''x', x'x$  and  $y''y$ , where  $T_k''$  is the tree corresponding to  $T_k$  in  $G(v_3)$ . Thus,  $T_i^*$ s,  $1 \leq i \leq k+1$ , are  $k$  internally disjoint  $S$ -tree.

Otherwise, that is,  $E(T_{k-1}) \cap E(G(v_1)[S']) \neq \emptyset$  and  $E(T_k) \cap E(G(v_1)[S']) \neq \emptyset$ . Then  $T_{k-1}$  and  $T_k$  must have the structures as shown in Figure 3a, b and c. If  $T_{k-1}$  and  $T_k$  have the structures as shown in Figure 3a, then we can obtain trees  $T_{k-1}^*, T_k^*$  and  $T_{k+1}^*$  as shown in Figure 3d. If  $T_{k-1}$  and  $T_k$  have the structures as shown in Figure 3b, then we can obtain trees  $T_{k-1}^*, T_k^*$  and  $T_{k+1}^*$  as shown in Figure 3e. If  $T_{k-1}$  and  $T_k$  have the structures as shown in Figure 3c, then we can obtain trees  $T_{k-1}^*, T_k^*$  and  $T_{k+1}^*$  as shown in Figure 3f.

**Case 2:** Two of  $x, y', z'$  are the same vertex in  $G(v_1)$ .

If  $y' = z'$ , since  $\kappa(G) > \kappa_3(G) = k$ , by Menger's Theorem, it is easy to construct  $k+1$  internally disjoint  $S$ -trees. See Figure 3g. The other cases ( $x = y'$  or  $x = z'$ ) can be proved with similar arguments.

**Case 3:**  $x, y', z'$  are the same vertex in  $G(v_1)$ .

Since  $\kappa(G) > \kappa_3(G) = k$ , by Menger's Theorem, it is easy to construct  $k+1$  internally disjoint  $S$ -trees. See Figure 3h.

□

We have the following observation by the argument in the proof of Proposition 3.1.

**Observation 3.1** *The  $k+1$  internally disjoint  $S$ -trees consist of three kinds of edges — the edges of original trees (or paths), the edges corresponding the edges of original trees (or paths) and two-type edges.*

Note that  $Q_n \cong P_2 \square P_2 \square \cdots \square P_2$ , where  $Q_n$  is the  $n$ -hypercube. We have the following corollary.

**Corollary 3.1** *Let  $Q_n$  be the  $n$ -hypercube with  $n \geq 2$ . Then  $\kappa_3(Q_n) = n - 1$ .*

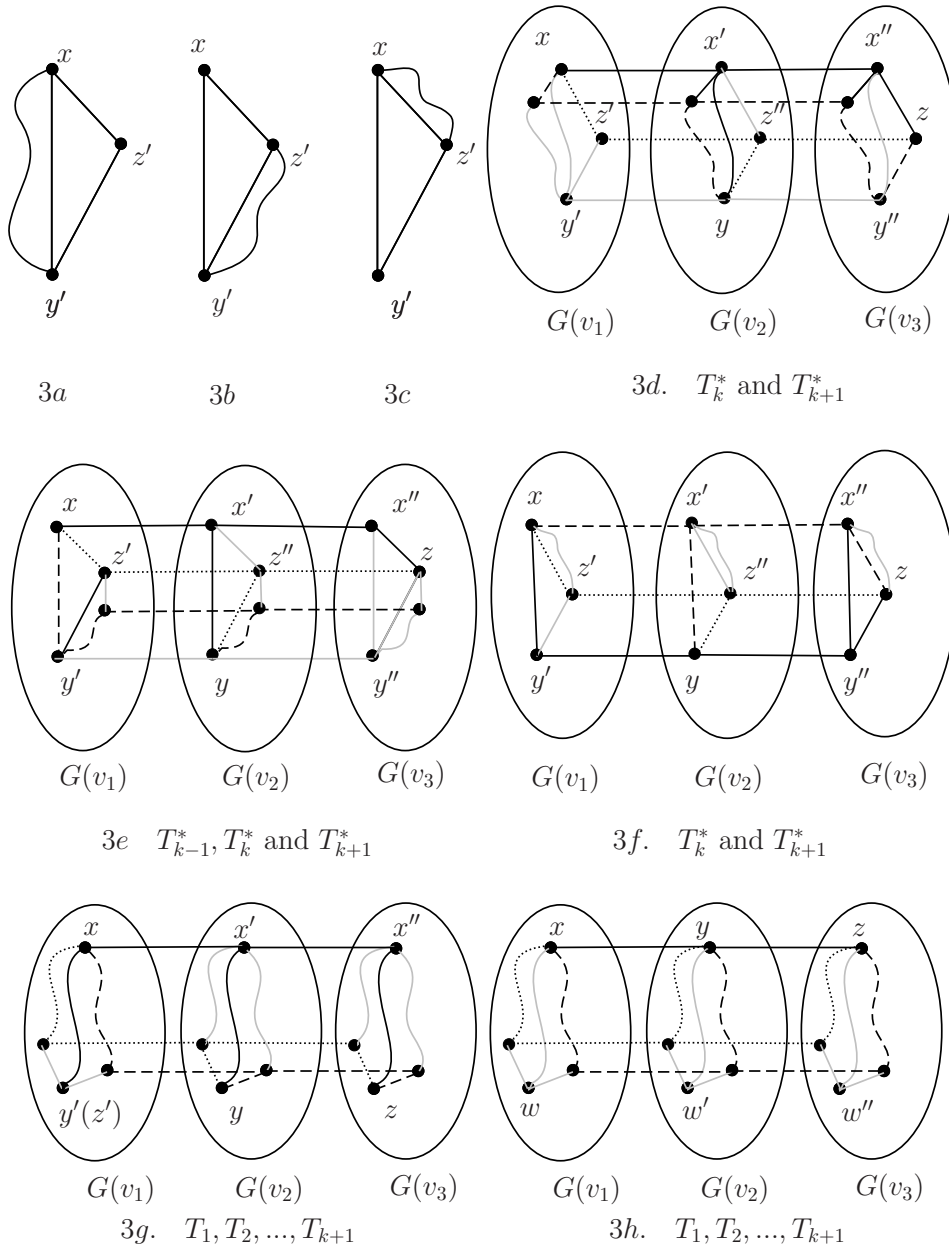
**Proof:** Recall that  $\kappa(Q_n) = n$  so that Proposition 3.1 (ii) inductively applies. It is easy to check that  $\kappa_3(Q_2) = 1$ . Suppose that the result holds for  $Q_{n-1}$ , where  $n \geq 3$ . We have  $\kappa_3(Q_n) \geq n - 1$  by Proposition 3.1. On the other hand, since  $Q_n$  is  $n$ -regular, we have  $\kappa_3(Q_n) \leq n - 1$  by Theorem 1.3. Thus  $\kappa_3(Q_n) = n - 1$ . □

**Example 3.1** *Let  $K_{2n}$  be the complete graph with vertex set  $V(K_{2n}) = \{u_1, u_2, \dots, u_{2n}\}$ , and let  $G_n$  be the graph obtained from  $K_{2n}$  by adding a new vertex  $u$  and edges  $uu_i, 1 \leq i \leq n$ .*

*For any  $S = \{x, y, z\} \subseteq V(G)$ , if  $u \notin S$ , then there exist  $k$  internally disjoint  $S$ -trees in  $G_n$  by Theorem 1.2. If  $u \in S$ , without loss of generality, assume  $x = u, y = u_1, z = u_2$ . Let  $T_1$  be the path  $u, u_1, u_{k+1}, u_2$ ,  $T_2$  be the path  $u, u_2, u_{k+2}, u_1$ , and  $T_i$  be the tree obtained from a path  $u, u_{n+i}, u_1$  by adding an edge  $u_{n+i}u_2$  for  $3 \leq i \leq n$ . Clearly,  $T_i, 1 \leq i \leq n$ , are  $n$  internally disjoint  $S$ -trees. So  $\kappa_3(G_n) \geq n$ . Since  $\delta(G_n) = n, \kappa_3(G_n) = n$  by Theorem 1.4. By Proposition 3.1,  $\kappa_3(G_n \square K_2) \geq n$ . Since  $G_n$  has two adjacent vertices of degree  $n+1$ ,  $\kappa_3(G_n) = n$  by Theorem 1.3. Moreover, clearly,  $\kappa(G) = n$ . Thus  $\kappa_3(G \square K_2) = \kappa_3(G) = n$ .*

**Remark 3.1** *We know that the bounds of (i) and (ii) in Theorem 3.1 are sharp by Example 3.1 and Corollary 3.1.*





**Fig. 3:** The edges (or paths) of a tree are shown by the same type of lines. The lightest lines stand for edges (or paths) not contained in  $T_i^*$ .

**Proposition 3.2** *Let  $G$  be a connected graph and  $T$  be a tree. The following assertions hold:*

- (i) *If  $\kappa_3(G) = \kappa(G) \geq 1$ , then  $\kappa_3(G \square T) \geq \kappa_3(G)$ . Moreover, the bound is sharp;*
- (ii) *If  $1 \leq \kappa_3(G) < \kappa(G)$ , then  $\kappa_3(G \square T) \geq \kappa_3(G) + 1$ . Moreover, the bound is sharp.*

**Proof:** Since the proofs of (i) and (ii) are similar, we only show (ii). It suffices to show that for any  $S = \{x, y, z\} \subseteq V(G \square H)$ , there exist  $k + 1$  internally disjoint  $S$ -trees. Set  $\kappa_3(G) = k$ ,  $V(G) = \{u_1, u_2, \dots, u_n\}$ , and  $V(T) = \{v_1, v_2, \dots, v_m\}$ .

Let  $x \in V(G(v_i)), y \in V(G(v_j)), z \in V(G(v_k))$  be three distinct vertices. If there exists a path in  $T$  containing  $v_i, v_j$  and  $v_k$ , then we are done from Proposition 3.1. If  $i, j$  and  $k$  are not distinct integers, such a path must exist. Thus, suppose that  $i, j$  and  $k$  are distinct integers, and that there exists no path containing  $v_i, v_j$  and  $v_k$ . By Observation 2.1, there exists a tree  $T'$  in  $T$  such that  $d_{T'}(v_i) = d_{T'}(v_j) = d_{T'}(v_k) = 1$  and all the vertices of  $V(T) \setminus \{v_i, v_j, v_k\}$  have degree 2 except for one vertex, say  $v_4$  with  $d_{T'}(v_4) = 3$ . Without loss of generality, we set  $i = 1, j = 2, k = 3$ . Furthermore, we assume  $v_i v_4 \in E(T')$ , where  $1 \leq i \leq 3$ . In the following argument, we can see that this assumption has no influence on the correctness of our proof.

Let  $P$  be the unique path in  $T'$  connecting  $v_1$  and  $v_2$ . By Proposition 3.1, we can construct  $k + 1$  internally disjoint  $\{x, y, z'\}$ -trees  $T_i$ ,  $1 \leq k + 1$ , in  $G \square P$ , where  $z'$  is the vertex corresponding to  $z$  in  $G(v_4)$ . By a similar method of Proposition 3.1, we can construct  $k + 1$  internally disjoint  $S$ -trees in  $G \square T$  on the basis of these trees.  $\square$

**Remark 3.2** *We know that the bounds of (i) and (ii) in Proposition 3.2 are sharp by Example 3.1 and Corollary 3.1.*

**Observation 3.2** *The  $k + 1$  internally disjoint  $S$ -trees consist of three kinds of edges — the edges of original trees (or paths), the edges corresponding the edges of original trees (or paths) and two-type edges.*

## 4 The Cartesian product of two general graphs

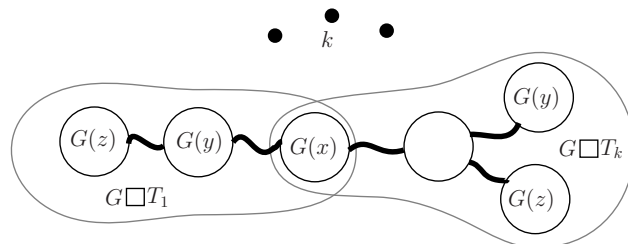
**Observation 4.1** *Let  $G$  and  $H$  be two connected graphs,  $x, y, z$  be three distinct vertices in  $H$ , and  $T_1, T_2, \dots, T_k$  be  $k$  internally disjoint  $\{x, y, z\}$ -trees in  $H$ . Then  $G \square \bigcup_{i=1}^k T_i = \bigcup_{i=1}^k (G \square T_i)$  has the structure as shown in Figure 4. Moreover,  $(G \square T_i) \cap (G \square T_j) = G(x) \cup G(y) \cup G(z)$  for  $i \neq j$ . In order to show the structure of  $G \square \bigcup_{i=1}^k T_i$  clearly, we take  $k$  copies of  $G(y)$ , and  $k$  copies of  $G(z)$ . Note that, these  $k$  copies of  $G(y)$  (resp.  $G(z)$ ) represent the same graph.*

**Example 4.1** *Let  $H$  be the complete graph of order 4. The structure of  $G \square (T_1 \cup T_2)$  is shown in Figure 5.*

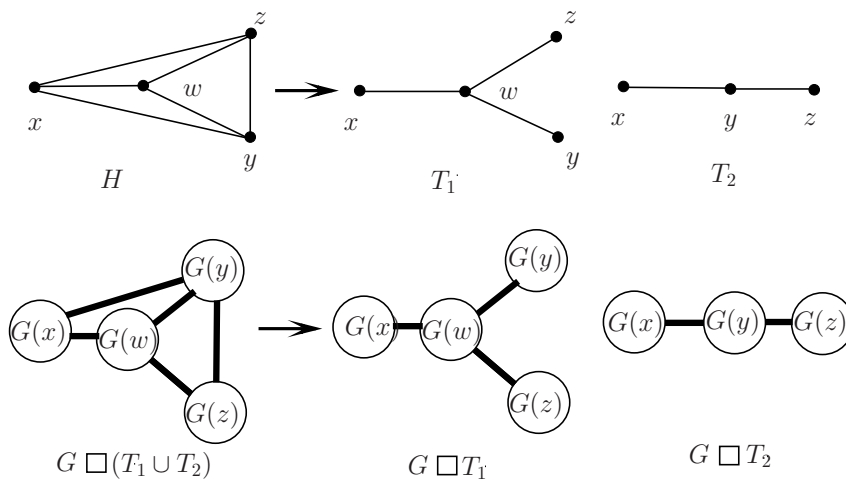
Now we are ready to prove Theorem 1.5.

**Proof of Theorem 1.5:** Since the proofs of (i) and (ii) are similar, we only show (ii). Without loss of generality, we set  $\kappa_3(G) := k, \kappa_3(H) := \ell$ . It suffices to show that for any  $S = \{x, y, z\} \subseteq V(G \square H)$ , there exist  $k + \ell$  internally disjoint  $S$ -trees. Assume  $V(G) = \{u_1, u_2, \dots, u_n\}$  and  $V(T) = \{v_1, v_2, \dots, v_m\}$ .

Let  $x \in V(G(v_i)), y \in V(G(v_j)), z \in V(G(v_k))$  be three distinct vertices in  $G \square H$ . We will do only the case that  $i, j, k$  are distinct integers. Other two possibilities are similar. Without loss of generality, set  $i = 1, j = 2, k = 3$ . Since  $\kappa_3(H) = \ell$ , there exist  $\ell$  internally disjoint  $\{v_1, v_2, v_3\}$ -trees  $T_i$ ,  $1 \leq i \leq \ell$ , in  $H$ . We use  $G_i$  to denote  $G \square T_i$ . By Observation 4.1, we know that  $G \square \bigcup_{i=1}^{\ell} T_i = \bigcup_{i=1}^{\ell} G_i$  and



**Fig. 4:** The structure of  $G \square \bigcup_{i=1}^k T_i$ .



**Fig. 5:** The structure of  $G \square (T_1 \cup T_2)$ .

$G_i \cap G_j = G(v_1) \cup G(v_2) \cup G(v_3)$  for  $i \neq j$ . Let  $y', z'$  be the vertices corresponding to  $y, z$  in  $G(v_1)$ , respectively.

If  $x, y', z'$  are distinct vertices in  $G(v_1)$ . Since  $\kappa_3(G(v_1)) = k$ , there exist  $k$  internally disjoint  $\{x, y', z'\}$ -trees  $T'_j$ ,  $1 \leq j \leq k$ , in  $G(v_1)$ . Let  $k_0, k_1, \dots, k_\ell$  be integers such that  $0 = k_0 < k_1 < \dots < k_\ell = k$ . Similar to the proofs of Proposition 3.1, we can construct  $k_i - k_{i-1} + 1$  internally disjoint  $S$ -trees  $T_{i,j_i}$ ,  $1 \leq j_i \leq k_i - k_{i-1} + 1$ , in  $(\bigcup_{j=k_{i-1}+1}^{k_i} T'_j) \square T_i$  for each  $i$ , where  $1 \leq i \leq \ell$ . By Observations 3.1 and 3.2,  $T_{i,j_i}$  and  $T_{r,j_r}$  are internally disjoint for  $i \neq r$ . Thus  $T_{i,j_i}$ ,  $1 \leq i \leq \ell, 1 \leq j_i \leq k_i - k_{i-1} + 1$  are  $k + \ell$  internally disjoint  $S$ -trees.

If exactly two of  $x, y', z'$  are the same vertex in  $G(v_i)$ . Without loss of generality, assume  $y' = z'$ . Since  $\kappa(G(v_1)) > k$ , there exist  $k + 1$  internally disjoint  $x - y'$  paths  $P_i$ ,  $1 \leq i \leq k + 1$ , in  $G(v_1)$  by Menger's Theorem. Note that at most one of them is a path of length 1. Let  $P_{k+1}$  be such a path if  $xy' \in E(G(v_1))$ , and let  $k_0, k_1, \dots, k_\ell$  be integers such that  $0 = k_0 < k_1 < \dots < k_\ell = k + 1$ . Similar to the proofs of Proposition 3.1, we can construct  $k_i - k_{i-1} + 1$  internally disjoint  $S$ -trees  $T_{i,j_i}$ ,  $1 \leq j_i \leq k_i - k_{i-1} + 1$ , in  $(\bigcup_{j=k_{i-1}+1}^{k_i} P_j) \square T_i$  for each  $i$ , where  $1 \leq i \leq \ell - 1$ , and  $k_\ell - k_{\ell-1}$  internally disjoint  $S$ -trees  $T_{\ell,j_\ell}$ ,  $1 \leq j_\ell \leq k_\ell - k_{\ell-1}$ , in  $(\bigcup_{j=k_{\ell-1}+1}^{k_\ell} P_j) \square T_\ell$ . By Observation 3.1 and 3.2,  $T_{i,j_i}$  and  $T_{r,j_r}$  are internally disjoint for  $i \neq r$ . Thus  $T_{i,j_i}$ ,  $1 \leq i \leq \ell, 1 \leq j_i \leq k_i - k_{i-1} + 1$  are  $k + \ell$  internally disjoint  $S$ -trees.

If all of  $x, y', z'$  are the same vertex in  $G(v_i)$ . Since  $\delta(G(v_1)) \geq \kappa(G(v_1)) > k$ ,  $x$  has  $k$  neighbors, say  $x_1, x_2, \dots, x_k$ , in  $G(v_1)$ . Let  $P_i$  be the path  $xx_i$ , and let  $k_0, k_1, \dots, k_\ell$  be integers such that  $0 = k_0 < k_1 < \dots < k_\ell = k$ . Similar to the proofs of Proposition 3.1, we can construct  $k_i - k_{i-1} + 1$  internally disjoint  $S$ -trees  $T_{i,j_i}$ ,  $1 \leq j_i \leq k_i - k_{i-1} + 1$ , in  $(\bigcup_{j=k_{i-1}+1}^{k_i} P_j) \square T_i$  for each  $i$ , where  $1 \leq i \leq \ell$ . By Observation 3.1 and 3.2,  $T_{i,j_i}$  and  $T_{r,j_r}$  are internally disjoint for  $i \neq r$ . Thus  $T_{i,j_i}$ ,  $1 \leq i \leq \ell, 1 \leq j_i \leq k_i - k_{i-1} + 1$  are  $k + \ell$  internally disjoint  $S$ -trees.

We now show that bounds of Theorem 1.5 are sharp. For (i), Example 3.1 is a sharp example. Let  $K_n$  be a complete graph with  $n$  vertices, and  $P_m$  be a path with  $m$  vertices, where  $m \geq 2$ . We have  $\kappa_3(P_m) = 1$ , and  $\kappa_3(K_n) = n - 2$  by Theorem 1.2. It is easy to check that  $\kappa_3(K_n \square P_m) = n - 2 + 1 = n - 1$ . Thus,  $K_n \square P_m$  is a sharp example for (ii).  $\square$

## Acknowledgments

We thank anonymous reviewers for their carefully reading of our work and their helpful suggestions.

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