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# Vertex-colouring edge-weightings with two edge weights

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An edge-weighting vertex colouring of a graph is an edge-weight assignment such that the accumulated weights at the vertices yields a proper vertex colouring. If such an assignment from a set  $S$  exists, we say the graph is  $S$ -weight colourable. It is conjectured that every graph with no isolated edge is  $\{1, 2, 3\}$ -weight colourable.

We explore the problem of classifying those graphs which are  $\{1, 2\}$ -weight colourable. We establish that a number of classes of graphs are  $S$ -weight colourable for much more general sets  $S$  of size 2. In particular, we show that any graph having only cycles of length  $0 \pmod 4$  is  $S$ -weight colourable for most sets  $S$  of size 2. As a consequence, we classify the minimal graphs which are not  $\{1, 2\}$ -weight colourable with respect to subgraph containment. We also demonstrate techniques for constructing graphs which are not  $\{1, 2\}$ -weight colourable.

**Keywords:** edge weighting, graph colouring

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## 1 Introduction

Let  $G$  be a simple graph and  $S$  be a set of real numbers. An  $S$ -edge-weighting of  $G$  is an assignment  $w : E(G) \rightarrow S$ . Given an  $S$ -edge-weighting, the *weighted degree* of a vertex  $v$ , denoted  $w(v)$ , is the sum of weights of the edges incident with  $v$ . An  $S$ -edge-weighting gives a *vertex colouring* if the weighted degrees of adjacent vertices are different. If an  $S$ -edge-weighting vertex colouring  $w$  exists, we also call  $w$  an  $S$ -weight colouring and we say  $G$  is  $S$ -weight colourable. For a positive integer  $k$ , we say  $G$  has a  $k$ -weight colouring or  $G$  is  $k$ -weight colourable if it is  $S$ -weight colourable for every set  $S$  of size  $k$ . The most commonly studied sets  $S$  are those of the form  $\{1, \dots, k\}$ .

**Problem 1** Given a graph  $G$  with no isolated edges, find the minimum  $k$  such that  $G$  is  $\{1, \dots, k\}$ -weight colourable.

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It is not hard to verify that  $K_4$  with a single leaf attached is  $\{1, 2\}$ -weight colourable but is not  $\{0, 1\}$ -weight colourable. It follows that the  $S$ -weight colourability of a graph is not only dependent on the size of  $S$  but also on the particular elements of  $S$ . However, if a graph  $G$  is  $S$ -weight colourable then there exists an  $i_0 = i_0(G, S)$  such that for all  $i > i_0$  the graph is also  $\{s + i : s \in S\}$ -weight colourable. One such value for  $i_0$ , though not necessarily the smallest, is  $i_0 = |S| \cdot \Delta(G) \cdot \max\{|s| : s \in S\}$ , where  $\Delta(G)$  is the maximum degree of  $G$ .

Let us start by considering the 2-weight colourability of a simple class of graphs – paths. If  $a$  and  $b$  are non-zero real numbers, then every path of length at least 2 has an  $\{a, b\}$ -weight colouring. Assigning the edge weights  $a, a, b, b, a, a, b, b, \dots$ , beginning with one leaf of the path, gives such a colouring. However, a path has a  $\{0, a\}$ -weight colouring if and only if it is not of length  $1 \pmod 4$ . The reader can easily check that paths of length 2, 3 and 4 have a  $\{0, a\}$ -weight colouring. However, if we let  $P = e_1, e_2, e_3, e_4, e_5$  be a path of length 5 (we omit vertex labels) then if  $w(e_2) = 0$  (or  $w(e_4) = 0$ ) then the ends of  $e_1$  ( $e_5$ ) will have equal weight. Thus the only way to achieve a  $\{0, a\}$ -weight colouring of  $P$  is if  $w(e_2) = w(e_4) = a$ . However, this implies that the ends of  $e_3$  will have the same weight, and hence a  $\{0, a\}$ -weight colouring cannot exist. These examples easily extend to longer paths; the details are left to the reader.

In general, it is unknown how difficult it is to decide if a given graph admits a  $\{1, 2\}$ -weight colouring, or more generally an  $\{a, b\}$ -weight colouring. As such, we present the following question:

**Problem 2** *Is it NP-complete to decide whether a given graph is 2-weight colourable?*

Returning to Problem 1, we state the following conjecture, due to Karoński, Łuczak, and Thomason [KŁT04], which motivates most of the known results on the  $\{1, \dots, k\}$ -weight colourability of graphs.

**Conjecture 1.1** *Every graph with no isolated edge is  $\{1, 2, 3\}$ -weight colourable.*

Karoński et al. [KŁT04] showed that the Conjecture 1.1 is true for 3-colourable graphs. They also proved that if  $S$  is any set of at least 183 real numbers which are linearly independent over the rational numbers then every graph with no isolated edge is  $S$ -weight colourable. Recently, Kalkowski et al. [KKP09] showed that every graph with no isolated edge is  $\{1, \dots, k\}$ -weight colourable for  $k = 5$ . This result is an improvement on the previous bounds on  $k$  established by Addario-Berry et al. [ABDM<sup>+</sup>07], Addario-Berry et al. [ABDR08], and Wang et al. [WY08], who obtained the bounds  $k = 30$ ,  $k = 16$ , and  $k = 13$ , respectively.

Our work in this paper is similarly motivated by Conjecture 1.1. However, where most others have attempted to lower the best known value of  $k$  as described above, our focus is on establishing which graphs are  $\{1, 2\}$ -weight colourable. Addario-Berry, Dalal and Reed [ABDR08] showed that asymptotically almost every graph is  $\{1, 2\}$ -weight colourable, however it is not known which ones are not. Chang et al and Lu et al ([CLWY10], [LYZ10]) have made some progress in determining which classes of graphs are  $\{1, 2\}$ -weight colourable, notably having shown that 3-connected bipartite graphs are one such class. A complete classification of such graphs would determine those graphs for which  $k = 3$  is the smallest possible solution in Problem 1, and would reduce Conjecture 1.1 to just those graphs.

The results that follow are, for the most part, concerned with a more general problem than that of finding  $\{1, 2\}$ -weight colourings, namely that of finding  $\{a, b\}$ -weight colourings for more general values of  $a$  and  $b$ . In such cases, the existence of a  $\{1, 2\}$ -weight colouring follows as an unstated corollary. In Section 2, we establish a wide range of basic graphs which admit  $\{a, b\}$ -weight colourings. We also establish classes of graphs which do not admit  $\{a, b\}$ -weight colourings, but which do admit an  $\{a, b\}$ -edge weighting which is almost a proper colouring. These results provide building blocks for our results on the weight

colourability of bipartite graphs in Section 3 and of other general classes of graphs, particularly direct products of graphs, in Section 4. Of note, we show in Section 3 that if every cycle of  $G$  is of length  $0 \pmod 4$ , then  $G$  is  $\{1, 2\}$ -weight colourable.

## 2 Building blocks: Weight colourings of basic graphs

We will use standard graph theory terminology; the reader may refer to [BM08] for clarification of any terms which are not specifically defined here.

The *length* of a path (walk) is defined to be the number of edges of the path (walk). A *thread* in a graph  $G$  is a walk connecting two vertices  $x$  and  $y$ , not necessarily distinct, such that the internal vertices are distinct from all others on the walk, all internal vertices have degree 2 in  $G$ , and  $\deg(x), \deg(y) \geq 3$ . If  $x$  and  $y$  are distinct, then the walk is in fact a path and in this case we may refer to the thread as an *ear*. If the condition that  $\deg(x), \deg(y) \geq 3$  is changed to  $\deg(x), \deg(y) \geq 2$  in either case, we have a *subthread* or *subear* respectively.

A *cut vertex* of a graph is one whose removal disconnects the graph. A graph is 2-connected if it has no cut vertex. A graph (not necessarily simple) is called *separable* if it can be decomposed into two nonempty subgraphs with exactly one vertex in common. A simple graph is separable if and only if it is not 2-connected. A maximal nonseparable subgraph of  $G$  is a *block* of  $G$ . Note that a block is isomorphic either to  $K_2$  or to a 2-connected graph. An *end block* of  $G$  is a block which contains at most one cut vertex of  $G$ .

A graph is *c-colourable* if the vertices can be coloured with  $c$  colours so that adjacent vertices get different colours.

$K_n$  and  $C_n$ , respectively, denote the complete graph and the cycle on  $n$  vertices. The *Cartesian product* of two graphs  $G$  and  $H$ , denoted by  $G \square H$ , is defined as the graph having vertex set  $V(G) \times V(H)$  where two vertices  $(u, u')$  and  $(v, v')$  are adjacent if and only if either  $u = v$  and  $u'$  is adjacent to  $v'$  in  $H$  or  $u' = v'$  and  $u$  is adjacent to  $v$  in  $G$ .

We present a few simple observations.

**Proposition 2.1** *Let  $a, b, t$  be nonzero real numbers and  $G$  a graph. Then*

- (i)  $G$  is  $\{a, b\}$ -weight colourable if and only if  $G$  is  $\{at, bt\}$ -weight colourable, and
- (ii) if  $G$  is  $\{a, b\}$ -weight colourable then  $G$  is  $\{p, q\}$ -weight colourable for any nonzero  $p, q \in \mathbb{R}$  which are linearly independent over  $\mathbb{Q}$ .

**Proof:** (i) This follows from the fact that  $w(u) \neq w(v)$  if and only if  $t \cdot w(u) \neq t \cdot w(v)$ . (ii) Note that if two adjacent vertices receive distinct linear combinations of  $a$  and  $b$  as weights, then the coefficients of these linear combinations will suffice for any two linearly independent nonzero reals.  $\square$

From Proposition 2.1 we deduce the following, adopting the convention that 0 and 1 are relatively prime integers:

**Corollary 2.2** *A graph  $G$  is 2-weight colourable if and only if  $G$  is  $\{a, b\}$ -weight colourable for every pair of relatively prime integers  $a$  and  $b$ .*

Proposition 2.1 allows us to reduce our proofs of positive results on the existence of  $\{a, b\}$ -weight colourings of a graph to relatively prime integers. Results in which we show that  $G$  does not admit an  $\{a, b\}$ -weight colouring will not rely on such assumptions – we will prove them for all real  $a, b$ .

**Proposition 2.3** *If  $G$  is  $d$ -regular and  $\{a, b\}$ -weight colourable for a fixed choice of  $a$  and  $b$  then (i) it is  $d$ -colourable, and (ii) it is 2-weight colourable.*

**Proof:** (i) The weighted degree of each vertex must be a number of the form  $ta + (d - t)b$  for some  $0 \leq t \leq d$ , and a vertex of weighted degree  $da$  cannot be adjacent to a vertex of weighted degree  $db$ . Thus putting the vertices of weighted degree  $da$  or  $db$  in the same colour class gives a  $d$ -colouring.

(ii) In an  $\{a, b\}$ -edge weighting of a  $d$ -regular graph, the accumulated weight at any vertex is in a one-to-one correspondence with the number of incident edges of weight  $a$ . Thus if one choice of  $a$  and  $b$  gives a vertex colouring, then any other choice of  $a$  and  $b$  will as well.  $\square$

**Corollary 2.4** *If  $\chi(G) = \Delta(G) + 1$  or, equivalently (by Brooks theorem), if  $G$  is an odd cycle or a complete graph then  $G$  is not  $S$ -weight colourable for any set  $S$  of size 2.*

Even though the complete graph is not  $S$ -weight colourable for any set of size 2, it has an  $S$ -edge-weighting that is very close to being an  $S$ -weight colouring. This specific weighting will be useful in constructing families of 2-weight colourable graphs and non-2-weight colourable graphs in Section 4.

**Lemma 2.5** *Given  $n \geq 2$  and  $a \neq b \in \mathbb{R}$ , there is an  $\{a, b\}$ -edge-weighting of  $K_n$  such that the weighted degrees of all the vertices are distinct except for 2 of them. Furthermore, in any such  $\{a, b\}$ -edge-weighting, the degree sequence of the subgraph induced by the edges of weight  $a$  (as well as the subgraph induced by the edges of weight  $b$ ) is either*

$$(1, 2, \dots, \lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1, \dots, n - 2, n - 1),$$

or

$$(0, 1, \dots, \lceil \frac{n}{2} \rceil - 2, \lceil \frac{n}{2} \rceil - 1, \lceil \frac{n}{2} \rceil - 1, \lceil \frac{n}{2} \rceil, \dots, n - 3, n - 2).$$

**Proof:** We prove the first part with an explicit construction. Choose any two vertices and assign weight  $a$  to the edge joining them. Choose a new vertex and assign weight  $b$  to all the edges joining this vertex to the previous two vertices. Choose another vertex and assign weight  $a$  to all the edges joining this vertex to the previous three vertices. By repeating this process until all vertices are exhausted, we achieve the desired edge-weighting since the two vertices chosen first will have the same weight while the remainder of the graph is properly coloured. Note that we achieve the same result by swapping  $a$  and  $b$  in this argument.

We prove the second part of the lemma by induction on  $n$ . Suppose  $w$  is such an edge-weighting of  $K_n$  and let  $w(u) = w(v)$ . It is easy to verify the claim for  $n = 2$  and  $n = 3$ . If  $w(x) \notin \{(n - 1)a, (n - 1)b\}$  for every vertex  $x$  then  $w(x)$  can only take  $n - 2$  values, a contradiction to the choice of  $w$ . If  $w(u) = w(v) \in \{(n - 1)a, (n - 1)b\}$  then by removing  $u$  and  $v$ ,  $w$  induces an  $\{a, b\}$ -weight colouring of  $K_{n-2}$ , a contradiction to Corollary 2.4. Thus there exists a vertex  $x \neq u, v$  such that  $w(x) \in \{(n - 1)a, (n - 1)b\}$ . The claim follows by induction on  $K_n - x$ .  $\square$

The following technical lemmata will be useful for the rest of the paper, since they establish useful tools for finding edge-weighting vertex colourings of graphs with specific structural properties.

**Lemma 2.6** *Suppose  $G$  has a vertex  $v$  with a set of leaf neighbours  $L$  where  $|L| \geq \lceil \deg(v)/2 \rceil$ . Let  $a \neq b$  be real numbers with  $ab > 0$ . If  $G \setminus L$  is  $\{a, b\}$ -weight colourable, then so is  $G$ .*

**Proof:** As mentioned, Proposition 2.1 allows us to only consider  $a, b \in \mathbb{Z}^+$ . Suppose  $w$  is an  $\{a, b\}$ -weight colouring of  $G \setminus L$ . The possible extensions of  $w$  to  $G$  give exactly  $|L| + 1$  possible weights for  $v$ . Since  $v$  has at most  $|L|$  neighbours in  $G \setminus L$ , in at least one of the extensions, the weighted degree of  $v$  is different from the weighted degrees of the neighbours of  $v$  in  $G \setminus L$ . The weighted degree of  $v$  is also different from the weighted degrees of the neighbours of  $v$  in  $L$ , since  $ab > 0$ .  $\square$

**Corollary 2.7** *Every tree with at least 3 vertices is  $\{a, b\}$ -weight colourable, where  $a \neq b$  are real numbers with  $ab > 0$ .*

**Proof:** The statement holds for any star,  $K_{1, n-1}$ , since the assignment of  $a$  to all edges achieves the desired result. As such the result holds for  $n = 3$  since the unique tree on 3 vertices is a star. Let  $T$  be a tree on  $n$  vertices which is not a star and assume the result holds for any tree with fewer than  $n$  vertices. Every tree has a vertex  $v$  that has at least  $\lceil \deg(v)/2 \rceil$  leaf neighbours. Since  $T$  is not a star, removing the leaf neighbours of  $v$  gives a subtree  $T'$  on at least 3 vertices. By the induction hypothesis  $T'$  has an  $\{a, b\}$ -weight colouring. By Lemma 2.6,  $T$  does as well.  $\square$

The following lemma establishes that we may contract long threads in a way that maintains weight colourability.

**Lemma 2.8** *Let  $G$  be a graph,  $P = v_0, e_1, v_1, e_2, v_2, e_3, v_3, e_4, v_4, e_5, v_5$  be a subthread of  $G$ , and  $a \neq b$  be any two real numbers. Let  $G' = G/\{e_1, e_2, e_3, e_4\}$  Then,*

- (i) *If  $w$  is an  $\{a, b\}$ -weight colouring of  $G$ , then  $w(e_1) = w(e_5) \neq w(e_3)$ .*
- (ii) *If  $G'$  is  $\{a, b\}$ -weight colourable, then so is  $G$ .*
- (iii) *If  $\deg(v_0) = 2$  or  $\deg(v_5) = 2$ , then  $G$  is  $\{a, b\}$ -weight colourable if and only if  $G'$  is  $\{a, b\}$ -weight colourable.*

**Proof:** (i) If  $w(e_1) \neq w(e_5)$  then either one of the two choices for  $w(e_3)$  results in an improper colouring at  $e_2$  or  $e_4$ . Hence  $w(e_1) = w(e_5)$  and  $w(e_3)$  must be distinct.

(ii) For convenience, we still denote the vertex obtained from the contraction by  $v_0$ . Suppose  $w'$  is an  $\{a, b\}$ -weight colouring of  $G'$ . Then  $w'(v_0) \neq w'(v_5)$ . Without loss of generality assume  $w'(v_0v_5) = a$ . Let  $w(e) = w'(e)$  for each  $e \notin \{e_1, e_2, e_3, e_4, e_5\}$ ,  $w(e_1) = w(e_5) = a$  and  $w(e_3) = b$ . There are two possibilities for the weights of  $e_2$  and  $e_4$ . Assigning  $w(e_2) = a$  and  $w(e_4) = b$  does not yield a proper vertex colouring of  $G$  if and only if either  $w(v_0) = 2a$  or  $w(v_5) = a + b$ . Similarly, defining  $w(e_2) = b$  and  $w(e_4) = a$  does not yield a proper vertex colouring of  $G$  if and only if either  $w(v_0) = a + b$  or  $w(v_5) = 2a$ . Suppose that neither weighting works. If the first possibility gives  $w(v_0) = 2a$ , then the second must give  $w(v_5) = 2a$ . If the first possibility gives  $w(v_5) = a + b$ , then the second gives  $w(v_0) = a + b$ . In either case  $w(v_0) = w(v_5)$ , a contradiction.

(iii) Assume  $\deg(v_0) = 2$  and let  $e_0$  be the other edge incident with  $v_0$ . Suppose  $w$  is an  $\{a, b\}$ -weight colouring of  $G$ . By (i) we have  $w(e_0) = w(e_4)$  and  $w(e_1) = w(e_5)$ . Hence  $w(v_0) = w(v_4) \neq w(v_5)$ . Thus, by assigning the common weight of  $e_1$  and  $e_5$  to the edge  $v_0v_5$ , we get an  $\{a, b\}$ -weight colouring of  $G'$ .  $\square$

The degree condition on the ends of  $P$  in Lemma 2.8 (iii) cannot be dropped. For example, by taking  $G$  to be the path of length 5,  $a = 1$ , and  $b = 2$ , Lemma 2.8 (iii) fails.

From this lemma we may deduce necessary and sufficient conditions for the existence of  $\{a, b\}$ -weight colourings of cycles.

**Proposition 2.9** *Let  $a$  and  $b$  be any distinct real numbers. Then  $C_n$  is  $\{a, b\}$ -weight colourable if and only if  $n \equiv 0 \pmod{4}$ .*

In lieu of a proof, we simply note that, by Lemma 2.8 (iii), the proof of this proposition may be reduced to the cases  $C_3$ ,  $C_4$ ,  $C_5$  and  $C_6$ . The details are left to the reader. There are  $\{a, b\}$ -edge weightings of other cycles of length  $4k + 1$ ,  $4k + 2$  and  $4k + 3$  which give vertex colourings with as few conflicts as possible. These results are largely technical, though not difficult to prove.

**Proposition 2.10** *Let  $a$  and  $b$  be any distinct real numbers. Then  $C_{2k+1}$  has an  $\{a, b\}$ -edge weighting  $w$  such that only one edge  $e = uv$  has the property that  $w(u) = w(v)$ .*

**Proposition 2.11** *Let  $a$  and  $b$  be any distinct real numbers. Then  $C_{4k+2}$  has an  $\{a, b\}$ -edge weighting  $w$  such that precisely two edges  $e = uv$  and  $e' = u'v'$  have the property that  $w(u) = w(v)$  and  $w(u') = w(v')$ . Furthermore,*

- *the distance between  $e$  and  $e'$  is even,*
- *$e$  and  $e'$  may be chosen to be any two edges at an even distance, and*
- *if  $f_1$  and  $f_2$  are the edges incident to  $e$ , then their weights are equal and can be chosen to be either  $a$  or  $b$  (similar for  $e'$ ).*

We present a specific consequence of Proposition 2.11 which we will find useful.

**Proposition 2.12** *Let  $k$  be an integer,  $k \geq 1$ . Then  $C_{4k+2}$  has an  $\{a, b\}$ -edge-weighting such that three consecutive vertices have equal weight and the rest of the cycle is properly coloured. Furthermore, the edge-weighting can be chosen so that the weights of the four edges which contribute to the weights of those three vertices will all be  $a$ , all  $b$ , or alternate between  $a$  and  $b$ .*

Let  $\Theta_{(m_1, \dots, m_d)}$ ,  $d \geq 3$ , be the graph constructed from  $d$  internally disjoint paths between distinct vertices  $x$  and  $y$ , where the  $i$ -th path has of length  $m_i$ . For simplicity, we assume  $m_1 \leq m_2 \leq \dots \leq m_d$ . Such graphs will be referred to as *theta graphs*. We present necessary and sufficient conditions for theta graphs to be 2-weight colourable.

**Theorem 2.13** *Let  $d \geq 3$  and let  $a, b$  be real numbers. The graph  $\Theta_{(m_1, m_2, \dots, m_d)}$  is 2-weight colourable if and only if it is not of the form  $\Theta_{(1, 4k_2+1, \dots, 4k_d+1)}$ .*

**Proof:** Let  $x$  and  $y$  be the two vertices of degree greater than two, and let  $\{P_i | 1 \leq i \leq d\}$  be the  $d$  internally disjoint paths between  $x$  and  $y$ .

Suppose  $w$  is an  $\{a, b\}$ -weight colouring of  $\Theta_{(1, 4k_2+1, \dots, 4k_d+1)}$ . By applying Lemma 2.8 (i) to each  $P_i$ , we observe that on any of the  $d$  disjoint paths between  $x$  and  $y$  the first and last edges must receive same weight. Thus  $w(x) = w(y)$ , a contradiction since  $x$  and  $y$  are adjacent. Hence  $\Theta_{(1, 4k_2+1, \dots, 4k_d+1)}$  is not  $\{a, b\}$ -weight colourable for any  $a, b$ .

Consider  $\Theta_{(m_1, m_2, \dots, m_d)} \not\cong \Theta_{(1, 4k_2+1, \dots, 4k_d+1)}$ . We can assume that  $|a| \geq |b|$ . Let  $n_j$  be the number of paths that have length equivalent to  $j \pmod{4}$ . Note that  $n_0 + n_1 + n_2 + n_3 = d = \deg(x) = \deg(y)$ . For each path  $P_i$ , weight each edge according to Lemma 2.8 so that the edges incident with  $x$  are weighted

$a$ . Then  $w(x) = da$  so it has no conflicts with its neighbours since  $d \geq 3$  and the condition on the magnitudes of  $a$  and  $b$  gives  $da \notin \{2a, a + b\}$ . Note that, if  $|P_i| \geq 2$ , there are two choices for the next edge's weight on  $P_i$  which determines the rest of the weights. Given one such weighting of a path  $P_i$ , the effects of switching to the alternate weighting where the edge incident to  $x$  receives weight  $a$  depend on the parity of the length of the path. If  $|P_i|$  is even, the weight of the edge incident to  $y$  and the vertex weights of the neighbours of  $x$  and  $y$  on  $P_i$  all change. If  $|P_i|$  odd, the weight of the edge incident to  $y$  remains unchanged, but the vertex weights of the neighbours of  $x$  and  $y$  on  $P_i$  do change. In all cases the only possible weights on path-neighbours of  $x$  or  $y$  are  $2b, a + b$  and  $2a$ . We prove, by cases, that there is an appropriate set of choices which make  $w(y)$  distinct from its neighbours.

$\mathbf{n}_0 + \mathbf{n}_2 \geq 4$ : Our choice of weightings for even  $P_i$ 's give at least 5 possible values for  $w(y)$ , so there is a choice such that  $w(y) \notin \{2b, a + b, 2a, da\}$ .

$\mathbf{n}_0 + \mathbf{n}_2 = 3$ : If no  $P_i$  has length 1,  $da$  is not a forbidden weight for  $y$ . Also, if  $n_3 \geq 1$  then there is an edge incident to  $y$  with weight  $b$ , and  $w(y) \neq da$ . In either case there is a choice of weightings so that  $w(y) \notin \{2b, a + b, 2a\}$ .

So, assume that  $m_1 = 1$  and  $n_3 = 0$ . If the initial weighting fails then we must have

$$\{2b, a + b, 2a, da\} = \{(d - 3)a + 3b, (d - 2)a + 2b, (d - 1)a + b, da\}$$

which implies that  $b = -(d - 3)a$  and  $d \geq 4$ . The fact that  $|a| \geq |b|$  gives that  $d = 4$ , implying  $n_1 = 1$  and  $b = -a$ . We weight all edges explicitly. The single edge on the path of length 1 receives weight  $a$ . If  $n_0 = 3$  then weight the edges of one even path  $a, a, \dots, -a, -a$  and the other two  $a, -a, \dots, -a, a$ . If  $n_0 = 2$  and  $n_2 = 1$  weight the edges of the paths of length  $0 \pmod 4$  with  $a, a, \dots, -a, -a$  and the other even path with  $a, a, \dots, a, a$ . If  $n_0 = 1$  and  $n_2 = 2$  weight the edges of the path of length  $0 \pmod 4$  with  $a, -a, \dots, -a, a$  and the two other even paths with  $a, -a, \dots, a, -a$ . Finally if  $n_0 = 0$  and  $n_2 = 3$  weight the edges of all even paths with  $a, -a, \dots, a, -a$ . Each weighting gives a vertex-colouring for its respective case.

$\mathbf{n}_0 + \mathbf{n}_2 = 2$ : If  $n_3 = 0$  and  $n_0 > 0$  then assign weights to the edges of one path which is length  $0 \pmod 4$  so that the weights of the first and last edges are both  $a$ . Weight the edges of the other even path so that the edge incident to  $x$  is weighted  $a$  and the edge incident to  $y$  is weighted  $b$ . If  $n_3 = n_0 = 0$  but either  $d > 3$  or  $b \neq 0$  then assign weights to the edges of both even paths so that their edges incident with  $x$  are weighted  $a$ , one of the edges incident with  $y$  is weighted  $a$  and the other is weighted  $b$ . In both cases weight the edges of the paths of length  $1 \pmod 4$  so the weights are, in order beginning with the edge incident with  $x$ ,  $a, a, \dots, b, a$  (if the path is a single edge, give it weight  $a$ ). In the case when  $n_3 = n_0 = 0$ ,  $d = 3$  and  $b = 0$  weight the edges of the two even paths  $a, 0, 0, \dots, a, 0$  and the single odd path with  $0, 0, a, a, \dots, a, 0$  (beginning with the edge incident with  $x$  in each case). The weighting given in each case gives a proper vertex colouring.

Assume  $n_3 \geq 1$ . If  $n_0 \neq n_2$  then choose weightings for each  $P_i$  so that  $w(x) = da$  and each remaining neighbour of  $y$  has accumulated weight  $a + b$ . Then  $w(y) = an_0 + an_1 + bn_2 + bn_3$ . Since  $n_3 \geq 1$  we have  $w(x) \neq w(y)$ , so the only possible conflict is if  $w(y) = a + b$ . In this case change both even  $P_i$ 's to



their alternate weighting, maintaining  $w(x) = da$  and producing a new weight at  $y$ :

$$\begin{aligned} w'(y) &= bn_0 + an_1 + an_2 + bn_3 \\ &= w(y) + (a - b)(n_2 - n_0) \\ &= a + b + (a - b)(n_2 - n_0) \\ &= \begin{cases} 3a - b & \text{if } n_0 = 0, n_2 = 2 \\ 3b - a & \text{if } n_0 = 2, n_2 = 0 \end{cases} \end{aligned}$$

In either case  $w'(y) \neq a + b$ . If  $n_0 = 2$  then  $y$  has neighbours with weights  $2b$ , and  $3b - a \neq 2b$ . Similarly if  $n_2 = 2$  then the weight at  $y$  avoids conflict with its neighbours with weight  $2a$ .

If  $n_0 = n_2 = 1$  we start again with choices from the basic strategy that leave all path-neighbours with weight  $a + b$ . We have  $w(x) = da \neq (n_1 + 1)a + (n_3 + 1)b = w(y)$ . Thus the only conflict can again be if  $w(y) = a + b$  or equivalently,  $an_1 + bn_3 = 0$ . In this case we weight the edges of  $P_i$ 's of lengths equivalent to 0, 1, 2, and 3 mod 4 with  $\{a, a, \dots, b, b\}$ ,  $\{a, b, \dots, a, a\}$ ,  $\{a, a, \dots, a, a\}$  and  $\{a, b, \dots, b, b\}$  respectively. We still have that  $w(y) = a + b \neq da = w(x)$  and no neighbour of  $y$  has weight  $a + b$ .

**$n_0 + n_2 = 1$ :** If  $n_3 = 0$  then weight the edges of the even path so that the edge incident with  $x$  receives weight  $a$  and the edge incident with  $y$  receives weight  $b$ . Weight the edges of the paths of length 1 mod 4 so the weights are, in order beginning with the edge incident with  $x$ ,  $a, a \dots b, a$  (if the path is a single edge, give it weight  $a$ ). This weighting gives a proper vertex colouring. Assume  $n_3 \geq 1$ . Again, weight the edges of each  $P_i$  so that  $w(x) = da$  and each neighbor of  $y$  (distinct from  $x$ ) has accumulated weight  $a + b$ . Since  $n_3 \geq 1$  we have that  $w(x) = da \neq w(y)$ . If  $w(y) \neq a + b$ , then  $w$  is an  $\{a, b\}$ -weight colouring. Suppose  $w(y) = a + b$ . Equivalently

$$(n_0 + n_1 - 1)a + (n_2 + n_3 - 1)b = 0. \quad (1)$$

Change the edge weights of the even length path to begin with  $b, a$ . Call this weighting  $w'$ . We now have  $w'(x) = (d - 1)a + b$  and  $w'(y) \neq a + b$ . All neighbours of  $y$  still have weight  $a + b$ , so the only possible conflicts are between  $x$  and its neighbours. We reduce all potential conflicts to one of four cases, which are solved explicitly.

If  $w'(x) = w'(y)$  then since  $w'(x) = (d - 1)a + b$ ,  $y$  is incident with precisely one edge with weight  $b$ . Since  $n_3 \geq 1$ , the edge with weight  $b$  comes from a path of length 3 mod 4. This gives  $n_0 = 0$ ,  $n_2 = 1$  and  $n_3 = 1$  and then Equation 1 and  $|a| \geq |b|$  gives either

- $n_0 = 0$ ,  $n_1 = 1$ ,  $n_2 = 1$ ,  $n_3 = 1$  and  $b = 0$  (**case iii.** below).
- $n_0 = 0$ ,  $n_1 = 2$ ,  $n_2 = 1$ ,  $n_3 = 1$  and  $b = -a$  (**case iv.** below).

The neighbours of  $x$  have accumulated weights either  $a + b$  or  $2a$ . If  $w(x) = (d - 1)a + b = a + b$  then this implies that  $d = 2$  but the hypotheses of the theorem include  $d \geq 3$ . If  $w(x) = (d - 1)a + b = 2a$  then  $b = -(d - 3)a$ . The fact that  $d \geq 3$  and  $|a| \geq |b|$  now give either

- $n_0 = 1$ ,  $n_1 = 0$ ,  $n_2 = 0$ ,  $n_3 = 2$  and  $b = 0$  which is dealt with in **case i.** below.
- $n_0 = 1$ ,  $n_1 = 1$ ,  $n_2 = 0$ ,  $n_3 = 2$  and  $b = -a$  which is dealt with in **case ii.** below.
- $n_0 = 0$ ,  $n_1 = 1$ ,  $n_2 = 1$ ,  $n_3 = 1$  and  $b = 0$  which is dealt with in **case iii.** below.

- $n_0 = 0, n_1 = 2, n_2 = 1, n_3 = 1$  and  $b = -a$  which is dealt with in **case iv.** below.

**case i.** In this case  $x$  and  $y$  are not adjacent. Weight the edges of the path of length equivalent to  $0 \pmod 4$  with  $0, 0, \dots, a, a$  and the two odd paths with  $a, 0, \dots, 0, 0$ .

**case ii.** In this case  $x$  and  $y$  are not adjacent. Weight the edges of the paths of lengths equivalent to  $0 \pmod 4, 1 \pmod 4$  and  $3 \pmod 4$  with  $-a, -a, \dots, a, a, a, -a, \dots, a, a$  and  $a, -a, \dots, -a, -a$  respectively.

**case iii.** In this case  $x$  and  $y$  may be adjacent. Weight the edges of the paths of lengths equivalent to  $1 \pmod 4, 2 \pmod 4$  and  $3 \pmod 4$  with  $a, a, \dots, 0, a, 0, 0, \dots, 0, 0$  and  $0, 0, \dots, 0, a$  respectively.

**case iv.** In this case  $x$  and  $y$  may be adjacent. Weight the edges of the paths of lengths equivalent to  $1 \pmod 4, 2 \pmod 4$  and  $3 \pmod 4$  with  $a, -a, \dots, a, a, -a, -a, \dots, -a, -a$  and  $a, -a, \dots, -a, -a$  respectively.

Each of these edge-weightings gives a proper vertex colouring.

$\mathbf{n}_0 + \mathbf{n}_2 = 0$ : Every weighting of the paths  $P_i$  which gives  $w(x) = da$  must give  $w(y) = an_1 + bn_3$ . If  $m_1 = 1$  then, since our graph is not  $\Theta_{(1,4k_2+1, \dots, 4k_d+1)}$ , we have  $n_3 \geq 1$  and thus  $w(x) \neq w(y)$ . Suppose  $m \neq 1$ . For each  $P_i$  we have two choices for  $y$ 's neighbour. Each choice leaves  $w(y)$  constant. Thus there is a choice for each path which gives an edge-weighting vertex-colouring.  $\square$

### 3 Bipartite graphs

We begin the section by noting that the property of being  $\{a, b\}$ -weight colourable is not one that is inherited by subgraphs, nor is the property of being non- $\{a, b\}$ -weight colourable. For example, the graph consisting of  $K_4$  with a leaf attached is  $\{1, 2\}$ -weight colourable, however  $K_4$  is not  $\{a, b\}$ -colourable for any choice of  $a$  and  $b$ . Similarly  $K_4$  contains the subgraph  $C_4$  which is 2-weight colourable.

We can, however, characterize the minimal graphs with respect to subgraph containment in the class of graph which are not  $\{a, b\}$ -weight colourable for many pairs  $\{a, b\}$  (in particular,  $\{1, 2\}$ ). In Theorem 3.9 we establish that any graph which is not  $\{a, b\}$ -weight colourable must contain  $C_{2k+1}$  or  $C_{4k+2}$  as a subgraph for some positive integer  $k$ .

**Definition 3.1** A graph  $G$  is round if every cycle of  $G$  has length  $0 \pmod 4$ .

The class of round graphs is much richer than merely those obtained by taking a graph and subdividing each edge into a path of length 4. For example,  $\Theta_{(2,2,2)} \cong K_{2,3}$  is a round graph which is not obtained in this way.

The following lemma establishes a useful subgraph condition of round graphs which we will use in our study of the  $\{a, b\}$ -weight colourability of round graphs.

**Proposition 3.2** If  $G$  is a round graph and  $\Theta_{(i,j,k)}$  is a subgraph of  $G$ , then  $i, j$  and  $k$  are even and  $i \equiv j \equiv k \pmod 4$ .

**Proof:** Let  $\Theta_{(i,j,k)}$  be a subgraph of  $G$  and let  $P_i, P_j$  and  $P_k$  be the corresponding paths of length  $i, j$  and  $k$  respectively. Since  $G$  is round,  $i + j \equiv i + k \equiv j + k \equiv 0 \pmod 4$ . The result follows.  $\square$

Before proceeding with our results on bipartite graphs we present the following definition which we adopt throughout this section (and this section only). In any  $\{a, b\}$ -edge-weighting of a graph, the weighted degree of every vertex is of the form  $ra + sb$  for some nonnegative integers  $r, s$ . We will call a weighted vertex *even* (*odd*) if its weighted degree is  $ra + sb$  with  $r$  even (odd). Note that the parity

of a weighted vertex does not necessarily refer to the parity of its weight. However, by Proposition 2.1, if  $a$  and  $b$  are not independent over  $\mathbb{Q}$  then we will assume that they are relatively prime integers, and so we will assume that  $a$  is an odd integer in this case. If  $b$  is even, which will be the case in a number of the following results, then the parity of a weighted vertex does coincide with the parity of its weight.

Since a number of our arguments rely on this notion of parity, we often exclude those pairs of numbers whose ratio may be reduced to a ratio of odd integers. We define the sets:

$$\mathcal{E} = \left\{ \{a, b\} \mid \frac{a}{b} = \frac{p}{q}, p, q \text{ odd integers} \right\}$$

$$\mathcal{N} = \left\{ \{a, b\} \mid \frac{a}{b} = \frac{p}{q}, p, q \in \mathbb{Z}, pq \leq 0 \right\}$$

We have already seen examples of bipartite graphs which are 2-weight colourable ( $C_{4k}$  for any  $k \geq 1$ , bipartite theta graphs except  $\Theta_{(1, 4k_2+1, \dots, 4k_d+1)}$ ) and some which are not ( $C_{4k+2}$  for any  $k \geq 1$ ). From these examples, we note that a bipartite graph  $G$  with both parts of odd size is not necessarily  $\{a, b\}$ -weight colourable. However, if  $G$  has one part of even size, we are able to prove  $G$  is  $\{a, b\}$ -weight colourable for particular values of  $a$  and  $b$ .

**Theorem 3.3** *Let  $a, b \in \mathbb{R}$  be such that  $\{a, b\} \notin \mathcal{E}$ . If  $G$  is a connected bipartite graph with at least one part being of even size, then  $G$  is  $\{a, b\}$ -weight colourable.*

**Proof:** Let  $V(G) = X \cup Y$  be a bipartition of the vertices of  $G$  with  $|X|$  even. By Corollary 2.2 and since  $\{a, b\} \notin \mathcal{E}$ , we may assume that  $a$  is an odd integer and  $b$  is an even integer. We assign the weight  $b$  to each edge of  $G$ . Clearly  $v$  is even for each  $v \in V(G)$ . Let  $V(X) = \{x_1, x_2, \dots, x_{2k}\}$  and let  $P_i$  be an  $x_{2i-1}x_{2i}$ -path in  $G$ . By changing every edge weight along  $P_1$  we only change the parity of  $x_1$  and  $x_2$ . By repeating this process for each  $P_i$  we have that every vertex of  $X$  has odd parity and every vertex of  $Y$  has even parity.

Call the resulting edge-weighting  $w$ . Suppose that  $w$  is not an  $\{a, b\}$ -weight colouring. Then there are adjacent vertices  $x$  and  $y$  such that  $w(x) = w(y)$ . Thus there exist integers  $r, r', s, s'$  such  $w(x) = ra + sb$  where  $r$  is odd,  $w(y) = r'a + s'b$  where  $r'$  is even, and  $ra + sb = r'a + s'b$ . If  $a$  and  $b$  are linearly independent over  $\mathbb{Q}$ , we must have  $r = r'$ , a contradiction. Hence  $b = (p/q)a$  for some  $p, q \in \mathbb{Z}$  with  $\gcd(p, q) = 1$ . Thus  $rq + sp = r'q + s'p$ . Since  $r$  is odd and  $r'$  is even,  $p$  even implies  $q$  must be even, a contradiction. Hence  $p$  is odd. Similarly,  $q$  is odd. Therefore,  $b/a = p/q$  with  $p, q$  odd, contradicting our choice of  $a$  and  $b$ . Thus  $w$  is an  $\{a, b\}$ -weight colouring of  $G$ .  $\square$

**Corollary 3.4** *Let  $a, b \in \mathbb{R}$  be such that  $\{a, b\} \notin \mathcal{E} \cup \mathcal{N}$ . Let  $G \neq K_2$  be a connected bipartite graph with a vertex of degree 1. Then  $G$  is  $\{a, b\}$ -weight colourable. In particular, trees are  $\{a, b\}$ -weight colourable.*

**Proof:** Let  $V(G) = X \cup Y$  be a bipartition of the vertices of  $G$ . Let  $x \in X$  be a vertex of degree 1 and let  $y \in Y$  be its neighbour. If  $|X|$  or  $|Y|$  is even, then  $G$  is  $\{a, b\}$ -weight colourable by Theorem 3.3. If  $|X|$  is odd, then  $G - x$  has an  $\{a, b\}$ -weight colouring by Theorem 3.3, say  $w'$ , such that vertices in  $X \setminus \{x\}$  are odd and vertices in  $Y$  are even. By assigning  $b$  to the edge  $xy$  we maintain the parity of all the vertices. Also, since  $\{a, b\} \notin \mathcal{N}$  we have  $w'(y) \neq 0$  and so  $x$  and  $y$  will receive different weights, thus giving an  $\{a, b\}$ -weight colouring of  $G$ .  $\square$

**Theorem 3.5** *Let  $a, b \in \mathbb{R}$  be such that  $\{a, b\} \notin \mathcal{E} \cup \mathcal{N}$ . Let  $G$  be a connected bipartite graph with a thread of even length  $P$  and let  $U$  be the internal vertices of  $P$ . If  $G - U$  is connected then  $G$  is  $\{a, b\}$ -weight colourable.*

**Proof:** We may assume that  $a$  is a positive odd integer and  $b$  is a positive even integer. If  $X \cup Y$  is the bipartition of  $V(G)$  and either  $|X|$  or  $|Y|$  is even, then  $G$  is  $\{a, b\}$ -weight colourable by Theorem 3.3. Assume both parts of  $G$  are of odd size. Let  $x$  and  $y$  be the ends of  $P$ . We first assume that  $x$  and  $y$  are distinct. By Lemma 2.8, we may assume that  $P$  is a path of length either 2 or 4.

Consider the case that  $P$  is of length 2, say  $P = xvy$ . Let  $G'$  be the bipartite graph obtained from  $G$  by deleting  $v$  and adding two leaves,  $v_1$  adjacent to  $x$  and  $v_2$  adjacent to  $y$ . Now  $G'$  is connected and bipartite with an even side, where  $v_1$  and  $v_2$  both belong to the even side. Theorem 3.3 gives an  $\{a, b\}$ -weight colouring of  $G'$ , say  $w'$ , so that  $v_1$  and  $v_2$  are both odd vertices. Hence  $xv_1$  and  $yv_2$  must both receive  $a$  as their weight. Let  $w$  be an  $\{a, b\}$ -edge-weighting of  $G$ , where  $w(xv) = w'(xv_1) = a$ ,  $w(yv) = w'(yv_2) = a$  and  $w(e) = w'(e)$  for all other edges  $e \in E(G)$ . If  $w$  is not an  $\{a, b\}$ -weight colouring of  $G$ , then either  $w(x) = 2a$  or  $w(y) = 2a$ . Without loss of generality, suppose  $w(x) = ra + sb = 2a$  (a similar argument will hold for  $y$ ). Since  $w(xv) = a$  and  $r$  even, we have  $r \geq 2$ . If  $r = 2$ , then  $sb = 0$  which implies  $s = 0$  or equivalently  $\deg_G(x) = 2$ , a contradiction. If  $r \geq 3$ , then  $sb < 0$  which gives  $b < 0$ , a contradiction. Thus  $w$  is an  $\{a, b\}$ -weight colouring of  $G$ .

Suppose  $|P| = 4$ . Let  $P = xv_1v_2v_3y$  and let  $G' = G - v_2$ . Now  $G'$  is bipartite with an even side  $X'$ , and  $x, y \in X'$ . Theorem 3.3 gives an  $\{a, b\}$ -weight colouring of  $G'$ , say  $w'$ , so that  $v_1$  and  $v_3$  are both even vertices. Hence  $xv_1$  and  $yv_3$  must both receive  $b$  as their weight. Let  $w$  be an  $\{a, b\}$ -edge-weighting of  $G$ , where  $w(v_1v_2) = w(v_2v_3) = a$  and  $w(e) = w'(e)$  for all other edges  $e \in E(G)$ . If  $w$  is not an  $\{a, b\}$ -weight colouring of  $G$ , then either  $w(x) = a + b$  or  $w(y) = a + b$ . Suppose  $w(x) = ra + sb = a + b$ . Then  $(r - 1)a = -(s - 1)b$ , and thus  $r$  is odd. Again, we have that  $a$  and  $b$  are positive integers. Thus either  $r - 1 < 0$  or  $s - 1 < 0$ . However, since  $w(xv_1) = b$ , we have  $s \neq 0$ , and since  $r$  is odd,  $r \neq 0$ . Thus  $w$  is an  $\{a, b\}$ -weight colouring of  $G$ .

Now, suppose  $x$  and  $y$  are not distinct (call this vertex  $x$ ). Then  $P$  is a cycle which is an end block of  $G$  and  $x$  is a cut vertex of  $G$ . Let  $z_1$  and  $z_2$  be the neighbours of  $x$  in  $P$ . Since  $G' = G - U$  is a connected bipartite graph with one part having even size, then by Theorem 3.3 there is an  $\{a, b\}$ -weight colouring of  $G'$ , say  $w'$ . We give an edge weighting  $w''$  of  $P$  as follows:

- if  $P$  has length 2 (mod 4), then by Proposition 2.12 we may define an  $\{a, b\}$ -weight colouring of  $P$ ,  $w''$ , so that  $w''(z_1) = w''(x) = w''(z_2) = 2a$  and  $P$  is properly coloured elsewhere;
- if  $P$  has length 0 (mod 4), then by Proposition 2.9 we may define an  $\{a, b\}$ -weight colouring of  $P$ ,  $w''$ , so that  $w''(x)$  is the larger of  $2a$  and  $2b$  and  $P$  is properly coloured.

Let  $w$  be the weighting obtained by combining  $w'$  and  $w''$ . Then  $w(x) > w(z_1), w(z_2)$  and  $x$  has the same parity under  $w$  as under  $w'$ . Hence the weight of  $x$  is distinct from its neighbours in  $G$ . Since all other vertices are properly coloured by  $w'$  or  $w''$ ,  $w$  gives an  $\{a, b\}$ -weight colouring of  $G$ .  $\square$

**Theorem 3.6** *If  $G$  is a 2-connected round graph which is not a cycle then  $G$  contains at least 2 even ears.*

**Proof:**

We first claim that  $G$  contains no proper 2-connected subgraph which contains all even ears of  $G$ . Toward a contradiction suppose  $H$  is a 2-connected maximal proper subgraph of  $G$  that contains all even ears of  $G$ . There exist two vertices of  $H$ , say  $x$  and  $y$ , which are connected by a path  $P$  such that  $H \cap P = \{x, y\}$ . Since  $H$  is 2-connected, there are also 2 edge disjoint paths  $P'$  and  $P''$  in  $H$  between  $x$  and  $y$ . Thus  $P \cup P' \cup P''$  is a theta graph, and by Lemma 3.2  $P$  must be of even length. Since  $H$  already contains all even ears of  $G$ ,  $H' = H \cup P$  must be a proper subgraph of  $G$  but  $H'$  is also 2-connected which contradicts the maximality of  $H$ .

Now, if  $G$  has no even ear, then any cycle of  $G$  is a 2-connected subgraph containing all the even ears and this is a contradiction as  $G$  is not a cycle. If  $G$  has only one ear, let  $T$  be the ear and let  $x$  and  $y$  be the two ends of  $T$ . Then there are 2 edge disjoint paths connecting  $x$  and  $y$ , one of which must be edge disjoint from  $T$ . This path together with  $T$  forms a cycle that contains all the even ears of  $G$ , a contradiction.  $\square$

**Corollary 3.7** *If  $G$  is a round graph and all threads of  $G$  are odd, then  $G$  has at least two leaves.*

We are now able to prove that round graphs can be edge-weight vertex-coloured with most sets of size 2.

**Theorem 3.8** *Every round graph is  $\{a, b\}$ -weight colourable for  $\{a, b\} \notin \mathcal{E} \cup \mathcal{N}$ .*

**Proof:** Let  $G$  be a round graph. Let  $B$  be an end block with vertex of attachment  $v$ . If  $B$  is isomorphic to  $K_2$ , then  $G$  is a bipartite graph with a leaf and thus is  $\{a, b\}$ -weight colourable by Corollary 3.4. If  $B$  is a cycle, then  $B$  is an even thread and  $G$  is  $\{a, b\}$ -weight colourable by Theorem 3.5. Otherwise, if  $B$  is a 2-connected graph which is not a cycle, then by Theorem 3.6,  $B$  has at least two even ears and thus  $B$  has at least one even ear, say  $P$ , which does not contain  $v$  as an internal vertex. Let  $U$  be the internal vertices of  $P$ . Since  $G - U$  is connected,  $G$  is  $\{a, b\}$ -weight colourable by Theorem 3.5.  $\square$

Theorem 3.8, together with Proposition 2.9, gives a class of minimal subgraphs with respect to containment which cannot be  $\{a, b\}$ -weight coloured for the pairs  $\{a, b\}$  on which we have focused.

**Corollary 3.9** *Let  $a$  and  $b$  be real numbers such that  $\{a, b\} \notin \mathcal{E} \cup \mathcal{N}$ . Any graph which is not  $\{a, b\}$ -weight colourable must contain a cycle of length  $1, 2$  or  $3 \pmod{4}$ .*

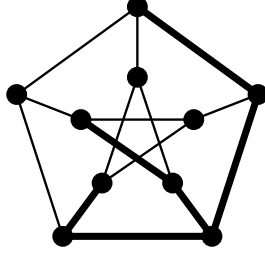
We end this section with the following problem.

**Problem 3** *Is it true that all bipartite graphs except  $C_{4k+2}$  and  $\Theta_{(1,4k_1+1,4k_2+1,\dots,4k_d+1)}$  are 2-weight colourable?*

## 4 More families of graphs with determined 2-weight colourability

We have given a number of examples of  $\{a, b\}$ -weight colourable graphs for values of  $a$  and  $b$  subject to particular restrictions. However we have seen few examples of graphs for which  $a$  and  $b$  can be any distinct real numbers. We note that the Petersen graph provides such an example of a 2-weight colourable graph. One such edge-weighting is given in Figure 1. By Proposition 2.3, note that any 2-weight colouring of the Petersen graph gives a 3-colouring of it, which is also an optimal proper vertex colouring.

In the rest of this section we describe more families of 2-weight colourable graphs as well as a class of nonbipartite graphs which are  $\{a, b\}$ -weight colourable when  $ab > 0$ . In particular we show that all



**Fig. 1:** An  $\{a, b\}$ -weight colouring of the Petersen graph. Bold edges are to receive weight  $b$ .

unicyclic graphs except cycles of length  $1, 2, 3 \pmod 4$  are 2-weight colourable. We also provide a number of results on Cartesian products of graphs. Finally, we explore techniques for constructing graphs which do not admit  $\{a, b\}$ -weight colourings for any choice of  $a$  and  $b$ .

We begin with our result on unicyclic graphs.

**Theorem 4.1** *Every connected unicyclic graph except  $C_{2k+1}$  and  $C_{4m+2}$  is  $\{a, b\}$ -weight colourable, where  $a$  and  $b$  are real numbers with  $ab > 0$ .*

**Proof:** We may assume that  $0 < a < b$ . By contradiction, let  $G$  be the smallest counterexample to our claim. Let  $C$  be the only cycle of  $G$ . We first note that by Lemma 2.6, we may assume that every vertex of  $G$  is either on  $C$  or is adjacent to a vertex of  $C$ . We may also assume that every vertex of  $G$  has degree at most 3. Next, we claim that there are at least two vertices of degree at least 3 on  $C$ . If not, let  $v$  be the only vertex of degree at least 3 on  $C$ . Let  $x$  and  $y$  be the neighbours of  $v$  on  $C$ . It is easy to find an edge-weighting  $w$  of  $C$  which yields a proper colouring on  $C - v$  and  $w(v) \geq w(x), w(y)$ . By assigning  $b$  to the other edge incident with  $v$ , we get an  $\{a, b\}$ -weight colouring of  $G$ , a contradiction.

Next, we claim that  $G$  has at most one ear of length at least 2. If not, then we choose some maximal path of degree 3 vertices on  $C$ ,  $x_1, \dots, x_k$ , and remove all leaves of  $G$  adjacent to those vertices. Call this subgraph  $G'$ . By minimality of our choice of  $G$ , we can assign an  $\{a, b\}$ -weight colouring  $w'$  to  $G'$ . Let  $w$  be the weighting of  $E(G)$  given by  $w(e) = w'(e)$  if  $e \in E(G')$  and  $w(e) = b$  otherwise. The only possible conflicts are between  $x_1$  and its neighbour on  $C$  which is not  $x_2$ , say  $y$  (or, similarly, between  $x_k$  and its neighbour on  $C$  which is not  $x_{k-1}$ ). However, since  $w(x_1) \geq a + b + w(x_1y)$  and  $w(y) \leq w(x_1y) + b$  (similar for  $x_k$ ),  $w$  is an  $\{a, b\}$ -weight colouring of  $G$  which contradicts our choice of  $G$ .

If  $G$  has exactly one ear of length at least 2, let  $e = rs$  and  $e' = r's'$  be the two edges that have exactly one endpoint of degree 2. Specifically, let  $\deg(r) = \deg(r') = 2$ ,  $\deg(s) = \deg(s') = 3$ . Note that  $r$  and  $r'$  need not be distinct, but, since there are at least 2 vertices on  $C$  of degree 3,  $s$  and  $s'$  are distinct. We construct an  $\{a, b\}$ -weight colouring of  $G$  based on the length of  $C \pmod 4$ .

- Suppose  $|C|$  is odd. By Proposition 2.10,  $C$  has an  $\{a, b\}$ -edge weighting  $w'$  which gives a proper vertex colouring except across  $rs$ . Let  $w(e) = w'(e)$  if  $e \in E(C)$ . If  $w'(r') - w'(s') = a$ , let  $w(e) = b$  for all  $e \in E(G) \setminus E(C)$ . Otherwise, let  $w(e) = a$  for all  $e \in E(G) \setminus E(C)$ . Clearly each leaf's neighbour has a weight strictly greater than its own. Since  $w'$  gives a proper colouring of  $C$  except for  $r$  and  $s$ , the only adjacent vertices of  $G$  which might not be properly coloured are  $r$

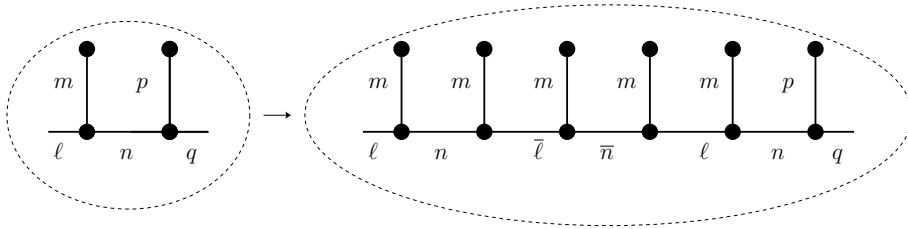
and  $s$  or  $r'$  and  $s'$ . However, our choice of weights for the leaves of  $G$  guarantees that  $r, s, r', s'$  are properly coloured as well. Thus  $w$  is an  $\{a, b\}$ -weight colouring of  $G$ .

- Suppose  $|C| \equiv 0 \pmod{4}$ . By Proposition 2.9,  $C$  has an  $\{a, b\}$ -weight colouring  $w'$  such that  $w(r) = 2a$  and  $w(s) = a + b$ . Let  $w(e) = w'(e)$  if  $e \in E(C)$ . If  $w'(r') - w'(s') = a$ , let  $w(e) = b$  for all  $e \in E(G) \setminus E(C)$ . Otherwise, let  $w(e) = a$  for all  $e \in E(G) \setminus E(C)$ . By the same argument as above,  $w$  is an  $\{a, b\}$ -weight colouring of  $G$ .
- Suppose  $|C| \equiv 2 \pmod{4}$ . Let  $t$  be the other neighbour of  $s$  on  $C$  and let  $t'$  be the other neighbour of  $r'$  on  $C$ . By Proposition 2.11, there is an  $\{a, b\}$ -edge weighting such that all vertices are properly coloured except  $r, s$  and  $t$ , and such that  $w(t'r') = w(r's') = a$ . Let  $w(e) = w'(e)$  for all  $e \in E(C)$ . Let  $f$  be the edge between  $s$  and its leaf, and let  $w(f) = a$ . For each  $e \in E(G) \setminus E(C) \setminus \{f\}$ , let  $w(e) = b$ . The only possible improperly coloured pairs of vertices are  $r$  and  $s$ ,  $s$  and  $t$  or  $r'$  and  $s'$ . However,

$$\begin{aligned} w(r) &= w'(r) = w'(s) < w(s) \\ w(s) &= w'(s) + a = w'(t) + a < w'(t) + b = w(t) \\ w(r') &= 2a < a + 2b = w(s') \end{aligned}$$

and so  $w$  is an  $\{a, b\}$ -weight colouring of  $G$ .

The only remaining case is that every vertex of  $C$  has degree 3. If  $|C|$  is even, assign the same weight to all the edges on the cycle and alternating weights to the leaf edges. The reader can verify that a solution for the cases when  $|C| = 3$  or  $|C| = 5$  exists. Each of these cases can be extended to larger odd cycle by making the replacement indicated in Figure 2. Note that the variables  $\bar{\ell}$  and  $\bar{n}$  refer to the weights different from  $\ell$  and  $n$ , respectively.



**Fig. 2:** Replacement operation to expand 2-weight colourings to larger cycles.

Thus, no minimal counterexample  $G$  exists. □

**Proposition 4.2** For  $n \geq 4$ , the graph  $K_2 \square K_n$  is 2-weight colourable.

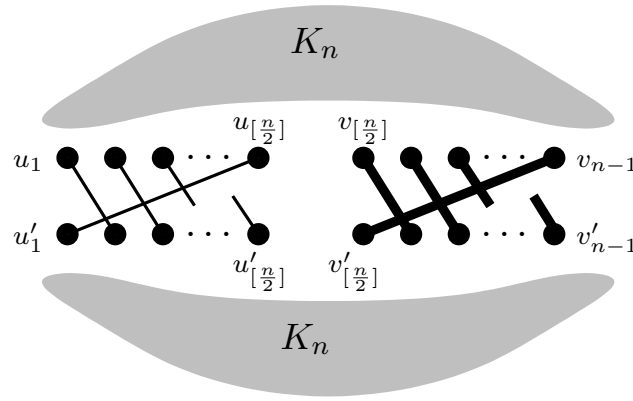
**Proof:** Let  $K_n$  and  $K'_n$  be the two copies of the complete graph. Denote the vertices of  $K_n$  and  $K'_n$ , respectively by

$$\begin{aligned} &\{u_1, u_2, \dots, u_{\lfloor n/2 \rfloor}, v_{\lfloor n/2 \rfloor}, v_{\lfloor n/2 \rfloor + 1}, \dots, v_{n-2}, v_{n-1}\}, \\ &\{u'_1, u'_2, \dots, u'_{\lfloor n/2 \rfloor}, v'_{\lfloor n/2 \rfloor}, v'_{\lfloor n/2 \rfloor + 1}, \dots, v'_{n-2}, v'_{n-1}\}. \end{aligned}$$

Let  $p$  be a derangement (permutation with no fixed points) of  $\{1, \lfloor n/2 \rfloor\}$  and  $\pi$  be a derangement of  $\{\lfloor n/2 \rfloor, n-1\}$ . Let  $u_i$  be adjacent to  $u'_{p(i)}$  for all  $1 \leq i \leq \lfloor n/2 \rfloor$  and  $v_i$  be adjacent to  $v'_{\pi(i)}$  for  $\lfloor n/2 \rfloor \leq i \leq n-1$ .

Since the graph is  $n$ -regular, if adjacent vertices have distinct weights then they have distinct numbers of incident edges having weight  $b$ . Using Lemma 2.5, we may weight the edges of  $K_n$  and  $K'_n$  so that the subscript of the vertex is precisely equal to the number of edges weighted  $b$  incident to that edge in  $K_n$ . Label  $u_i u'_{p(i)}$  with  $a$  for all  $1 \leq i \leq \lfloor n/2 \rfloor$  and weight  $v_i v'_{\pi(i)}$  with  $b$  for  $\lfloor n/2 \rfloor \leq i \leq n-1$ . Then any two vertices that are adjacent have a distinct number of incident edges weighted  $b$  and thus  $K_2 \square K_n$  is 2-weight colourable.  $\square$

Figure 3 gives an illustration of this construction.



**Fig. 3:** An  $\{a, b\}$ -weight colouring of  $K_2 \square K_n$ . Bold edges are to receive weight  $b$ .

**Proposition 4.3** *The graph  $K_2 \square C_n$  is 2-weight colourable if and only if  $n \geq 4$  and  $n \neq 5$ .*

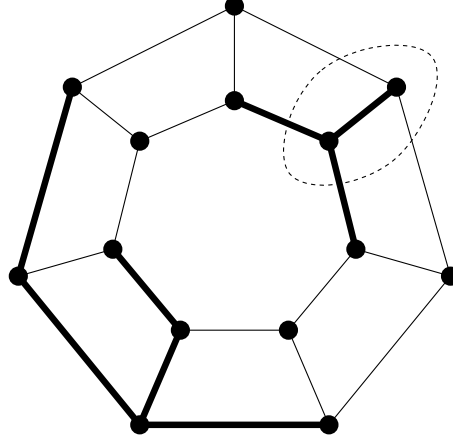
**Proof:** If  $n$  is even, then give every edge of one copy of  $C_n$  weight  $a$  and every edge of the other copy weight  $b$ . By alternating the weights of the images of  $K_2$  between  $a$  and  $b$  along the cycles, we have the desired  $\{a, b\}$ -weight colouring.

An example of an  $\{a, b\}$ -weight colouring of  $K_2 \square C_7$  is given in Figure 4. It can be extended to an  $\{a, b\}$ -weight colouring of  $K_2 \square C_9$  by replacing the left subgraph in Figure 5 with the right graph. Note that the right subgraph contains the left one, and thus this operation may be repeated as many times as needed to give an  $\{a, b\}$ -weight colouring for any  $K_2 \square C_{2k+1}$  ( $k \geq 3$ ). The reader may verify that no  $\{a, b\}$ -weight colouring of  $K_2 \square C_3$  or  $K_2 \square C_5$  exists.  $\square$

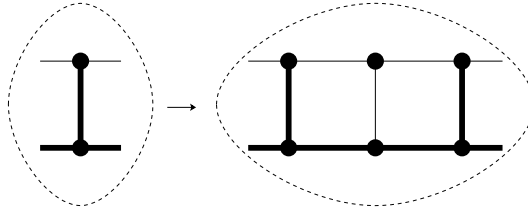
**Theorem 4.4** *Let  $G$  be a graph and  $H$  be a regular bipartite graph. If  $G \square K_2$  is 2-weight colourable, then  $G \square H$  is 2-weight colourable.*

**Proof:** Let  $w$  be an  $\{a, b\}$ -weight colouring of  $G \square K_2$ . Denote the two copies of  $G$  by  $G_1$  and  $G_2$  and denote the vertices of  $K_2$  by  $t_1$  and  $t_2$ . Since  $H$  is regular (say  $d$ -regular) and bipartite, Hall's Theorem guarantees a perfect matching  $M$  of  $H$ . Let  $X$  and  $Y$  be the parts of  $V(H)$ .





**Fig. 4:** An  $\{a, b\}$ -weight colouring of  $K_2 \square C_7$ . Bold edges are to receive weight  $b$ .



**Fig. 5:** Replacement operation for obtaining an  $\{a, b\}$ -weight colouring of  $K_2 \square C_{2k+1}$  for  $k \geq 4$ .

Define an edge-weighting of  $G \square H$  as follows. For each edge  $e = xy \in M$  where  $x \in X$  and  $y \in Y$ , weight the edges of the subgraph  $G \square e$  by  $w$  so that each vertex  $(u_G, x) \in V(G \square H)$  has weight  $w(u_G, t_1)$  and  $(u_G, y) \in V(G \square H)$  has weight  $w(u_G, t_2)$ . Assign every other edge of  $G \square H$  weight  $a$ . Call this weighting  $\phi$ .

We have that  $\phi(u) = w(u_G, t_1) + (d-1)a$  if  $u_H \in X$  and  $\phi(u) = w(u_G, t_2) + (d-1)a$  if  $u_H \in Y$ . Two vertices are adjacent if either their  $H$ -coordinates agree and they are adjacent in a copy of  $G$  or if their  $G$ -coordinates agree and they are adjacent in a copy of  $H$ . In the former case, their weights are distinct under  $\phi$  since they are distinct under  $w$ . In the latter, consider two adjacent vertices  $u = (u_G, u_H)$  and  $u' = (u_G, u'_H)$  where  $u_H \in X, u'_H \in Y$ . Then,  $w(u_G, t_1) \neq w(u_G, t_2)$  by choice of  $w$ , which implies that  $\phi(u) \neq \phi(u')$ . Thus  $\phi$  is an  $\{a, b\}$ -weight colouring of  $G \square H$ .  $\square$

**Corollary 4.5** *If  $G$  and  $H$  are regular bipartite graphs, then the following graphs are 2-weight colourable:*

- (i)  $K_n \square H$ , if  $n \geq 4$
- (ii)  $C_n \square H$  if  $n \geq 4, n \neq 5$
- (iii)  $G \square H$

**Proof:** Applying Theorem 4.4 to Propositions 4.2 and 4.3 immediately gives results (i) and (ii) respectively. For (iii), since  $K_2 \square K_2 \cong C_4$ ,  $K_2 \square K_2$  is 2-weight colourable by Proposition 2.9. By Theorem 4.4,  $K_2 \square H$  is 2-weight colourable; applying Theorem 4.4 again gives us that  $G \square H$  is 2-weight colourable.  $\square$

In order to construct non-2-weight colourable graphs below, we make use of a class of ‘‘gadget’’ graphs. These gadgets are themselves 2-weight colourable, but they have the property that in any of their 2-weight colourings, certain edges receive a predetermined weight.

Define the graph  $\widehat{K}_n$  to be the graph obtained from  $K_n$  by subdividing one edge exactly once.

**Proposition 4.6** *For  $n \geq 4$ , the graph  $\widehat{K}_n$  is 2-weight colourable. Moreover, in any 2-weight colouring of  $\widehat{K}_n$ , the edges incident to its degree 2 vertex must receive the same colour.*

**Proof:** Let  $x$  be the vertex of  $\widehat{K}_n$  of degree 2 and let  $u, v$  be its neighbours. An  $\{a, b\}$ -weight colouring of  $\widehat{K}_4$  is given in Figure 6. So assume  $n \geq 5$ . Let  $K_n$  be obtained by adding the edge  $uv$  to  $\widehat{K}_n - x$ . By Lemma 2.5, there exists an edge-weighting  $w$  of  $K_n$  such that all the vertices have distinct weighted degrees except for  $u$  and  $v$ . Moreover,  $w(u) = w(v) = ra + (n-1-r)b$ , where  $r \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil - 1\}$ . Assign the weight  $w(uv)$  from  $K_n$  to the edges  $xu$  and  $xv$  in  $\widehat{K}_n$ . Note that  $w$  is an  $\{a, b\}$ -weight colouring as long as  $w(u) = w(v) \neq w(x)$ . We have  $w(x) \in \{2a, 2b\}$ . Since  $a$  and  $b$  may be swapped in Lemma 2.5, we assume that  $w(x) = 2a$ . If  $w(u) \neq 2a$ , we are done. Suppose  $w(u) = 2a$ . We consider two cases:

- If  $n$  is odd, then the edge weighting  $w'$  given by swapping every edge’s weight gives  $w'(u) = w(u) = 2a \neq 2b = w'(x)$ .
- If  $n$  is even then, by the construction of the weighting in Lemma 2.5,  $w(u) = \frac{n}{2}a + (\frac{n}{2} - 1)b$ . So,  $2a = \frac{n}{2}(a+b) - b$ . If the edge weighting  $w'$  given by swapping every edge’s weight gives a conflict between  $u$  and  $x$ , then  $2b = \frac{n}{2}(a+b) - a$ . Together, these imply that  $a = b$ , a contradiction.

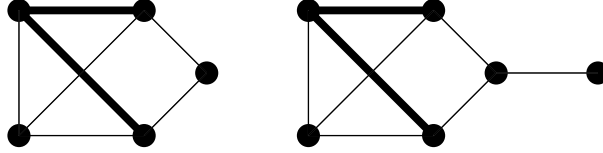
Thus  $\widehat{K}_n$  admits an  $\{a, b\}$ -weight colouring.

To prove the second part, toward a contradiction, suppose  $\widehat{K}_n$  is the smallest counterexample for which there exists an  $\{a, b\}$ -weight colouring  $w$  such that  $w(xu) \neq w(xv)$ . By inspection, we may check that  $\widehat{K}_4$  does not admit such edge-weighting. So assume  $n \geq 5$ . Note that there exists no vertex  $y \neq u, v$  such that  $w(y) \in \{(n-1)a, (n-1)b\}$ , otherwise  $\widehat{K}_n - y$  would be a smaller counterexample. Therefore, since  $w$  induces a vertex colouring and all the weighted degrees (except for  $x$ ) are of the form  $ra + (n-1-r)b$  for some  $0 \leq r \leq n-1$ , we must have  $w(u), w(v) \in \{(n-1)a, (n-1)b\}$ . But then by removing  $u, v$ , and  $x$  we get an  $\{a, b\}$ -weight colouring of  $K_{n-2}$ , a contradiction to Corollary 2.4.  $\square$

**Corollary 4.7** *Given a graph  $G$ , let  $G'$  be obtained from identifying a vertex of  $G$  with the degree 2 vertex of  $\widehat{K}_n$ . Then in any 2-weight colouring of  $G'$ , edges in  $\widehat{K}_n$  incident to its degree 2 vertex must receive same colour.*

**Proof:** Since the proof of Proposition 4.6 did not depend in any way on the accumulated weight at vertex  $x$ , then regardless of graph joined to  $\widehat{K}_n$  at  $x$ , the two edges incident with  $x$  in  $\widehat{K}_n$  must still receive the same weight.  $\square$

An example is given on the left of Figure 6. In the case  $G = K_2$  and  $n = 4$ , the weight of the leaf's edge is forced to be equal to that of its incident edges; this is another useful gadget. It is shown on the right of Figure 6.



**Fig. 6:** The graphs  $\hat{K}_4$  and  $\hat{K}_4$  with a leaf are 2-weight colourable. Bold edges represent one weight-class.

We use Proposition 4.6, which established the weight colourability of  $\hat{K}_n$ , to construct the following examples of non 2-weight colourable graphs.

**Example 4.8** *The following graphs cannot be 2-weight coloured:*

- (i) *Join two copies of  $\hat{K}_4$  by an edge attached at their vertices of degree 2.*
- (ii) *Join  $2n + 1$  copies of  $\hat{K}_4$  to a  $C_{2n+1}$  by an edge attaching the degree 2 vertex in each copy of  $\hat{K}_4$  to a distinct cycle vertex.*

To see why the graph defined in (ii), which we denote  $H$ , cannot be  $S$ -weight coloured for any set  $S$  of size 2, consider the accumulated weight at one of the cycle vertices, say  $v$ . Since  $H$  is 3-regular graph,  $w(v) \in \{3a, 2a+b, a+2b, 3b\}$ . If  $w(v) = 3a$ , then the noncycle edge,  $e$ , incident with  $v$  must have weight  $a$  and, as shown in Figure 6, so must the two edges in the copy of  $\hat{K}_4$  joined to  $v$  by an edge. Thus both endpoints of  $e$  would have weight  $3a$ . A similar argument shows that  $w(v) \neq 3b$ . Thus the only possible accumulated weights on cycle vertices are  $2a + b$  and  $2b + a$ . Since an odd cycle cannot be properly 2-coloured, we see that  $H$  cannot be 2-weight coloured.

Our next family of gadget graphs are described below.

**Proposition 4.9** *Let  $0 \leq a \in \mathbb{Z}$  and  $d \mid a$ . Let  $H$  be a graph and  $G$  be a graph obtained from identifying a vertex  $u$  of  $H$  with a vertex of a  $K_n$  (all other vertices of  $H$  and  $K_n$  being disjoint). If*

$$\deg_H(u) < \left( \frac{d}{a+d} \right) \left\lfloor \frac{n-1}{2} \right\rfloor,$$

*then  $G$  is not  $\{a, a+d\}$ -weight colourable. Furthermore, if*

$$\deg_H(u) = \left( \frac{d}{a+d} \right) \left\lfloor \frac{n-1}{2} \right\rfloor,$$

*then in any  $\{a, a+d\}$ -weight colouring of  $G$ , all edges in  $H$  incident to  $u$  must receive weight  $a+d$ .*

**Proof:** We first prove the statement for  $d = 1$ . Toward a contradiction, suppose  $w$  is an  $\{a, a+1\}$ -weighting colouring of  $G$ . Every vertex of  $K_n - u$  has weighted degree  $ra + (n-1-r)(a+1) = (n-1)(a+1) - r$  for some  $0 \leq r \leq n-1$  and both of the weights  $(n-1)a$  and  $(n-1)(a+1)$  cannot

appear simultaneously on  $K_n - u$ . If  $w(u) < (n-1)(a+1)$  then there are only  $n-1$  colours available for the vertices of  $K_n$ , a contradiction. So  $w(u) \geq (n-1)(a+1)$ .

Let  $w|_{K_n}$  be the edge-weighting of  $K_n$  induced by  $w$ . By Corollary 2.4,  $K_n$  is not  $\{a, a+1\}$ -edge-weight colourable. Thus, there must be exactly two vertices of  $K_n$  with the same weight given by  $w|_{K_n}$  and  $u$  must be one such vertex. By Lemma 2.5, we get  $w|_{K_n}(u) = ra + (n-1-r)(a+1) = (n-1)(a+1) - r$ , where  $r \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil - 1\}$ . Note that  $r \geq \lfloor (n-1)/2 \rfloor$ . If  $u$  is incident with  $s$  edges of weight  $a$  in  $H$ , then we have

$$(n-1)(a+1) \leq w(u) = ra + (n-1-r)(a+1) + sa + (\deg_H(u) - s)(a+1),$$

which simplifies to  $\deg_H(u) \geq \left(\frac{1}{a+1}\right)(r+s)$ . Hence

$$\deg_H(u) \geq \left(\frac{1}{a+1}\right) \left( \left\lfloor \frac{n-1}{2} \right\rfloor + s \right).$$

This is a contradiction since  $\deg_H(u) < \left(\frac{1}{a+1}\right) \lfloor \frac{n-1}{2} \rfloor$ . Also, if  $\deg_H(u) = \left(\frac{1}{a+1}\right) \lfloor \frac{n-1}{2} \rfloor$  then we must have  $s = 0$ , proving the second claim.

Now, let  $d$  be any positive divisor of  $a$ . By Proposition 2.1,  $G$  has an  $\{a, a+d\}$ -weight colouring if and only if it has a  $\{\frac{a}{d}, \frac{a}{d}+1\}$ -weight colouring. If  $\deg_H(u) < \left(\frac{1}{a/d+1}\right) \lfloor \frac{n-1}{2} \rfloor$  then  $G$  has no  $\{\frac{a}{d}, \frac{a}{d}+1\}$ -weight colouring by the above argument. Hence, if  $\deg_H(u) < \left(\frac{d}{a+d}\right) \lfloor \frac{n-1}{2} \rfloor$  then  $G$  has no  $\{a, a+d\}$ -weight colouring. The second result follows similarly.  $\square$

We may use Proposition 4.9 to construct many graphs which are not  $\{a, a+1\}$ -weight colourable and so, in particular, are not  $\{1, 2\}$ -weight colourable. In fact, if  $H$  is any graph and  $u$  any vertex of  $H$ , then there is an  $n$  large enough so that attaching  $K_n$  to  $u$  (and only to  $u$ ) gives a graph which is not  $\{a, a+1\}$ -weight colourable. We can also use the equality condition to construct graphs with no  $\{a, a+1\}$ -weight colouring (for a specific  $a$ ). For example, the graph obtained by joining two copies of  $K_n$ ,  $n \geq 5$ , with a path of length 3, say  $e_1, e_2, e_3$ , is not  $\{a, a+1\}$ -weight colourable for  $a = \lfloor \frac{n-1}{2} \rfloor - 1$  since the weights of  $e_1$  and  $e_3$  are forced to be  $a+1$ , and thus the ends of  $e_2$  will receive the same weight.

We finish this section with a problem whose solution would be useful in the study of  $\{1, 2, 3\}$ -weight colourable graphs.

**Problem 4** *Does there exist a graph which is either*

- *uniquely  $\{1, 2, 3\}$ -weight colourable (up to isomorphism), or*
- *$\{1, 2, 3\}$ -weight colourable where any such colouring forces certain edges of  $G$  to receive a particular weight?*

*Moreover, does there exist a graph with either of these properties which maintains that property when it is attached in some way to another graph?*

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