



**HAL**  
open science

# On the number of maximal independent sets in a graph

David R. Wood

► **To cite this version:**

David R. Wood. On the number of maximal independent sets in a graph. *Discrete Mathematics and Theoretical Computer Science*, 2011, Vol. 13 no. 3 (3), pp.17–19. 10.46298/dmtcs.543 . hal-00990494

**HAL Id: hal-00990494**

**<https://inria.hal.science/hal-00990494v1>**

Submitted on 13 May 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On the number of maximal independent sets in a graph

David R. Wood <sup>†</sup>

*Department of Mathematics and Statistics, The University of Melbourne, Melbourne, Australia*

*received 17<sup>th</sup> August 2011, accepted 30<sup>th</sup> August 2011.*

Miller and Muller (1960) and independently Moon and Moser (1965) determined the maximum number of maximal independent sets in an  $n$ -vertex graph. We give a new and simple proof of this result.

**MSC:** 05C30 Enumeration in graph theory; 05C69 Dominating sets, independent sets, cliques

**Keywords:** graph, independent sets

Let  $G$  be a (simple, undirected, finite) graph. A set  $S \subseteq V(G)$  is *independent* if no edge of  $G$  has both its endpoints in  $S$ . An independent set  $S$  is *maximal* if no independent set of  $G$  properly contains  $S$ . Let  $\text{MIS}(G)$  be the set of all maximal independent sets in  $G$ . Miller and Muller (1960) and Moon and Moser (1965) independently proved that the maximum, taken over all  $n$ -vertex graphs  $G$ , of  $|\text{MIS}(G)|$  equals

$$g(n) := \begin{cases} 3^{n/3} & \text{if } n \equiv 0 \pmod{3} \\ 4 \cdot 3^{(n-4)/3} & \text{if } n \equiv 1 \pmod{3} \\ 2 \cdot 3^{(n-2)/3} & \text{if } n \equiv 2 \pmod{3} \end{cases} .$$

This result is important for various reasons. For example,  $g(n)$  bounds the time complexity of various algorithms that output all maximal independent sets (Bron and Kerbosch, 1973; Lawler et al., 1980; Tsukiyama et al., 1977; Tomita et al., 2006; Johnson et al., 1988; Eppstein, 2003; Eppstein et al., 2010). Here we give a new and simple proof of this upper bound on  $|\text{MIS}(G)|$ .

**Theorem 1 ((Miller and Muller, 1960; Moon and Moser, 1965))** *For every  $n$ -vertex graph  $G$ ,*

$$|\text{MIS}(G)| \leq g(n) .$$

**Proof:** We proceed by induction on  $n$ . The base case with  $n \leq 2$  is easily verified. Now assume that  $n \geq 3$ . Let  $G$  be a graph with  $n$  vertices. Let  $d$  be the minimum degree of  $G$ . Let  $v$  be a vertex of degree

<sup>†</sup>woodd@unimelb.edu.au. Supported by a QEII Research Fellowship from the Australian Research Council.

$d$  in  $G$ . Let  $N[v]$  be the closed neighbourhood of  $v$ . If  $I \in \text{MIS}(G)$  then  $I \cap N[v] \neq \emptyset$ , otherwise  $I \cup \{v\}$  would be an independent set. Moreover, if  $w \in I \cap N[v]$  then  $I \setminus \{w\} \in \text{MIS}(G - N[v])$ . Thus

$$|\text{MIS}(G)| \leq \sum_{w \in N_G[v]} |\text{MIS}(G - N_G[w])| ,$$

Since  $\deg(w) \geq d$  and  $g$  is non-decreasing, by induction,

$$|\text{MIS}(G)| \leq (d+1) \cdot g(n-d-1) .$$

Note that

$$4 \cdot 3^{(n-4)/3} \leq g(n) \leq 3^{n/3} .$$

If  $d \geq 3$  then

$$|\text{MIS}(G)| \leq (d+1) \cdot 3^{(n-d-1)/3} \leq 4 \cdot 3^{(n-4)/3} \leq g(n) .$$

If  $d = 2$  then

$$|\text{MIS}(G)| \leq 3 \cdot g(n-3) = g(n) .$$

If  $d = 1$  and  $n \equiv 1 \pmod{3}$  then since  $n-2 \equiv 2 \pmod{3}$ ,

$$\text{MIS}(G) \leq 2 \cdot g(n-2) \leq 2 \cdot 2 \cdot 3^{(n-2-2)/3} = 4 \cdot 3^{(n-4)/3} = g(n) .$$

If  $d = 1$  and  $n \equiv 0 \pmod{3}$  then since  $n-2 \equiv 1 \pmod{3}$ ,

$$\text{MIS}(G) \leq 2 \cdot g(n-2) \leq 2 \cdot 4 \cdot 3^{(n-2-4)/3} < 3^{n/3} = g(n) .$$

If  $d = 1$  and  $n \equiv 2 \pmod{3}$  then since  $n-2 \equiv 0 \pmod{3}$ ,

$$\text{MIS}(G) \leq 2 \cdot g(n-2) \leq 2 \cdot 3^{(n-2)/3} = g(n) .$$

This proves that  $|\text{MIS}(G)| \leq g(n)$ , as desired.  $\square$

For completeness we describe the example by Miller and Muller (1960) and Moon and Moser (1965) that proves that Theorem 1 is best possible. If  $n \equiv 0 \pmod{3}$  then let  $M_n$  be the disjoint union of  $\frac{n}{3}$  copies of  $K_3$ . If  $n \equiv 1 \pmod{3}$  then let  $M_n$  be the disjoint union of  $K_4$  and  $\frac{n-4}{3}$  copies of  $K_3$ . If  $n \equiv 2 \pmod{3}$  then let  $M_n$  be the disjoint union of  $K_2$  and  $\frac{n-2}{3}$  copies of  $K_3$ . Observe that  $|\text{MIS}(M_n)| = g(n)$ .

Note that Vatter (2011) gave another proof of Theorem 1, and also described a connection between this result and the question, ‘‘What is the largest integer that is the product of positive integers with sum  $n$ ?’’ Also note that Dieter Kratsch proved that  $|\text{MIS}(G)| \leq 3^{n/3}$  using a similar proof to that presented here; see Gaspers (2010, page 177). Thanks to the authors of (Vatter, 2011; Gaspers, 2010) for pointing out these references.

## References

- C. Bron and J. Kerbosch. Finding all cliques of an undirected graph. *Comm. ACM*, 16(9):575–577, 1973. doi:10.1145/362342.362367.
- D. Eppstein. Small maximal independent sets and faster exact graph coloring. *J. Graph Algorithms Appl.*, 7(2):131–140, 2003. <http://jgaa.info/accepted/2003/Eppstein2003.7.2.pdf>.
- D. Eppstein, M. Löffler, and D. Strash. Listing all maximal cliques in sparse graphs in near-optimal time. In O. Cheong, K.-Y. Chwa, and K. Park, editors, *Proc. of 21st International Symposium on Algorithms and Computation (ISAAC 2010), Lecture Notes in Computer Science 6056:403–414*, 2010. doi:10.1007/978-3-642-17517-6\_36.
- S. Gaspers. *Exponential Time Algorithms: Structures, Measures, and Bounds*. VDM Verlag Dr. Müller, 2010. [http://www.kr.tuwien.ac.at/drm/gaspers/SergeBookETA2010\\_screen.pdf](http://www.kr.tuwien.ac.at/drm/gaspers/SergeBookETA2010_screen.pdf).
- D. S. Johnson, M. Yannakakis, and C. H. Papadimitriou. On generating all maximal independent sets. *Inform. Process. Lett.*, 27(3):119–123, 1988. doi:10.1016/0020-0190(88)90065-8.
- E. L. Lawler, J. K. Lenstra, and A. H. G. R. Kan. Generating all maximal independent sets: NP-hardness and polynomial-time algorithms. *SIAM J. Comput.*, 9(3):558–565, 1980. doi:10.1137/0209042.
- R. E. Miller and D. E. Muller. A problem of maximum consistent subsets. IBM Research Report RC-240, Thomas J. Watson Research Center, New York, USA, 1960.
- J. W. Moon and L. Moser. On cliques in graphs. *Israel J. Math.*, 3:23–28, 1965. doi:10.1007/BF02760024.
- E. Tomita, A. Tanaka, and H. Takahashi. The worst-case time complexity for generating all maximal cliques and computational experiments. *Theoret. Comput. Sci.*, 363(1):28–42, 2006. doi:10.1016/j.tcs.2006.06.015.
- S. Tsukiyama, M. Ide, H. Ariyoshi, and I. Shirakawa. A new algorithm for generating all the maximal independent sets. *SIAM J. Comput.*, 6(3):505–517, 1977. doi:10.1137/0206036.
- V. Vatter. Maximal independent sets and separating covers. *American Mathematical Monthly*, 118:418–423, 2011. <http://www.math.ufl.edu/~vatter/publications/sss-mis/>.

