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# An improved bound on the largest induced forests for triangle-free planar graphs<sup>†</sup>

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We proved that every planar triangle-free graph of order  $n$  has a subset of vertices that induces a forest of size at least  $(71n + 72)/128$ . This improves the earlier work of Salavatipour (2006). We also pose some questions regarding planar graphs of higher girth.

**Keywords:** Induced forest, Forest number, Decycling number

## 1 Introduction

The maximum size of acyclic induced subgraphs is studied in several different ways. If only connected subgraphs are considered, the problem is to find the order of maximum induced tree of a graph  $G$ , denoted by  $t(G)$ . The problem was initiated by Erdős, Saks, and Sós in 1986 [ESS86]. Some latest results are due to Matoušek and Šámal [MŠ08], and also due to Fox, Loh, and Sudakov [FLS09].

On the other hand, if the maximum acyclic induced subgraph is not necessarily connected, the task is to find the maximum induced forest. There are two equivalent approaches to obtain the maximum forest of a graph. The former is determining the *decycling number*  $\nabla(G)$  of a graph  $G$ , which is the least number of vertices whose deletion results in an induced forest. In [Kar72] it was shown that determining this invariant is NP-hard even for planar graphs. An interested reader can find more results on the decycling number in a survey of Punim [Pun08].

The latter approach is finding a maximum set  $S$  of vertices of graph  $G$  such that the graph  $G[S]$  induced on  $S$  is a forest. The size of such a set  $S$  is denoted by  $a(G)$  and it is referred to as a *forest number*. Note that  $a(G) + \nabla(G) = |V(G)|$ . We call the ratio between the forest number and the order of a graph a *forest ratio* and denote it by  $\gamma(G)$ . Large induced forests in graphs recently attracted attention in various graph classes. In 1979, Albertson and Berman [AB79] raised a conjecture regarding planar graphs and initiated the study of this topic:

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**Conjecture 1 (Albertson & Berman)** *Every planar graph has an induced forest on at least half of its vertices.*

Notice that the conjecture implies that every planar graph of order  $n$  has an independent set of size at least  $\frac{n}{4}$ . This fact is known to be true only as a consequence of the Four Color Theorem. In 1987, Akiyama and Watanabe [AW87] posed a similar conjecture on bipartite planar graphs:

**Conjecture 2 (Akiyama & Watanabe)** *Every bipartite planar graph has an induced forest on at least  $\frac{5}{8}$  of its vertices.*

Note that Conjectures 1 and 2, if true, are sharp by  $K_4$  and  $Q_3$ . Motivated by Conjecture 2, Alon [Alo03] proved the following result for sparse bipartite graphs:

**Theorem 1** *There exists an absolute positive constant  $b$  such that for every bipartite graph  $G$  of order  $n$  and average degree at most  $d$  holds the inequality*

$$a(G) \geq \left(\frac{1}{2} + e^{-bd^2}\right)n.$$

Additionally, Alon proved that the exponential dependence on  $d$  cannot be replaced by a polynomial one. In [AMT01], Alon, Mubayi, and Thomas proved a result for triangle-free graphs:

**Theorem 2** *Let  $G$  be a subcubic triangle-free graph of order  $n$ . Then  $a(G) \geq \frac{5}{8}n$ , and the bound is sharp whenever  $n$  is divisible by 8.*

Furthermore, Alon, Mubayi, and Thomas proved that for every triangle-free graph  $G$  of order  $n$  and size  $m$  the forest number is at least  $n - \frac{m}{4}$ . For planar triangle-free graphs this bound implies that the forest number is at least  $\frac{n}{2} + 1$  due to Euler's formula. Salavatipour [Sal06] improved the bound for planar graphs and showed that every triangle-free planar graph of order  $n$  has the forest number  $a(G) \geq \lceil \frac{17n+24}{32} \rceil$ . Using his approach, we improve this bound and prove the following theorem:

**Theorem 3** *Let  $G$  be a planar graph of order  $n$  and size  $m$  with girth  $g(G) \geq 4$ . Then,  $a(G) \geq \frac{119n-24m-24}{128}$ .*

From Theorem 3 and the fact that there is at most  $2n - 4$  edges in planar graphs of order  $n$  and girth at least 4, the following corollary immediately follows.

**Corollary 4** *Let  $G$  be a planar graph of order  $n$  and girth  $g(G) \geq 4$ . Then,  $a(G) \geq \lceil \frac{71n+72}{128} \rceil$ .*

As mentioned above, the investigation of triangle-free graphs was motivated by Conjecture 2. However, one could also ask what is the forest number of graphs with girth at least 5. We pose the following problem:

**Conjecture 3** *For every planar graph of order  $n$  and girth at least 5, the forest number  $a(G)$  is at least  $\frac{7}{10}n$ .*

The conjecture, if true, is sharp by the dodecahedron, and it was inspired by the fact that the dodecahedron has the minimal edge to vertex ratio among all connected graphs of girth at least 5. Note that we can also obtain planar graphs of girth 5 and with forest ratio  $\frac{7}{10}$  by carefully connecting multiple copies of dodecahedra. By this fact, it is natural to ask:

**Question 1** *Is there a planar graph of girth at least 5 with forest ratio precisely  $\frac{7}{10}$ , which does not contain dodecahedron as a subgraph?*

Now, we pose two simple observations regarding vertices of degree 2.

**Proposition 5** *For every connected graph  $G$  distinct from a cycle, there exists a maximum induced forest which contains every vertex of degree 2 in  $G$ .*

**Proof:** Let  $F$  be a maximum induced forest in a graph  $G$  with as many vertices of degree 2 as possible. Suppose  $v$  is a vertex of degree 2 which is not in  $F$ . Obviously,  $v$  is an element of a cycle, since  $F$  is maximum. Let  $e_1$  and  $e_2$  be the edges incident with  $v$ . Let  $u_1$  be a vertex of degree at least 3 such that the shortest path  $p_1$  between  $v$  and  $u_1$  contains  $e_1$  and  $u_1$  is the only vertex of degree at least 3 on  $p_1$ . Note that such  $u_1$  is unique and it always exists, since  $G$  is not a cycle. Similarly, we define  $u_2$  and the shortest path  $p_2$  which should contain  $e_2$ . Notice that  $u_1 = u_2$  is possible, but then  $v$  belongs to a cycle where  $u_1 (= u_2)$  is the only vertex of degree at least 3.

Now, if  $u_1 \neq u_2$ , they are both contained in  $F$ , otherwise  $v$  could be added to  $F$ . In case when  $u_1 = u_2$ , similarly  $u_1$  is in  $F$ . We now replace  $u_1$  by  $v$  in  $F$  and obtain a contradiction to the choice of  $F$ .  $\square$

Before stating the next observation, we define graph  $G^*$  obtained from the graph  $G$  by contracting all 2-vertices. Notice that  $G^*$  may have parallel edges and loops.

**Proposition 6** *For any graph  $G$  with  $n_2$  vertices of degree 2 the following equality holds:*

$$a(G) = a(G^*) + n_2.$$

**Proof:** First, we prove that  $a(G) \leq a(G^*) + n_2$ . Let  $V_2$  be the set of all vertices of degree 2 in  $G$ , so  $|V_2| = n_2$ . By Proposition 5 there exists a maximum induced forest  $F$  in  $G$  that contains all vertices from  $V_2$ . Obviously,  $F - V_2$  is an induced (not necessarily maximum) forest in  $G^*$ .

Now, we prove that  $a(G) \geq a(G^*) + n_2$ . Let  $F^*$  be the maximum induced forest in  $G^*$ . Obviously,  $F^*$  is an induced forest in  $G$ . Next, observe that by adding vertices from  $V_2$  we do not introduce any cycles, thus  $F^* \cup V_2$  is also an induced forest in  $G$ .  $\square$

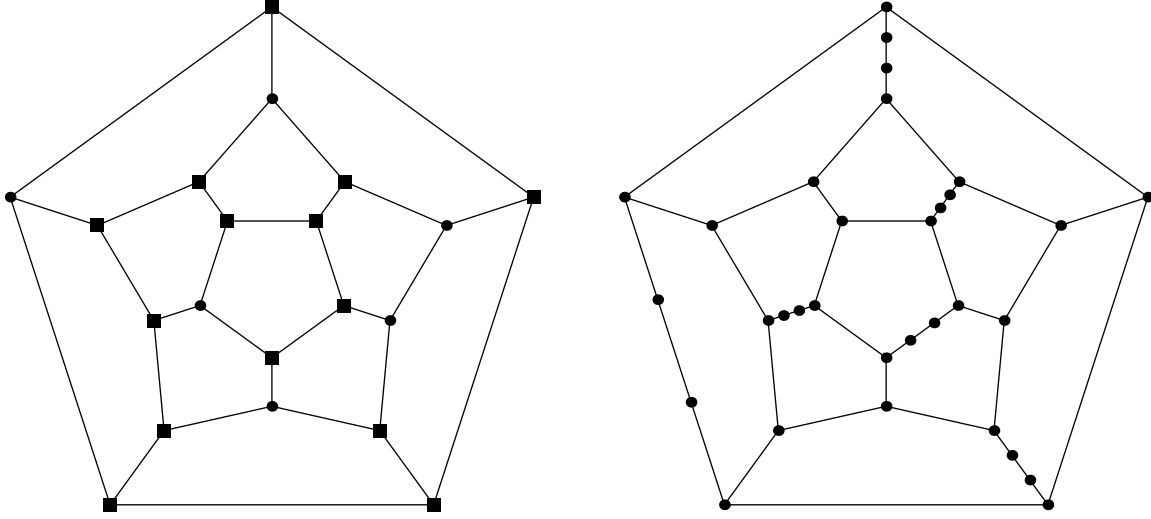
The above observations imply that if the dodecahedron is the graph with the smallest forest ratio, it follows that when considering graphs of higher girth, the graphs with the smallest forest ratio are dodecahedra with some edges subdivided. In particular, we can easily state such graphs for girth 6, 7, and 8. Let  $M$  be the minimum set of edges in dodecahedron  $D$ , such that each face is incident with an edge in  $M$ . Note that  $|M| = 6$ . We define  $D_k$  to be a graph obtained by subdividing each edge in  $M$  by  $k$ -vertices. It is easy to see that  $D_k$  has girth  $5 + k$  for  $k \in \{1, 2, 3\}$ , and  $\gamma(D_k) = \frac{7+3k}{10+3k}$ .

Observe that the maximum induced forest in  $D_k$  contains all the vertices of degree 2 and the vertices which form the induced forest of the dodecahedron.

## 2 Proof of Theorem 3

In the proof, we mostly follow the notation from [Sal06]. We call a vertex of degree  $k$  a  $k$ -vertex, and a neighbor of degree  $k$  of vertex  $v$  a  $k$ -neighbor of  $v$ . For a given cycle  $C$  in a plane embedding of graph  $G$  we define  $\text{int}(C)$  to be the graph induced by the vertices lying strictly in the interior of  $C$ . Similarly,  $\text{ext}(C)$  is the graph induced by the vertices lying strictly in the exterior of  $C$ . A *separating cycle* is a cycle  $C$  such that  $\text{int}(C) \neq \emptyset$  and  $\text{ext}(C) \neq \emptyset$ .

We will prove Theorem 3 by contradiction, i.e. we will prove that the minimal counter-example  $G$  does not exist. It is easy to see that  $G$  is connected. In what follows, we prove several lemmas, which determine



**Fig. 1:** The dodecahedron with 14 square vertices that induce a forest, and the graph  $D_2$ .

some structure of  $G$ . Note that by  $G$  we always mean a plane embedding of  $G$ , moreover  $n$ ,  $m$ , and  $f$  are the order, size and the number of faces in  $G$ , respectively.

**Lemma 7** *Graph  $G$  does not contain a bridge, i.e. it is 2-edge-connected.*

**Proof:** Assume  $G$  contains a bridge  $uv$ . Let  $G_u$  and  $G_v$  be the connected components that contain  $u$  and  $v$ , respectively, in  $G - uv$ . By the minimality of  $G$ , there is a set of vertices  $R_u$  in  $G_u$  and  $R_v$  in  $G_v$ , respectively, that induces a forest of size at least  $\frac{119|V(G_u)|-24|E(G_u)|-24}{128}$  and  $\frac{119|V(G_v)|-24|E(G_v)|-24}{128}$ , respectively. Then  $R_u \cup R_v$  induces a forest in  $G$  of size at least  $\frac{119n-24(m-1)-48}{128} = \varphi$ , which is a contradiction.  $\square$

**Lemma 8** *The maximum degree of  $G$  is at most 4, i.e.  $\Delta(G) \leq 4$ .*

**Proof:** Let  $v$  be a  $\geq 5$ -vertex of  $G$ . By minimality of  $G$ , there is an induced forest of size at least  $\frac{119(n-1)-24(m-5)-24}{128}$  in  $G - v$ , which is the required size  $\phi$  for an induced forest of  $G$ .  $\square$

**Lemma 9** *The minimum degree of  $G$  is at least 3, i.e.,  $\delta(G) \geq 3$ .*

**Proof:** Suppose that the lemma is false and that  $G$  has 2-vertices. First, we claim that *no 2-vertex is adjacent to a 4-vertex*. Let  $u$  be a 4-neighbor of a 2-vertex  $v$ . By minimality of  $G$ , we have that  $G - \{u, v\}$  has an induced forest of size at least  $\frac{119(n-2)-24(m-5)-24}{128} \geq \varphi - 1$  induced by a set  $R'$ . Then  $R' \cup \{v\}$  induces a forest of size at least  $\phi$  in  $G$ .

We now claim that *no 2-vertex has both a 2-neighbor and a 3-neighbor*. Let  $u$  be a 2-neighbor, and let  $w$  be a 3-neighbor of a 2-vertex  $v$ . Let  $R'$  be a subset of vertices in  $G - \{u, v, w\}$  that induces a forest of size  $\frac{119(n-3)-24(m-5)-24}{128} \geq \varphi - 2$ . Then  $R' \cup \{u, v\}$  induces a forest of size at least  $\phi$  in  $G$ .

By the above two claims, we obtain that  $G$  is either a cycle or it does not contain a pair of adjacent 2-vertices. If  $G$  is a cycle, it has an induced tree of size  $n - 1 \geq \varphi$ . Hence, we conclude that *both neighbors of a 2-vertex in  $G$  are of degree 3*.

Now, we claim that *every 3-vertex in  $G$  has at most one 2-neighbor*. Let us consider a 3-vertex  $w$  adjacent to 2-vertices  $u$  and  $v$ . Let  $R'$  be a subset of vertices in  $G - \{u, v, w\}$  that induces a forest of size at least  $\frac{119(n-3)-24(m-5)-24}{128} \geq \varphi - 2$ . By introducing vertices  $u$  and  $v$  to  $R'$ , we obtain an induced forest of size at least  $\phi$  in  $G$ , which establishes the claim.

Let  $v$  be one of the 2-vertices of  $G$ . By the third observation above, it has two 3-neighbors, say  $u$  and  $w$ . We will consider few possibilities regarding the number of their common neighbors, and each time we will obtain a contradiction. This will establish the lemma.

- *$u$  and  $w$  have one common neighbor, namely  $v$ .* Let  $G' = G + uw - v$ . Observe that  $G'$  has girth at least 4, since the only 2-path between  $u$  and  $w$  in  $G$  contains  $v$ . By the minimality of  $G$ , there is a subset  $R'$  that induces a forest of size at least  $\frac{119(n-1)-24(m-1)-24}{128} \geq \varphi - 1$  in  $G'$ . So,  $R' \cup \{v\}$  induces a forest of size at least  $\phi$  in  $G$ . Notice that adding  $v$  to the forest does not introduce a cycle, since  $u$  and  $w$  were adjacent in  $G'$ , i.e. if  $u, w \in R'$  then an edge in the forest is subdivided, and if at most one of  $u, w$  was in the forest, then  $v$  is a leaf or an isolated vertex.
- *$u$  and  $w$  have precisely two common neighbors.* Let these two neighbors be  $v$  and  $z$ , and let  $x$  be the third neighbor of  $u$ . By the last claim, we know that  $z$  and  $x$  have degree at least 3. Since  $x$  and  $z$  are non-adjacent by the girth assumption, there are at most  $m - 9$  edges in  $G - \{u, v, w, x, z\}$ . By the induction, it has a subset  $R'$  of vertices that induces a forest of size at least  $\frac{119(n-5)-24(m-9)-24}{128} \geq \varphi - 3$ , and so,  $R' \cup \{u, v, w\}$  induces a forest of size at least  $\phi$  in  $G$ .
- *$u$  and  $w$  have three common neighbors.* Let these three vertices be  $v, z$ , and  $x$ . Notice again that  $z$  and  $x$  are not adjacent due to the girth assumption. If one of  $z$  and  $x$  is a 4-vertex, similarly as above, we obtain an induced forest of size at least  $\frac{119(n-5)-24(m-9)-24}{128} \geq \varphi - 3$  in  $G - \{u, v, w, x, z\}$ , and by adding vertices  $u, v$ , and  $w$  to the forest, we obtain a forest of size at least  $\phi$  in  $G$ . So, we can assume that  $z$  and  $x$  are 3-vertices.

Now, let  $y$  be the neighbor of  $z$  distinct from  $u$  and  $w$  (note that  $x \neq y$ ). If  $y$  is a 2-neighbor of  $x$ , then  $G$  is a graph on six vertices, and the vertices  $u, v, x, z$  induce a forest of size  $4 > \phi = \frac{119 \cdot 6 - 24 \cdot 8 - 24}{128} = \frac{498}{128}$ . However, if  $y$  is a  $\geq 3$ -vertex, not necessarily adjacent to  $x$ , then there is a subset of vertices  $R'$  in  $G - \{u, v, w, x, y, z\}$  that induces a forest of size at least  $\frac{119(n-6)-24(m-9)-24}{128} \geq \varphi - 4$ , and hence  $R' \cup \{u, v, x, z\}$  induces a forest of size  $\geq \phi$  in  $G$ . Finally, if  $y$  is a 2-vertex not adjacent to  $x$ , then there are at most  $m - 9$  edges in  $G' = G - \{u, v, w, x, y, z\}$ , and  $G'$  has a forest  $F'$  of size  $\phi - 4$ . So, by adding the vertices  $u, v, x, z$  to  $F'$  we infer an induced forest of size at least  $\phi$  in  $G$ .

□

**Lemma 10** *Let  $v$  be a 3-vertex adjacent to a 4-vertex  $u$ . Then the other two neighbors of  $v$  have a common neighbor distinct from  $v$ .*

**Proof:** Let  $w$  and  $z$  be the other two neighbors of  $v$ . Suppose that  $v$  is the only neighbor of  $w$  and  $z$ . Consider the graph  $G' = G + wz - \{u, v\}$ . Note that  $G'$  has girth at least 4. By minimality of  $G$ , there is a subset of vertices  $R'$  in  $G'$  that induce a forest of size at least  $\frac{119(n-2)-24(m-5)-24}{128} \geq \varphi - 1$  in  $G'$ , thus  $R' \cup \{v\}$  induces a forest of size at least  $\phi$  in  $G$ .  $\square$

In the following two lemmas the indices are considered modulo 4.

**Lemma 11**  *$G$  does not contain a 4-cycle  $C = v_0v_1v_2v_3$  which has at least two 4-vertices and a 3-vertex  $v_i$  such that*

- (a)  $v_{i+2}$  is a 3-vertex; or
- (b)  $v_{i+2}$  is connected to both  $\text{int}(C)$  and  $\text{ext}(C)$ .

**Proof:** By minimality of  $G$ , there is a set  $R'$  of size at least  $\frac{119(n-4)-24(m-10)-24}{128} \geq \varphi - 2$  which induces a forest  $F'$  in  $G - V(C)$ . Note that  $v_i$  has at most one neighbor in  $R'$  and  $v_{i+2}$  has either at most one neighbor in  $R'$  (when  $\deg(v_{i+2}) = 3$ ) or it is connected to two distinct trees in  $F'$  (when  $\deg(v_{i+2}) = 4$ ). It follows that  $R' \cup \{v_i, v_{i+2}\}$  induces a forest of size at least  $\phi$  in  $G$ .  $\square$

**Lemma 12** *There is no separating 4-cycle  $C = v_0v_1v_2v_3$  which has*

- (a) *at least two 3-vertices; or*
- (b) *precisely one 3-vertex  $v_i$  and precisely one neighbor of  $v_{i+2}$  in  $\text{int}(C)$ .*

**Proof:** Let  $C = v_0v_1v_2v_3$  be a separating 4-cycle. Note that (b) follows from Lemma 11. We split the proof of (a) in several cases regarding the number of 3-vertices of  $C$ .

**Case 1:**  *$C$  has at least three 3-vertices.* Let  $v_0, v_1$ , and  $v_2$  be three such vertices.

Suppose first that vertices  $v_0, v_1$ , and  $v_2$  are all connected to one of  $\text{ext}(C)$  and  $\text{int}(C)$ , say  $\text{int}(C)$ . Then by 2-edge-connectivity of  $G$  and the fact that  $C$  is a separating cycle, we have that  $v_3$  is a 4-vertex connected to  $\text{ext}(C)$  with two edges. Let  $u_0, u_1$ , and  $u_2$  be the neighbors of  $v_0, v_1$ , and  $v_2$  from  $\text{int}(C)$ , respectively. By Lemma 10 we infer that  $u_0u_1, u_2u_1 \in E(G)$ , and from that, by the girth assumption,  $u_0u_2 \notin E(G)$ . Notice that  $u_0$  and  $u_2$  may coincide. In that case, the graph  $G' = G - V(C) - u_0$  has an induced forest  $F$  of size at least  $\frac{119(n-5)-24(m-10)-24}{128} \geq \varphi - 3$ , so by adding  $v_0, v_1, v_2$  to  $F$ , we obtain an induced forest of size at least  $\phi$  in  $G$ . Hence, we may assume  $u_0 \neq u_2$ .

If all three vertices  $u_0, u_1, u_2$  are of degree 3, then we add  $u_0, u_1$ , and  $v_0$  to an induced forest  $F'$  in  $G - \{v_0, v_1, u_0, u_1, u_2\}$  of size at least  $\frac{119(n-5)-24(m-10)-24}{128} \geq \varphi - 3$ . So the induced forest in  $G$  is of size at least  $\phi$ . On the other hand, if at least one of these three neighbors is of degree 4, the induced forest in  $G$  is obtained from a forest in the graph  $G - V(C) - \{u_0, u_1, u_2\}$  of size at least  $\frac{119(n-7)-24(m-14)-24}{128} \geq \varphi - 4$  by introducing vertices  $v_0, v_1, v_2$ , and  $u_1$ .

Suppose now that not all of  $v_0, v_1$ , and  $v_2$  are connected to  $\text{ext}(C)$  or to  $\text{int}(C)$ . Without loss of generality, two vertices from  $\{v_0, v_1, v_2\}$  are connected to  $\text{int}(C)$  and the third one is connected to  $\text{ext}(C)$ . By symmetry, we can assume  $v_0$  is connected to  $\text{int}(C)$ , and just one of  $v_1, v_2$  with  $\text{ext}(C)$ . Let  $u$  be a neighbor of  $v_0$  in  $\text{int}(C)$ . There is an induced forest  $F'$  of size at least  $\frac{119(n-5)-24(m-10)-24}{128} \geq \varphi - 3$  in the graph  $G - V(C) - u$ . By adding  $v_0, v_1$ , and  $v_2$  to  $F'$ , we obtain an induced forest  $F$  of size at least  $\phi$ .

in  $G$ . Note that  $F$  is acyclic, since the path  $v_0v_1v_2$  is connected to  $F'$  by at most two edges which are not incident with the same tree in  $F'$ .

**Case 2:**  $C$  has exactly two 3-vertices. Note that by Lemma 11 we can assume that the two 3-vertices of  $C$  are consecutive, say  $v_0$  and  $v_1$ . Let  $u_0$  be the neighbor of  $v_0$  distinct from  $v_1$  and  $v_3$ . By symmetry we can assume that  $u_0 \in \text{ext}(C)$  for otherwise one can change the plane embedding of  $G$ . By Lemma 10, the vertices  $v_1$  and  $u_0$  have another common neighbor  $u_1$ , beside the vertex  $v_0$ .

Observe that by Lemma 11 and the fact that  $C$  is a separating cycle, at least one of the vertices  $v_2$  and  $v_3$  has two neighbors in  $\text{int}(C)$ . By symmetry, we can assume that  $v_2$  has two neighbors in  $\text{int}(C)$ . By Lemma 11,  $v_3$  has either both neighbors in  $\text{int}(C)$  or both in  $\text{ext}(C)$ . In the former case consider the graph  $G - \{v_0, v_1, v_3, u_0\}$ . It has at most  $m - 10$  edges, so it has an induced forest  $F'$  of size at least  $\frac{119(n-4)-24(m-10)-24}{128} \geq \varphi - 2$ . By adding the vertices  $v_0$  and  $v_1$  to  $F$ , we obtain an induced forest of size at least  $\varphi$  in  $G$ . It is easy to see that no cycle is introduced, since  $\text{int}(C)$  and  $\text{ext}(C)$  are not connected. It follows that  $v_3$  is connected to  $\text{ext}(C)$  with two edges.

Now we note that if  $v_3$  is adjacent with  $u_1$  then  $\deg(u_1) = 4$  for otherwise  $u_1v_1v_0v_3$  is a separating 4-cycle with three 3-vertices and we can proceed as in Case 1. It follows that the set of vertices  $S = \{u_1, v_1, v_0, v_3\}$  is always incident with at least 10 edges. Hence in the graph  $G' = G - S$  there is a subset of vertices  $R'$  which induces a forest of size at least  $\frac{119(n-4)-24(m-10)-24}{128} \geq \varphi - 2$ . In graph  $G'$  there is no path from  $v_2$  to  $u_0$  since in  $G$  vertex  $v_2$  has two neighbors in  $\text{int}(C)$ . It follows that  $R' \cup \{v_0, v_1\}$  induces a forest of size at least  $\varphi$  in  $G$ , even if both  $v_2$  and  $u_0$  are in  $R'$ .  $\square$

**Lemma 13** *The graph  $G$  has no 4-face with four 3-vertices.*

**Proof:** Let  $C = v_0v_1v_2v_3$  be a 4-face in  $G$  incident only with 3-vertices. If  $v_0$  and  $v_2$  have a common neighbor  $u$  in  $G - V(C)$ , then we have a separating 4-cycle  $v_0v_1v_2u$  with at least three 3-vertices, which is a contradiction by Lemma 12. A similar argument applies if  $v_1$  and  $v_3$  have a common neighbor. Thus, each vertex of  $C$  has a distinct neighbor in  $G - V(C)$ . Let  $u_0, u_1, u_2$ , and  $u_3$  be the third neighbors of  $v_0, v_1, v_2$ , and  $v_3$ , respectively. As we argued,  $u_0, u_1, u_2$ , and  $u_3$  are pairwise distinct. Note also that by the planarity, at most one of  $u_0u_2$  and  $u_1u_3$  is in  $G$ .

Suppose first that at least two consecutive edges,  $v_i v_{i+1}$  and  $v_{i+1} v_{i+2}$ , of  $C$  are incident to  $\geq 5$ -faces (indices are considered modulo 4). Note that in case, when  $u_i$  and  $u_{i+2}$  are adjacent, at least one of  $v_{i+2} v_{i+3}$  and  $v_{i+3} v_i$  is incident to a  $\geq 5$ -face (and  $u_{i+1}, u_{i+3}$  are non-adjacent as stated above), due to the girth assumption. Hence, there always exist edges  $v_j v_{j+1}, v_{j+1} v_{j+2}$  incident to  $\geq 5$ -faces such that  $u_j$  and  $u_{j+2}$  are non-adjacent. Say  $v_0 v_1$  and  $v_1 v_2$  are such edges. Now, consider the graph  $G'$  obtained from  $G$  by removing vertices of  $C$ , and adding a new vertex  $x$ . As  $u_0$  and  $u_2$  are non-adjacent, let  $x$  be adjacent to  $u_0, u_1$ , and  $u_2$ , i.e.  $G' = G \cup \{x, xu_0, xu_1, xu_2\} - V(C)$ . The resulting graph  $G'$  has girth at least 4, since  $u_0, u_1$ , and  $u_2$  are pairwise non-adjacent by Lemma 12. By minimality of  $G$ , there is a vertex set  $R'$  that induces a forest of size at least  $\frac{119(n-3)-24(m-5)-24}{128} \geq \varphi - 2$  in  $G'$ . If  $x \notin R'$ , then  $R' \cup \{v_1, v_3\}$  induces a forest of size at least  $\phi$  in  $G$ . On the other hand, if  $x \in R'$ , we consider the vertex set  $R' \setminus \{x\} \cup \{v_0, v_1, v_2\}$ , inducing a forest of size at least  $\phi$  in  $G$ .

Now, we may assume that  $C$  has at most two edges incident to  $\geq 5$ -faces, and they are non-consecutive on  $C$ . We will consider several cases regarding the number of 3-vertices in  $U = \{u_0, u_1, u_2, u_3\}$ . Note that any two vertices of  $U$  incident with the same  $\geq 5$ -face, which is also incident with  $C$ , are non-adjacent due to Lemma 12.



**Case 1:** All the vertices in  $U$  are of degree 3 and  $C$  is incident with at least three 4-faces. In case when  $C$  is incident with four 4-faces,  $G$  is the cube, so there is an induced forest of size  $\phi = \frac{119 \cdot 8 - 24 \cdot 12 - 24}{128} = 5$ .

Suppose now that  $C$  is incident to precisely three 4-faces, and let  $u_0$  and  $u_1$  be the two vertices incident with the only  $\geq 5$ -face. Since  $G$  is not a cube, there exists a vertex  $x$  adjacent to  $u_0$  and distinct from  $v_0, u_1, u_3$ . By the minimality, there is a subset of vertices  $R'$  in  $G - V(C) - \{u_0, u_1, u_2, u_3, x\}$  which induces a forest in  $G'$  of size at least  $\frac{119(n-9) - 24(m-14) - 24}{128} \geq \varphi - 6$ . Consider the set  $R = R' \cup \{u_0, u_1, u_2, v_0, v_1, v_3\}$ . The tree induced by  $\{u_0, u_1, u_2, v_0, v_1, v_3\}$  might be connected with  $R'$  only by one edge that is incident with  $u_1$ . So,  $R$  induces a forest of size at least  $\phi$  in  $G$ .

**Case 2:**  $U$  has one 4-vertex and  $C$  is incident with at least three 4-faces. First, note that  $C$  cannot be incident with four 4-faces due to 2-edge-connectivity, so we may assume that  $C$  is incident to precisely one  $\geq 5$ -face  $f$ . Then  $U$  must have a 3-vertex  $x$  which is incident to two of the three 4-faces, and which has a 4-neighbor  $y$  in  $U$ . We may assume  $x = u_0$ . Observe that  $u_2$  is a 3-vertex. Also note that the two vertices of  $U$  incident to  $f$  are not adjacent by Lemma 12. Now, by minimality of  $G$  there is a set of vertices  $R'$  in  $G - V(C) - \{u_0, u_1, u_2, u_3\}$  that induces a forest of size at least  $\frac{119(n-8) - 24(m-14) - 24}{128} \geq \varphi - 5$ . The set  $R' \cup \{u_0, u_2, v_0, v_1, v_3\}$  induces a forest of size at least  $\phi$  in  $G$ . Obviously, no cycles are introduced, since the vertices  $u_0, v_0, v_1, v_3$  induce a tree which is not connected to  $R'$  and  $u_2$  has at most one neighbor in  $R'$ .

**Case 3:**  $U$  has at most one 4-vertex and  $C$  is incident with precisely two 4-faces. Recall that these two faces are not consecutive around  $C$ , so we may assume the 4-faces incident with  $C$  are bounded by the cycles  $v_0u_0u_1v_1$  and  $v_2u_2u_3v_3$ . Note that  $u_0u_2, u_1u_3 \notin E(G)$  due to 2-edge-connectivity of  $G$ . By symmetry, we may also assume that the possible 4-vertex in  $U$  is  $u_1$ . Consider the graph  $G - V(C) - \{u_0, u_1, u_2, u_3\}$  and its vertex set  $R'$  that induces the forest  $F'$  of size at least  $\frac{119(n-8) - 24(m-14) - 24}{128} \geq \varphi - 5$ . The set of vertices  $R' \cup \{u_0, u_2, v_0, v_1, v_3\}$  induces a forest of size at least  $\phi$  in  $G$ .

**Case 4:**  $U$  has two 4-vertices. If  $C$  is incident to four 4-faces, then there is a separating 4-cycle  $u_0u_1u_2u_3$ , which is reducible by Lemma 12.

Suppose now  $C$  is incident with precisely one  $\geq 5$ -face  $f$ . We consider two possibilities. First, let both 4-vertices of  $U$  be incident to  $f$ . We may assume that these two vertices are  $u_0$  and  $u_1$ . Note that  $u_0$  and  $u_1$  are non-adjacent by Lemma 12. Again, there exists an induced forest of size at least  $\frac{119(n-8) - 24(m-15) - 24}{128} \geq \varphi - 5$  in  $G - V(C) - \{u_0, u_1, u_2, u_3\}$ . It is easy to see that inserting the vertices  $u_2, u_3, v_0, v_1$ , and  $v_3$  into the forest does not introduce any cycles, so we obtain a forest of size at least  $\phi$  in  $G$ . Thus we can assume that at least one 3-vertex from  $U$  is incident to  $f$ , say  $u_0$ . There exists a vertex set  $R'$  that induces a forest of size at least  $\frac{119(n-7) - 24(m-14) - 24}{128} \geq \varphi - 4$  in  $G - V(C) - \{u_1, u_2, u_3\}$ . By inserting vertices  $u_2, v_1, v_2$ , and  $v_3$  to  $R'$  we obtain an induced forest of size at least  $\phi$  in  $G$ .

Finally, suppose  $C$  is incident with two (non-consecutive) 4-faces. Again, we may assume that  $u_0$  and  $u_1$  are incident with the same  $\geq 5$ -face, and  $u_0$  is a 3-vertex. Let  $R'$  be the vertex set that induces a forest in  $G' = G - V(C) - \{u_0, u_1, u_3\}$ . By the minimality, this set is of size at least  $\frac{119(n-7) - 24(m-14) - 24}{128} \geq \varphi - 4$  unless  $u_1$  and  $u_3$  are adjacent. But in the exceptional case,  $u_0$  and  $u_2$  are not adjacent by the planarity, so we redefine  $G' = G - V(C) - \{u_0, u_1, u_2\}$  to obtain an induced forest of size at least  $\frac{119(n-7) - 24(m-14) - 24}{128} \geq \varphi - 4$ . Now, we add vertices  $u_0, v_0, v_1$ , and  $v_3$  (resp.  $u_1, v_0, v_1$ , and  $v_2$ ) to  $R'$ . We obtain an induced forest in  $G$  of size at least  $\phi$ .

**Case 5:**  $U$  has three 4-vertices. By Lemma 12, we infer that there are no four 4-faces incident with

$C$ , since  $u_0u_1u_2u_3$  is a separating 4-cycle with one 3-vertex and its opposite vertex is adjacent with the internal and external component.

Suppose now that there are three 4-faces incident with  $C$ . By symmetry, assume  $u_0$  is the 4-vertex not incident to a  $\geq 5$ -face. Next, let  $R'$  be the vertex set which induces the forest of size at least  $\frac{119(n-7)-24(m-14)-24}{128} \geq \varphi - 4$  in  $G - V(C) - \{u_0, u_1, u_3\}$ . To obtain an induced forest of size at least  $\phi$  in  $G$ , just introduce the vertices  $u_0, v_0, v_1$  and  $v_3$  to  $R'$ . Again, the introduced claw could be connected to the forest only by  $u_0$ , thus no cycle is introduced.

Finally, suppose that  $C$  is incident to two 4-faces. Here, let  $u_0$  be the only 3-vertex in  $U$ . It is easy to see that by defining  $R'$  as in the case above and by adding the vertices  $u_0, v_0, v_1$ , and  $v_3$  to  $R'$  we obtain the induced forest of size at least  $\phi$  in  $G$ .

**Case 6:**  $U$  has four 4-vertices. Note first that by planarity if  $u_0$  and  $u_2$  are adjacent then  $u_1$  and  $u_3$  are non-adjacent. By symmetry, we may assume  $u_0u_2 \notin E(G)$ . Next, let  $R'$  be the vertex set that induces a forest of size at least  $\frac{119(n-6)-24(m-14)-24}{128} \geq \varphi - 3$  in  $G - V(C) - \{u_0, u_2\}$ . It is easy to see that  $R' \cup \{v_0, v_1, v_2\}$  induces a forest of size at least  $\phi$  in  $G$ .  $\square$

**Lemma 14** *The graph  $G$  has no 4-face with precisely two 3-vertices.*

**Proof:** Let the cycle  $C = v_0v_1v_2v_3$  be a 4-face with precisely two 3-vertices. By Lemma 11 we obtain that the 3-vertices are adjacent, so we can assume  $v_0$  and  $v_1$  are of degree 3. Let  $u_0$  be the third neighbor of  $v_0$ , and let  $u_1$  be the third neighbor of  $v_1$ . By Lemma 10, we have that  $u_0$  and  $u_1$  are adjacent, moreover, Lemma 12 implies that the cycle  $v_0v_1u_1u_0$  bounds a face and that  $u_0v_2, u_1v_3 \notin E(G)$ . Moreover, at least one of  $u_0$  and  $u_1$  is a 4-vertex, otherwise  $v_0v_1u_1u_0$  is a 4-face with four 3-vertices, which is reducible by Lemma 13. By symmetry, we may assume that  $\deg(u_0) = 4$ .

If  $u_1$  is a 3-vertex, there is a set of vertices  $R'$  of size at least  $\frac{119(n-6)-24(m-14)-24}{128} \geq \varphi - 3$  in  $G - V(C) - \{u_0, u_1\}$ . The set  $R' \cup \{u_1, v_1, v_0\}$  induces a forest of size at least  $\phi$  in  $G$ . It is easy to see that no cycle is introduced by this set.

Finally, assume  $u_1$  is a 4-vertex. Note that if  $u_0$  and  $v_2$  have a common neighbor, then by the planarity and the girth assumption,  $u_1$  and  $v_3$  do not have a common neighbor. By symmetry, we may assume that  $u_0$  and  $v_2$  have no common neighbor (and recall that  $u_0$  and  $v_2$  are non-adjacent). Then there is a subset of vertices  $R'$  in  $G + u_0v_2 - \{u_1, v_0, v_1, v_3\}$  which induces a forest of size at least  $\frac{119(n-4)-24(m-10)-24}{128} \geq \varphi - 2$ . The vertex set  $R' \cup \{v_0, v_1\}$  induces a forest of size at least  $\phi$  in  $G$ ; observe that by adding these two vertices no cycle is introduced, since if  $\{u_0, v_2\} \subseteq R'$ , we only subdivide an edge in the forest.  $\square$

**Lemma 15** *The graph  $G$  has no 5-face incident only with 3-vertices.*

**Proof:** Assume  $C = v_0v_1v_2v_3v_4$  is a 5-face incident only with 3-vertices. Let  $u_i$  be the third neighbor of  $v_i$ , where  $i \in \{0, 1, 2, 3, 4\}$ . By minimality of  $G$ , there is a subset of vertices  $R'$  in a graph  $G'$  obtained by removing  $C$  and adding the vertices  $x, y$  and the edges  $xu_0, xu_1, xy, yu_2$ , and  $yu_3$  to  $G$ , i.e.  $G' = G - V(C) + \{x, y, xu_0, xu_1, xy, yu_2, yu_3\}$ , that induces a forest  $F'$  of size  $\frac{119(n-3)-24(m-5)-24}{128} \geq \varphi - 2$ . Note that by adding these edges we do not violate the girth assumption, since there is no 4-cycle with two 3-vertices in  $G$  by Lemma 11.

Now, we distinguish few possibilities regarding whether  $x$  and  $y$  are in  $R'$ . If none of them is in  $R'$ , then adding  $v_0$  and  $v_2$  to  $F'$  does not introduce a cycle and the size of the resulting forest is at least  $\phi$  in

$G$ . If precisely one of  $x$  and  $y$  is in  $R'$ , say  $x$ , we need to add three vertices to  $F'$  to assure that its size is at least  $\phi$ , since  $x$  is not a vertex of  $G$ . However, as  $x$  is in the forest, adding  $v_0$  and  $v_1$  to  $F' - x$  does not introduce a cycle. The third vertex,  $v_3$ , is connected to the forest with at most one edge.

Finally, if  $x$  and  $y$  are both in  $R'$ , then we replace them with four vertices of  $C$ . Namely, we add vertices  $v_0, v_1, v_2$ , and  $v_3$ . It is easy to see that no cycle is introduced, so we obtain an induced forest of size at least  $\phi$  in  $G$ .  $\square$

**Lemma 16** *The graph  $G$  is not cubic.*

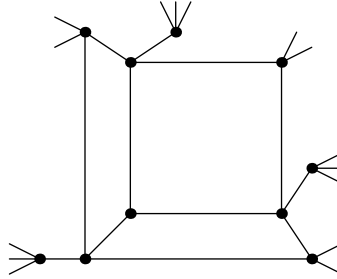
**Proof:** Suppose for a sake of contradiction that  $G$  is cubic. By the girth assumption and Lemmas 13 and 15, we infer that faces in  $G$  have length at least 6. On the other hand, Lemma 9 implies that the minimum degree in  $G$  is at least 3, so by Euler's formula  $G$  contains a face of length at most 5, a contradiction.  $\square$

**Lemma 17** *Every 3-vertex of  $G$  has three 4-neighbors.*

**Proof:** Suppose the claim is false. By Lemma 16,  $G$  is not cubic, thus it has at least one 4-vertex. Therefore if there is any 3-vertex in  $G$ , there is also a 3-vertex  $v$  adjacent to at least one 4-vertex  $u$ . Let  $w$  and  $z$  be the other two neighbors of  $v$ . By Lemma 10, we know that  $w$  and  $z$  have a common neighbor  $x$  distinct from  $v$ . However, if  $w$  or  $z$  is a 3-vertex, a separating 4-cycle or a 4-face with two 3-vertices is introduced, but such configurations are reducible by Lemmas 12 and 14.  $\square$

**Lemma 18** *The graph  $G$  does not contain 4-faces with precisely one 3-vertex.*

**Proof:** Suppose for a sake of contradiction that the cycle  $C = v_0v_1v_2v_3$  is a 4-face in  $G$  with exactly one 3-vertex  $v_0$ . Let  $u_0$  be the neighbor of  $v_0$  distinct from  $v_1$  and  $v_3$ . By Lemma 17, it follows that  $u_0$  is a 4-vertex. Moreover, by Lemma 10,  $u_0$  and  $v_1$  have a common neighbor  $x$  distinct from  $v_0$ , and also  $u_0$  and  $v_3$  have a common neighbor  $y$  distinct from  $v_0$ . Lemmas 12 and 14 imply that  $d(x) = d(y) = 4$ . Now we show in few steps that the vertices of Fig. 2 are all pairwise distinct.

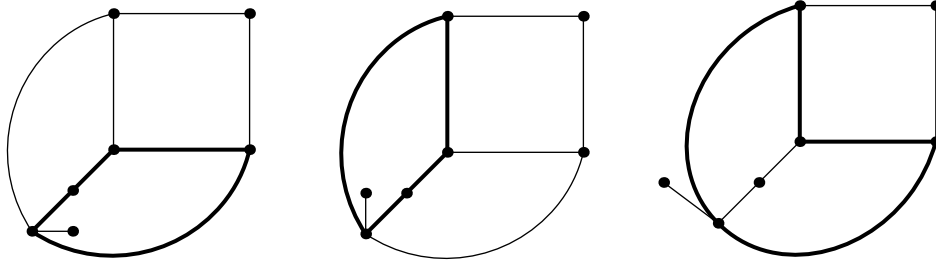


**Fig. 2:** The neighborhood of vertices  $v_0, v_1, v_2$ , and  $v_3$ .

First we claim that  $x \neq v_2$ . Suppose contrary that  $x = v_2$ . Let  $u_2$  be the neighbor of  $v_2$  which is distinct from  $v_1, v_3$ , and  $u_0$ . Let  $C_1$  be the separating 4-cycle  $v_0v_3v_2u_0$ . If  $u_2$  is not in the same component of

$G - V(C_1)$  as  $v_1$ , then we have a separating 4-cycle with one 3-vertex and its opposite vertex adjacent to two components. Such a cycle is reducible by Lemma 12. On the other hand, if  $u_2$  is in the same component, consider instead the reducible separating 4-cycle  $v_0v_1v_2u_0$ , a contradiction. We similarly show that  $y \neq v_2$ .

Now, we claim that  $x \neq y$ . Again, suppose contrary that  $x = y$ . Let  $x_1$  be the neighbor of  $x$  distinct from  $u_0, v_1$ , and  $v_3$ . Consider the 4-cycles  $C_1 = xu_0v_0v_1$ ,  $C_2 = xu_0v_0v_3$ , and  $C_3 = xv_3v_0v_1$ . Note that among these three cycles we can always choose a cycle  $C_i$  such that  $x_1$  and the neighbor of  $x$  not incident to  $C_i$  are in different parts  $\text{int}(C_i)$  and  $\text{ext}(C_i)$  (see Fig. 3). Hence,  $C_i$  is a separating cycle. Since  $C_1, C_2$ , and  $C_3$  all contain a 3-vertex  $v_0$ ,  $C_i$  is reducible by Lemma 12(b), a contradiction. This establishes the claim.



**Fig. 3:** We can always find such a 4-cycle  $C_i$  that  $x_1$  and the other neighbor of  $x$  not in incident with  $C_i$  are in different parts  $\text{int}(C_i)$  and  $\text{ext}(C_i)$ .

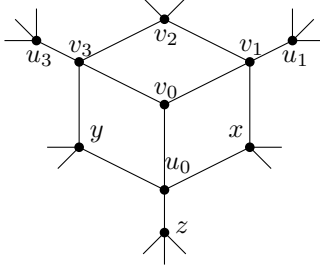
Consider now the neighbor  $z$  of  $u_0$  distinct from  $v_0, x$ , and  $y$ . We claim that  $z \neq v_2$ , for otherwise we consider the separating 4-cycles  $u_0v_0v_3v_2$  and  $u_0v_0v_1v_2$ . At least one of them satisfies the assumptions of Lemma 12(b), a contradiction.

Next, let  $u_1$  be the neighbor of  $v_1$  distinct from  $x, v_0$ , and  $v_2$ . Also, let  $u_3$  be the neighbor of  $v_3$  distinct from  $y, v_0$ , and  $v_2$ . We claim that  $u_1 \neq u_3$ . Suppose contrary that  $u_1 = u_3$ . Note that  $xu_1$  and  $yu_1$  are not the edges in  $G$ , due to the girth assumption. Let  $R'$  be the subset of vertices that induce the forest in  $G - V(C) - \{u_0, u_1, x, y\}$  of size at least  $\frac{119(n-8)-24(m-19)-24}{128} \geq \varphi - 4$ . Now, the set  $R' \cup \{u_0, v_0, v_1, v_3\}$  induces a forest of size at least  $\phi$  in  $G$ . This establishes the claim that  $u_1$  and  $u_3$  are distinct.

We also claim that  $x \neq u_3$ . Suppose contrary that  $x = u_3$ . By planarity,  $y = u_1$ . Let  $G' = G - \{v_0, v_1, v_3, u_0, x, y\}$ . There exists a set of vertices  $R'$  in  $G'$  that induces the forest of size at least  $\frac{119(n-6)-24(m-15)-24}{128} \geq \varphi - 3$ . Inserting the vertices  $v_0, v_3$ , and  $u_0$  in  $R'$  infers a forest of size at least  $\phi$  in  $G$ . Observe that no cycle is introduced, since  $z$  and  $v_2$  are in different parts of the plane regarding the separating cycle  $v_0v_1xv_3$ . We show similarly that  $y \neq u_1$ .

As we established that all vertices from Fig. 2 are distinct, we continue by considering the adjacency of the vertices  $z, u_1$ , and  $u_3$ . If neither of them are adjacent, then there exists a set of vertices  $R'$  of size at least  $\frac{119(n-6)-24(m-15)-24}{128} \geq \varphi - 3$  in  $G \cup \{w, wz, wu_1, wu_3\} - \{v_0, v_1, v_2, v_3, u_0, x, y\}$ , where  $w$  is a new vertex. If  $w \notin R'$ , then the vertices  $v_1, v_3$ , and  $u_0$  are added to  $R'$  to induce a forest  $F$  of size at least  $\phi$ . On the other hand, if  $w \in R'$ , then such a forest is induced by adding  $v_0, v_1, v_3$ , and  $u_0$  to  $F$ . Obviously in both cases no cycle is introduced.

By the above paragraph, we may assume that some two vertices from  $\{z, u_1, u_3\}$  are adjacent, however, there exist a pair which is not, due to the girth assumption. Observe that without loss of generality, we may assume that  $zu_3 \notin E(G)$  and  $u_1u_3 \in E(G)$ , since we may use the symmetry of the neighborhood of vertex  $v$  (see Fig. 4).



**Fig. 4:** The symmetric neighborhood of vertex  $v_0$ .

Next, we claim that both of the vertices  $u_1$  and  $u_3$  are of degree 4. Suppose for a contradiction that  $u_1$  is a 3-vertex. Then, by Lemma 17,  $u_3$  is a 4-vertex, and by Lemma 10,  $u_3$  is adjacent to  $x$ .

So, consider the separating 5-cycle  $g = u_1v_1v_0v_3u_3$  and an induced forest  $F'$  of  $G - V(g) - y$ . It is of size at least  $\frac{119(n-6)-24(m-16)-24}{128} \geq \varphi - 3$ . Observe that after adding vertices  $v_0, v_3$ , and  $u_1$  to  $F'$  no cycles are introduced, thus we obtain an induced forest of size at least  $\phi$  in  $G$ , a contradiction.

Finally, consider two subcases regarding the degree of  $z$ :

- $z$  is a 4-vertex. In this case, by planarity at most one of the edges  $u_1y$ , and  $u_3x$  exists. By symmetry, suppose that  $u_3x$  does not. Then there is a set of vertices  $R'$  in  $G - V(C) - \{u_0, u_3, z, x, y\}$  which induces a forest of size at least  $\frac{119(n-9)-24(m-24)-24}{128} \geq \varphi - 4$ . The vertex set  $R' \cup \{u_0, v_0, v_1, v_3\}$  induces a forest of size at least  $\phi$  in  $G$ .
- $z$  is a 3-vertex. Let  $s$  and  $t$  be the neighbors of  $z$  distinct from  $u_0$ . By Lemma 10 we infer that  $s$  and  $t$  have a common neighbor  $p$ . In addition, by Lemma 10, we may assume that  $ty, sx \in E(G)$ . Note that  $t \neq u_3$  and  $s \neq u_1$  by the girth assumption. Moreover, if  $t = u_1$ , then either  $p = y$  or  $p = u_3$ . Both cases are violating the planarity, thus  $t \neq u_1$ . Similarly we show that  $s \neq u_3$ .

Let  $q$  be the neighbor of  $x$  distinct from  $v_1, u_0$ , and  $s$ , and similarly let  $r$  be the neighbor of  $y$  distinct from  $v_3, u_0$ , and  $t$ . Note that  $q = r$  is possible only when  $q$  is a 4-vertex. Otherwise, if  $q$  is a 3-vertex, we obtain a separating 4-cycle  $u_0xqy$  which is reducible by Lemma 12(b).

Next, by the planarity at least one of the edges  $qt$  and  $rs$  does not exist, say  $qt \notin E(G)$ . Then, there exists a set  $R'$  in  $G - \{v_0, v_1, v_3, u_0, x, y, z, s, t, q\}$  which induces a forest  $F'$  of size at least  $\frac{119(n-10)-24(m-24)-24}{128} \geq \varphi - 5$ . By introducing vertices  $u_0, v_0, x, y$ , and  $z$  to  $R'$ , we obtain a set of vertices which induces a forest of size at least  $\phi$  in  $G$ . It is easy to see that the new vertices do not introduce any cycles, since only the edge  $yu_0$  could be incident with  $F'$ .

This analysis establishes the lemma. □

From Lemmas 13, 14, and 16–18 immediately follows the corollary below:

**Corollary 19** *Every 4-face (resp. 5-face) of  $G$  is incident with four (resp. three) 4-vertices.*

Finally we are ready to establish the theorem with the following short application of Euler's formula.

**Proof (of Theorem 3):** Let  $n_d$  be the number of vertices of degree  $d$  and let  $f_l$  be the number of faces of length  $l$  in  $G$ . By Corollary 19 we infer

$$4n_4 \geq 4f_4 + 3f_5.$$

Using this inequality and Euler's Formula we obtain

$$\begin{aligned} -12 &= 6m - 6n - 6f \\ &= 2 \sum_{v \in V(G)} d(v) + \sum_{f \in F(G)} l(f) - 6n - 6f \\ &= \sum_{d \geq 3} (2d - 6)n_d + \sum_{l \geq 4} (l - 6)f_l \\ &\geq 2n_4 - 2f_4 - f_5 \\ &\geq 0. \end{aligned}$$

Hence, we obtain a contradiction which shows that the minimal counter-example does not exist and establish Theorem 3.  $\square$

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