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**RESEARCH
REPORT**

N° 8535

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Project-Team COATI



Applying clique-decomposition for computing Gromov hyperbolicity*

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Aurélien Lancin^{‡§}

Project-Team COATI

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Abstract: The shortest-path metric d of a connected graph G is δ -hyperbolic if, and only if, it satisfies $d(u, v) + d(x, y) \leq \max\{d(u, x) + d(v, y), d(u, y) + d(v, x)\} + 2\delta$, for every 4-tuple u, x, v, y of G . We investigate some relations between the hyperbolicity of a graph and the hyperbolicity of its *atoms*, that are the subgraphs resulting from the clique-decomposition invented by Tarjan [34, 45]. More precisely, we prove that the maximum hyperbolicity taken over all the atoms is at least the hyperbolicity of G minus one. We also give an algorithm to slightly modify the atoms, which is at no extra cost than computing the atoms themselves, and so that the maximum hyperbolicity taken over all the resulting graphs is *exactly* the hyperbolicity of G . An experimental evaluation of our methodology is provided for large collaboration networks. Finally, we deduce from our theoretical results the first *linear-time* algorithm to compute the hyperbolicity of an outerplanar graph.

Key-words: Hyperbolicity, algorithm, graph, decomposition.

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Décomposition par des cliques-séparatrices pour le calcul de l'hyperbolicité de Gromov

Résumé : La métrique d des plus courts chemins d'un graphe connexe G est δ -hyperbolique si, et seulement si, elle satisfait $d(u, v) + d(x, y) \leq \max\{d(u, x) + d(v, y), d(u, y) + d(v, x)\} + 2\delta$, pour tout quadruplet u, x, v, y de G . Nous étudions la relation entre l'hyperbolicité d'un graphe et celle de chacun de ses *atomes*. Ces derniers sont les sous-graphes résultant de la décomposition d'un graphe par des cliques-séparatrices [34, 45]. Plus précisément, nous montrons que l'hyperbolicité d'un atome est au plus l'hyperbolicité de G moins un. Nous proposons un algorithme pour modifier les atomes de sorte que la valeur maximale de l'hyperbolicité de ces atomes modifiés soit exactement l'hyperbolicité de G . La complexité de cet algorithme est la même que celle de la décomposition du graphe par des cliques-séparatrices. Nous évaluons expérimentalement cette méthode sur des graphes de collaborations (co-auteurs). Enfin, nous proposons un algorithme pour calculer en temps linéaire l'hyperbolicité des graphes planaires extérieurs.

Mots-clés : Hyperbolicité, algorithme, graphe, décomposition.

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1 Introduction

In this paper, we primarily aim to study graph hyperbolicity, from an algorithmic point of view. This parameter was first introduced by Gromov in the context of automatic groups [26], then extended to more general metric spaces, including the shortest-path metrics of simple graphs. Hyperbolicity of unweighted graphs, and its tight relations to other metric parameters, has received growing attention over the last decades, especially due to its practical applications in approximation algorithms [13], routing [9], network security [29] and bioinformatics [12, 22], to name a few. The reader may refer to [1, 21] for a recent survey.

However, the computational cost of hyperbolicity has received less attention. So far, the best-known algorithm for computing the hyperbolicity of a graph [23] is impractical for large-scale graphs such as the graph of the Autonomous Systems of the Internet, road maps, etc. This both comes from its challenging implementation, relying on fast square matrix multiplications, and its time complexity which is strictly more than cubic. While practical advances have been made, improving the computational cost on certain graph classes (see [14, 31]), a recent theoretical work [15] suggests that an algorithm for the problem with a significant speed-up for *all graphs* is unlikely to exist. This motivates the study of structural properties that may help to decrease the running time of the computation of hyperbolicity.

Our approach relies on divide-and-conquer techniques, especially *graph decompositions*. Roughly, we aim to reduce the computation of the hyperbolicity of a graph to the computation of the hyperbolicity of (some of) its subgraphs, so that we can decrease the size of the input graphs to deal with. A first step toward this direction was the result in [41], where the author proved that the hyperbolicity of a connected graph is the maximum hyperbolicity taken over all the subgraphs obtained either via the modular decomposition [24], or with the split-decomposition [16].

We here address a similar question for the subgraphs obtained via the *clique-decomposition*, also known as *atoms*. This decomposition was first introduced by Tarjan in [45], then made unique by Leimer in [34]. On the theoretical side, it is already known that clique-decomposition can be applied to speed-up the computation of many graph parameters, including metric parameters that are related to hyperbolicity, such as tree-length [20]. Also, complex networks are expected to be decomposable w.r.t. clique-decomposition, especially when they are related to phylogenetical data, text-data mining, or distance data [6, 18, 30]. This makes our approach practical for large-scale graphs.

We will prove that while the hyperbolicity of a graph cannot be guessed from the hyperbolicity of its atoms (Section 3), it yields an approximation with additive constant 1 of this parameter (Section 4). Additionally, we characterize the cases for which this small additive distortion might happen. In Section 5, we will show how each atom can be modified in order to compute exactly the hyperbolicity, and provide a complexity analysis of the procedure. Experiments in Section 7 show the benefit of our method in terms of size of the graph, when applied to some real collaboration networks.

Finally, we will see in Section 6 that our method is beneficial for the class of outerplanar graphs, as it gives a linear-time algorithm to compute the hyperbolicity of these graphs.

Definitions and notations used in this paper will be introduced in Section 2.

2 Definitions and notations

We essentially rely on the graph terminology of [10, 19]. All graphs considered in this paper are finite, unweighted and simple. We here only emphasize some notions related to metric graph theory. The reader may refer to [3] for a survey about this domain. Given two vertices u, v , a uv -path of length $l \geq 0$ is a sequence of vertices $(u = v_0, v_1, \dots, v_l = v)$, satisfying $\{v_i, v_{i+1}\}$ is an edge for every $0 \leq i \leq l - 1$. In particular, G is *connected* if there exists a uv -path for all pair $u, v \in V$, and in such a case the *distance* $d_G(u, v)$ is defined as the minimum length of a uv -path in G . Note that it yields a discrete metric space (V, d_G) , also known as the shortest-path metric space of G . Also, we will write d instead of d_G whenever G is clear from the context. In the sequel, we will abuse of the distance notation to denote the distance $d(u, X) = \min_{x \in X} d(u, x)$ between a vertex u and a subset X .

Studying the properties of the metric space (V, d) requires stronger notions than the classical one of *induced subgraph*, as in general those subgraphs fail to be distance-preserving. Indeed, the wheel W_n is a good example, as it contains the cycle C_n as an induced subgraph, yet the shortest-path metric of the cycle is not induced by the shortest-path metric of the wheel (*e.g.* there is a shortcut between cyclic vertices, by using the central vertex of the wheel). We here restrict our study to *isometric subgraphs*. Formally, H is an isometric subgraph of the graph G if, and only if, it is connected and we have $d_H(u, v) = d_G(u, v)$, for all $u, v \in V(H)$.

δ -hyperbolic graphs. Graph hyperbolicity provides tight bounds on the worst additive distortion of the distances in a (connected) graph when its vertices are embedded into a weighted tree. Several definitions exist, some of them considering graph metrics that are slightly different than the usual shortest-path metric, but they are known to be equivalent up to a linear-function [5, 17, 26]. Moreover, 0-hyperbolic graphs are exactly the connected graphs whose given metric is a *tree metric*, which makes hyperbolicity a tree-likeness parameter. Especially, the shortest-path metric of a graph is 0-hyperbolic if, and only if, it is a *block-graph*, that is a graph whose biconnected components are complete subgraphs [4, 28]. This class of graphs includes trees and cliques, and a block-graph can be recognized in linear $O(n + m)$ -time.

The four-point definition of hyperbolicity is now defined as follows.

Definition 1 (4-points Condition, [26]). *Let G be a connected graph. For every 4-tuple u, x, v, y of G , we define $\delta(u, v, x, y)$ as half of the difference between the two largest sums amongst $S_1 = d(u, v) + d(x, y)$, $S_2 = d(u, x) + d(v, y)$, and $S_3 = d(u, y) + d(v, x)$.*

The graph hyperbolicity, denoted by $\delta(G)$, is equal to $\max_{u, x, v, y} \delta(u, v, x, y)$. Moreover, we say that G is δ -hyperbolic, for every $\delta \geq \delta(G)$.

It is straightforward, by the above definition, to compute graph hyperbolicity in $\theta(n^4)$ -time. Currently, the best-known *theoretical* algorithm for the problem runs in $O(n^{3.69})$ -time [23], and the best-known *practical* algorithm has $O(n^4)$ -time complexity [14]. Recognizing graphs with small hyperbolicity upper-bounded by $\frac{1}{2}$ is computationally equivalent to decide whether there is a chordless cycle of length 4 in a graph, and it can be done in $O(n^{3.256689})$ -time by using fast rectangular matrix multiplication [15, 33]. Note also that the hyperbolicity of a connected graph is the maximum hyperbolicity taken over all its biconnected components.

Atoms and clique-separators. Given a connected graph $G = (V, E)$, we name *separator* a subset of vertices $X \subset V$ such that the removal of X disconnects the graph. When the induced subgraph $G[X]$ is a complete graph, then we say that X is a *clique-separator*. More generally, we

say that X is a separator of $S \subseteq V$ if S intersects (at least) two distinct connected components of $G \setminus X$, and S is then said to be *separable*. The clique-decomposition of a graph (see Figure 1) is the collection of all its maximal sets of vertices that do not admit a clique-separator (we will call them *atoms*) [45]. The decomposition is unique [34], and it can be computed in $O(nm)$ -time [34, 45].

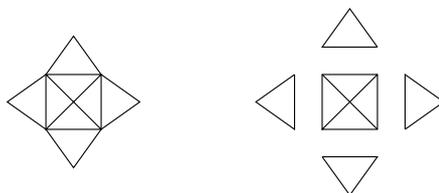


Figure 1: Clique-decomposition of a graph into five atoms.

Notations. Let us fix some notations for the proofs. Given two subsets A, B of vertices, we say that a separator X is an $(A|B)$ -separator if it disconnects any $a \in A \setminus X$ from any $b \in B \setminus X$. In particular, any separator X containing A or B is an $(A|B)$ -separator, and in such a case we call X a *trivial* $(A|B)$ -separator.

We also denote by $(a|b_1, b_2, b_3)$ a 4-tuple such that $a \in A$ and $b_1, b_2, b_3 \in B$. In the same way, we denote by $(a_1, a_2|b_1, b_2)$ a 4-tuple such that $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

Finally, a clique-separator is called a *clique-minimal separator* if there exists some pair u, v such that it is an inclusionwise minimal, non-trivial $(u|v)$ -clique-separator. It has to be noted, perhaps counter-intuitively, that a clique-minimal separator may be contained into another one.

3 Hyperbolicity and clique-separators

We analyse in this section the relationship between the hyperbolicity of a graph and a given clique-separator, leading to the approximation with additive constant of Theorem 11. It begins with an observation about $(a_1, a_2|b_1, b_2)$ 4-tuples and the diameter $\text{diam}(X) = \max_{u, v \in X} d_G(u, v)$ of an $(A|B)$ -separator X .

3.1 Hyperbolicity of $(a_1, a_2|b_1, b_2)$ 4-tuples

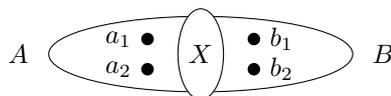


Figure 2: Illustration of an $(A|B)$ -separator.

Lemma 2. *Let X be a an $(A|B)$ -separator of a connected graph G . For every $(a_1, a_2|b_1, b_2)$ 4-tuple, we have $\delta(a_1, a_2, b_1, b_2) \leq \text{diam}(X)$.*

Proof. Recall that we have:

$$\begin{aligned} S_1 &= d(a_1, a_2) + d(b_1, b_2), \\ S_2 &= d(a_1, b_1) + d(a_2, b_2), \\ S_3 &= d(a_1, b_2) + d(a_2, b_1). \end{aligned}$$

We can assume without loss of generality (w.l.o.g.) that $S_2 \geq S_3$. Writing $d(v, X) = \min_{x \in X} d(v, x)$, we also know that for all $i, j \in \{1, 2\}$:

$$\begin{aligned} d(a_i, b_i) &\geq d(a_i, X) + d(b_i, X) \\ d(a_i, a_j) &\leq d(a_i, X) + \text{diam}(X) + d(a_j, X) \end{aligned}$$

If $S_1 \geq S_2$, we get:

$$\begin{aligned} S_2 &= d(a_1, b_1) + d(a_2, b_2) \\ &\geq d(a_1, X) + d(b_1, X) + d(a_2, X) + d(b_2, X) \\ &\geq [d(a_1, X) + d(a_2, X) + \text{diam}(X)] + \\ &\quad [d(b_1, X) + d(b_2, X) + \text{diam}(X)] - 2 \text{diam}(X) \\ &\geq S_1 - 2 \text{diam}(X) \end{aligned}$$

Hence we have that $\delta(a_1, a_2, b_1, b_2) \leq (S_1 - S_2)/2 \leq \text{diam}(X)$. It can be shown similarly that $S_3 \geq S_2 - 2 \text{diam}(X)$, and when $S_2 \geq S_1$ that $S_1 \geq S_2 - 2 \text{diam}(X)$. Consequently, $\delta(a_1, a_2, b_1, b_2) \leq \text{diam}(X)$ in all cases. \square

Corollary 3. *Let X be an $(A|B)$ -clique-separator of a connected graph G . For every $(a_1, a_2|b_1, b_2)$ 4-tuple, we have $\delta(a_1, a_2, b_1, b_2) \leq 1$.*

3.2 Hyperbolicity of $(a|b_1, b_2, b_3)$ 4-tuples

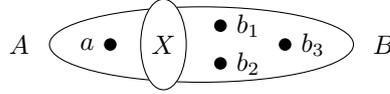


Figure 3: Illustration of an $(a|b_1, b_2, b_3)$ -separator.

Note that X being a clique, each vertex $a \in A$ is at distance at least $d(u, X)$ and at most $d(u, X) + 1$ from any vertex of X . We now show how this can be used with respect to the hyperbolicity.

Lemma 4. *Let X be an $(A|B)$ -clique-separator of a connected graph G , and let $a \in A$. If we add to G a vertex a^* adjacent to $\{x \in X : d(a, x) = d(a, X)\}$, then for every $b_1, b_2, b_3 \in B$ we have $\delta(a, b_1, b_2, b_3) = \delta(a^*, b_1, b_2, b_3)$.*

Proof. $\forall x \in X, d(a, x) \in \{d(a, X), d(a, X) + 1\}$ holds as X is a clique. Consequently, for every $b \in B, d(a, b) = d(a^*, b) + d(a, X) - 1$ and replacing a with a^* does not change the hyperbolicity of a, b_1, b_2, b_3 . \square

Lemma 5. *Let X be an $(A|B)$ -clique-separator of a connected graph G . Given an $(a|b_1, b_2, b_3)$ 4-tuple, let $x \in X$ be such that $d(a, x) = d(a, X)$. We have $\delta(a, b_1, b_2, b_3) \leq \delta(x, b_1, b_2, b_3) + 1/2$.*

Proof. Let us assume w.l.o.g. that $S_1 \geq S_2 \geq S_3$, where $S_1 = d(a, b_1) + d(b_2, b_3)$, $S_2 = d(a, b_2) + d(b_1, b_3)$ and $S_3 = d(a, b_3) + d(b_1, b_2)$.

We can assume that every vertex $a \in A$ has been replaced by an equivalent vertex a^* as defined in Lemma 4 and so, that all vertices of A have a neighbor in X . In such a situation, any

$b \in B$ satisfies $d(x, b) \leq d(a, b) \leq d(x, b) + 1$. Similarly, any sum $S'_i = d(x, b_i) + d(b_j, b_k)$, where $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$, satisfies $S'_i \leq S_i \leq S'_i + 1$. Thus for every $i \in \{2, 3\}$:

$$\begin{aligned} \delta(a, b_1, b_2, b_3) &\leq (S_1 - S_i)/2 \\ &\leq (S'_1 + 1 - S'_i)/2 \\ &\leq (S'_1 - S'_i)/2 + 1/2 \end{aligned}$$

In particular, if $S'_1 \neq \max\{S'_1, S'_2, S'_3\}$, then we have $\delta(a, b_1, b_2, b_3) \leq \frac{1}{2}$ by the choice of $S'_i = \max\{S'_1, S'_2, S'_3\}$. Otherwise, we have $\delta(a, b_1, b_2, b_3) \leq \delta(x, b_1, b_2, b_3) + \frac{1}{2}$ by the choice of $S'_i = \max\{S'_2, S'_3\}$. \square

3.3 Separable sets

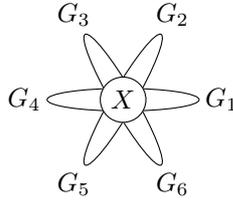


Figure 4: Illustration of separable sets.

Theorem 6. *Let X be a clique-separator of a connected graph G , and let C_1, \dots, C_l be the connected components of $G \setminus X$. We define $G_i = G[C_i \cup X]$. We have :*

$$\max\{\delta(G_1), \dots, \delta(G_l)\} \leq \delta(G) \leq \max\left\{\frac{1}{2}, \delta(G_1), \dots, \delta(G_l)\right\} + 1/2$$

Proof. Note that since a complete subgraph is isometric, then every subgraph G_i is isometric as well. Hence, the lower-bound follows from the four-point definition.

Let us now prove that $\delta(a, b, c, d) \leq \max\{\frac{1}{2}, \delta(G_1), \dots, \delta(G_l)\} + 1/2$ holds for any $a, b, c, d \in V$. We consider a connected component C_i maximizing $|C_i \cap \{a, b, c, d\}|$.

- If $|C_i \cap \{a, b, c, d\}| = 4$ we are done as $\delta(a, b, c, d) \leq \delta(G_i)$.
- If $|C_i \cap \{a, b, c, d\}| = 3$ we can assume that a, b, c, d is an $(a|b_1, b_2, b_3)$ 4-tuple, for the choices of $B = C_i \cup X$ and $A = V \setminus C_i$. By Lemma 5 it follows that $\delta(a, b, c, d) \leq \delta(G_i) + 1/2$.
- Finally, if $|C_i \cap \{a, b, c, d\}| \leq 2$ we can assume that a, b, c, d is an $(a_1, a_2|b_1, b_2)$ 4-tuple, for an appropriate merging of the sets $C_j \cup X$ into two subsets A, B . We know by Corollary 3 that $\delta(a_1, a_2, b_1, b_2) \leq 1$ in this case. \square

The upper-bound of Theorem 6 is actually tight. It can be shown using the graph in Figure 5, constructed from a cycle C_7 of length 7 to which we add a triangle.

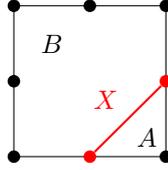


Figure 5: X is an $(A|B)$ -clique-separator: we have $\delta(G) = 3/2$, while $\delta(G[B]) = 1$, and $\delta(G[A]) = 0$.

4 Hyperbolicity and clique-decomposition

In Section 3, we explained how a single clique-separator can be used to approximate hyperbolicity. However, this result cannot be used on a whole clique-decomposition as the successive approximations would add up. We thus need to find additional properties to approximate the hyperbolicity of a graph from computations on its atoms in order to prove Theorem 11.

4.1 Relating atoms and 4-tuples with large hyperbolicity

Firstly, we aim to relate to every 4-tuple a, b, c, d with a sufficiently large hyperbolicity, some atom by which all the paths between a, b, c, d go through. Our result involves basic knowledge about *tree-decomposition* (see [8]). We remind the reader that a tree-decomposition of a connected graph G is a tree T whose nodes are labeled by subsets of $V(G)$ (also known as *bags*), and which satisfies the following properties:

- every edge of G is contained into some bag;
- for every vertex $u \in V(G)$, the subgraph induced by the bags containing u is a subtree of T .

Lemma 7. *Let a, b, c, d be a 4-tuple satisfying $\delta(a, b, c, d) \geq \frac{3}{2}$ in a connected graph G . There exists an atom A such that $\forall u \in \{a, b, c, d\} \setminus A$, there is a clique-separator $X_u \subseteq A$ which separates u from $\{a, b, c, d\} \setminus \{u\}$.*

Proof. If the vertices a, b, c, d are contained into a common atom A , then we are done as $\{a, b, c, d\} \setminus A$ is empty. Suppose on the contrary that there is no atom containing the 4-tuple. The authors of [38] proved that there exists a tree-decomposition T_G of G whose bags are exactly the atoms of G . As there is no bag containing a, b, c, d by the hypothesis, there is a unique smallest subtree T of T_G containing a, b, c, d . Note that T is not reduced to a single vertex, and it has at most 4 leaves.

It can be checked that there exists a bag A in T (i.e. an atom of G) such that no connected component of $T \setminus \{A\}$ (and thus no connected component of $G \setminus A$) contains more than two elements among a, b, c, d . Moreover, if a connected component C of $G \setminus A$ contains exactly two elements of a, b, c, d , then $N_G(C) \cap A$ is a clique of G separating two elements of a, b, c, d from the two others, which is impossible by Corollary 3 as we supposed $\delta(a, b, c, d) \geq \frac{3}{2}$.

Thus, for every $u \in \{a, b, c, d\} \setminus A$, the connected component C_u of $G \setminus A$ containing u yields a clique $N_G(C_u) \cap A$, which separates u from $\{a, b, c, d\} \setminus \{u\}$. \square

As an illustration, one may notice that the central atom in Figure 6 satisfies the property of Lemma 7 with respect to the 4-tuple v_0, v_1, v_2, v_3 . Indeed, none of the four vertices is contained into this atom, but each of them is simplicial and can be separated from the three others by its two neighbours.

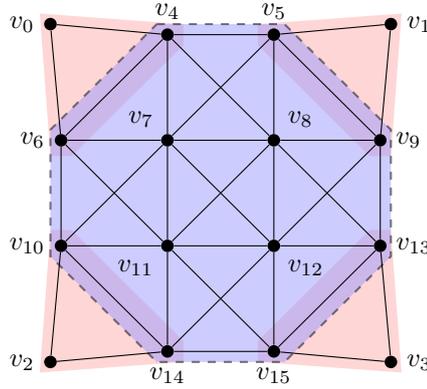


Figure 6: A 2-hyperbolic graph with five atoms: four are 0-hyperbolic, one is 1-hyperbolic.

Moreover, we note that our above result heavily relies on the tree-like structure of the atoms (*e.g* we use a tree-decomposition whose bags are the atoms), and that such a tree-like structure does not exist in general if separators with diameter at least 2 are involved. In fact, our results in this section cannot be extended to decompositions with separators of bounded diameter, as there exist infinitely many graphs that are completely separable with isometric separators of diameter at most 2, and whose hyperbolicity is arbitrarily large [32].

4.2 Substitution method

From Lemma 7 we can associate a specific atom to a 4-tuple of large hyperbolicity. Four applications of Lemma 5 are then sufficient to prove that the hyperbolicity of this 4-tuple and the hyperbolicity of the atom differ by at most 2. The purpose of this section is to prove that this difference is at most 1. To do this, we refine the results of Section 3.2.

Lemma 8. *Let X be an $(A|B)$ -clique-separator of a connected graph G . Given an $(a|b_1, b_2, b_3)$ 4-tuple, write:*

$$S_1 = d(a, b_1) + d(b_2, b_3),$$

$$S_2 = d(a, b_2) + d(b_1, b_3),$$

$$S_3 = d(a, b_3) + d(b_1, b_2).$$

Assume w.l.o.g. that $S_1 \geq S_2 \geq S_3$, and let $x_2 \in X$ be such that $d(a, b_2) = d(a, x_2) + d(x_2, b_2) = d(a, X) + d(x_2, b_2)$. If $\delta(a, b_1, b_2, b_3) > \delta(x_2, b_1, b_2, b_3)$, then we have

- $S_1 > S_2 = S_3$.
- $d(a, b_1) = d(a, x_2) + d(x_2, b_1)$.

Proof. Recall that by the substitution of Lemma 4, we can assume w.l.o.g. that for every i we have $d(a, x_i) = d(a, X) = 1$, where x_i is a vertex of X located on an ab_i -shortest path.

Now, let $\varepsilon_i = d(x_2, b_i) - d(x_i, b_i)$. Observe that $\varepsilon_i \in \{0, 1\}$, and that $\varepsilon_i = 0$ if, and only if, x_2 lies on an ab_i -shortest path. In particular we have $\varepsilon_2 = 0$.

Let us denote by S'_i the sum $d(x_2, b_i) + d(b_j, b_k)$, where $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$. We aim to exhibit a relation between S_i and S'_i , which would yield in turn a relation between the values

$\delta(a, b_1, b_2, b_3)$ and $\delta(x_2, b_1, b_2, b_3)$. First, we have $d(a, b_i) = d(x_2, b_i) + 1 - \varepsilon_i$ and so,

$$\begin{aligned} S_i &= d(a, b_i) + d(b_j, b_k) \\ &= d(x_2, b_i) + d(b_j, b_k) + 1 - \varepsilon_i \\ &= S'_i + 1 - \varepsilon_i \end{aligned}$$

Since we assume here that $S_1 \geq S_2 \geq S_3$ and $\delta(a, b_1, b_2, b_3) > 0$, we have $S'_1 \geq S_1 - 1 \geq \max\{S_2, S_3\} \geq \max\{S'_2, S'_3\}$. More precisely:

- If $S'_2 \geq S'_3$, then $\delta(a, b_1, b_2, b_3) = \delta(x_2, b_1, b_2, b_3) - \varepsilon_1/2 \leq \delta(x_2, b_1, b_2, b_3)$.
- If $S'_3 > S'_2$, then $\varepsilon_3 = 1$ because $S_2 \geq S_3$, which implies $S_2 = S_3$. This, in turn, implies that $\delta(x_2, b_1, b_2, b_3) = (S'_1 - S'_3)/2 = (S_1 - 1 + \varepsilon_1 - S_3)/2 = (S_1 - S_2)/2 - (1 - \varepsilon_1)/2 = \delta(a, b_1, b_2, b_3) - (1 - \varepsilon_1)/2 \leq \delta(a, b_1, b_2, b_3)$.

In such a case, $\delta(x_2, b_1, b_2, b_3) < \delta(a, b_1, b_2, b_3)$ if, and only if, we have $\varepsilon_1 = 0$, that is x_2 lies on an ab_1 -shortest path. □

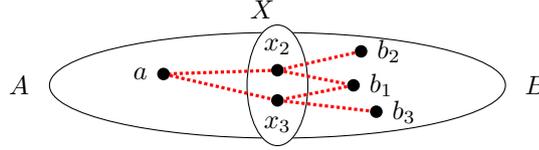


Figure 7: An illustration of the metric property. The dashed lines represent shortest paths.

The metric property of Lemma 8 is illustrated with Figure 7. We use it to extend Lemma 5 as follows.

Lemma 9. *Let a, b, c, d be a 4-tuple of a connected graph G , and two sets $X_a, X_d \subseteq V$ satisfying:*

- X_a is a $(a|b, c, d)$ -clique-separator;
- X_d is a $(d|a, b, c)$ -clique-separator;
- $X_a \setminus X_d$ lies in the same connected component of $G \setminus X_d$ as a, b, c .

Then there exist $x_a \in X_a$ and $x_d \in X_d$ such that

$$\delta(a, b, c, d) \leq \delta(x_a, b, c, x_d) + 1/2.$$

Proof. Let u_1, u_2, u_3 be such that $\{u_1, u_2, u_3\} = \{b, c, d\}$, $T_1 \geq T_2 \geq T_3$, where $T_i = d(a, u_i) + d(u_j, u_k)$ and $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$. We distinguish two cases:

- First, assume that there is $x_a \in X_a$ satisfying $\delta(a, b, c, d) \leq \delta(x_a, b, c, d)$. Then we are done, as Lemma 5 applied to the $(d|x_a, b, c)$ -clique separator X_d yields a vertex $x_d \in X_d$ satisfying $\delta(x_a, b, c, d) \leq \delta(x_a, b, c, x_d) + \frac{1}{2}$, hence $\delta(a, b, c, d) \leq \delta(x_a, b, c, x_d) + \frac{1}{2}$.
- Second, assume that no vertex of X_a satisfies the above property. In particular, let $x_a \in X_a$ be such that $d(a, x_a) = d(a, X_a)$, and x_a lies on an au_2 -shortest path. Note that $\delta(a, b, c, d) \leq \delta(x_a, b, c, d) + \frac{1}{2}$ by Lemma 5 and so, $\delta(a, b, c, d) = \delta(x_a, b, c, d) + \frac{1}{2}$. Moreover,

we deduce from Lemma 8 that $T_1 > T_2 = T_3$, and x_a also lies on an au_1 -shortest-path. Let $T'_i = d(x_a, u_i) + d(u_j, u_k)$. It thus follows that we have:

$$T'_1 = T_1 - d(a, X_a), T'_2 = T_2 - d(a, X_a) \text{ and } T'_3 = T_3 - d(a, X_a) + 1.$$

So, we have $T'_1 \geq T'_3 > T'_2$. By the contrapositive of Lemma 8 applied to the 4-tuple $(d|x_a, b, c)$, and noticing that the two least sums amongst T'_1, T'_2, T'_3 are different, there exists a vertex $x_d \in X_d$ satisfying $\delta(x_a, b, c, d) \leq \delta(x_a, b, c, x_d)$. \square

Corollary 10. *Let a, b, c, d be a 4-tuple of a connected graph G satisfying $\delta(a, b, c, d) \geq 3/2$. There exists an atom A of G such that $\delta(a, b, c, d) \leq \delta(G[A]) + 1$*

Proof. The atom A is obtained by applying Lemma 7. For every vertex $u \in \{a, b, c, d\} \setminus A$, let $X_u \subseteq A$ be a clique-separator disconnecting u from $\{a, b, c, d\} \setminus u$. We claim that $A \setminus X_u$ lies in the same connected component of $G \setminus X_u$ as $\{a, b, c, d\} \setminus u$, because otherwise X_u would separate two elements of $\{a, b, c, d\}$ from the two others, which by Corollary 3 implies that $\delta(a, b, c, d) \leq 1$, a contradiction.

One can then apply Lemmas 9 and 5 to find four vertices $a', b', c', d' \in A$ such that $\delta(a, b, c, d) \leq \delta(a', b', c', d') + 1$. Note that the sets $\{a, b, c, d\}$ and $\{a', b', c', d'\}$ are not necessarily disjoint. \square

4.3 Additive approximation for hyperbolicity

Theorem 11. *Let A_1, \dots, A_l be the atoms of a connected graph G . Then:*

$$\max_i \delta(G[A_i]) \leq \delta(G) \leq \max_i \delta(G[A_i]) + 1$$

Proof. As for Theorem 6, the lower-bound of Theorem 11 follows from the fact that the subgraphs $G_i = G[A_i]$ are isometric subgraphs of G . The upper-bound trivially holds when $\delta(G) \leq 1$. We can thus suppose that $\delta(G) \geq 3/2$ and so, that there exist four vertices a, b, c, d such that $\delta(a, b, c, d) = \delta(G) \geq 3/2$. Corollary 10 then yields an atom A such that $\delta(G) \leq \delta(G[A]) + 1$, which proves the second part of our claim. \square

Note that the upper-bound is reached by the graph of Figure 6, and by the 1-hyperbolic chordal graph from Figure 1 whose atoms have hyperbolicity 0.

5 Substitution method for an exact computation

As shown with Theorem 11, the maximum hyperbolicity taken over all the atoms may be slightly lesser than the hyperbolicity of the graph. In the two previous sections, we characterized under which situations both values might differ. We now propose to extend the usual clique-decomposition, by replacing the atoms by some *substitute graphs*, so that both values are always equal. The main drawback of our method is that it requires the hyperbolicity to be at least 1.

Outline of the method. We recall that a *simplicial* vertex is a vertex whose neighborhood induces a complete subgraph. In the spirit of Lemma 4, we add simplicial vertices to the atoms, in order to mimic the maximum $(a|b_1, b_2, b_3)$ 4-tuples that may result from a disconnection. We first introduce our method of substitution in the simpler context of a *single* disconnection. Then we focus on technical details related to the implementation.

5.1 Substitute graphs

5.1.1 Basic step: single disconnection

Let us consider the following algorithm. Given a connected graph G , let A, B be a covering of $V(G)$ satisfying $X = A \cap B$ is an $(A|B)$ -clique separator of G .

- Starting from $G_1 = G[A]$, for every vertex b of $B \setminus X$, we consider the subset X_b of vertices which are the closest to b in X . For every such a subset, we add in G_1 a simplicial vertex s_{X_b} whose neighborhood is X_b .
- Starting from $G_2 = G[B]$, for every vertex a of $A \setminus X$, we consider the subset X_a of vertices which are the closest to a in X . For every such a subset, we add in G_2 a simplicial vertex s_{X_a} whose neighborhood is X_a .

The two resulting graphs are denoted G_1^*, G_2^* , and they are called the *substitute graphs* of G_1, G_2 .

Lemma 12. *Let G be a connected graph satisfying $\delta(G) \geq 1$, and A, B be a covering of the vertices satisfying $A \cap B = X$, where X denotes an $(A|B)$ -clique separator of G . We define $G_1 = G[A]$, $G_2 = G[B]$. We have:*

$$\delta(G) = \max\{1, \delta(G_1^*), \delta(G_2^*)\}.$$

Proof. First, we prove the upper-bound. By construction, the subgraph G_i is an isometric subgraph of G_i^* , for every $i \in \{1, 2\}$. It thus follows that $\delta(G_i^*) \geq \delta(G_i)$. Let us now consider the 4-tuples of G which are separated by the disconnection. We recall that by Corollary 3, the hyperbolicity of any $(a_1, a_2|b_1, b_2)$ 4-tuple of G is bounded by 1. In addition, for every $a \in A \setminus X$, and for every $b_1, b_2, b_3 \in B$, there exists by construction a simplicial vertex a^* of $V(G_2^*) \setminus B$ that is adjacent to $\{x \in X : d_G(a, x) = d_G(a, X)\}$; hence, $\delta(a, b_1, b_2, b_3) = \delta(a^*, b_1, b_2, b_3) \leq \delta(G_2^*)$ by Lemma 4. The proof is symmetric for $b \in B \setminus X$ and $a_1, a_2, a_3 \in A$. As a result, $\delta(G) \leq \max\{1, \delta(G_1^*), \delta(G_2^*)\}$.

To prove the lower-bound, let us consider w.l.o.g. an arbitrary 4-tuple of G_2^* . If such a 4-tuple does not contain a simplicial vertex of $V(G_2^*) \setminus B$, then we are done as it exists in the isometric subgraph G_2 of G . If it contains exactly one simplicial vertex $a^* \in V(G_2^*) \setminus B$, then by construction we can replace a^* with a vertex $a \in A \setminus X$, satisfying $N(a^*) = \{x \in X : d_G(a, x) = d_G(a, X)\}$; this operation does not modify the hyperbolicity of the 4-tuple by Lemma 4. Finally, if it contains at least two simplicial vertices of $V(G_2^*) \setminus B$, then it is an $(a_1, a_2|b_1, b_2)$ 4-tuple of G_2^* and so, the hyperbolicity is bounded by 1. \square

We emphasize that some simple rules can be applied, in order to add fewer vertices into the substitute graphs. In particular, we can remove every pendant vertex, as the hyperbolicity of a connected graph is the maximum hyperbolicity taken over all its biconnected components (this well-known result is a particular case of our proofs in Section 3, as a cut-point is a clique-separator containing a single vertex).

5.1.2 Extension to the atoms

The substitution operation can be naturally extended to the whole clique-decomposition, by mimicking each step of it and applying the basic substitution operation that we describe above at each of these steps. We formalize it by first introducing the following definition of an atom tree.

Definition 13 ([6, 7, 34]). *Let G be a connected graph. An atom tree of G is a labeled binary rooted tree T , satisfying the following recursive definition:*

- if G is prime w.r.t. clique-decomposition, then T is reduced to a node labeled with V ;
- otherwise, the root of T is labeled with a clique-minimal separator X , and there exists a connected component C of $G \setminus X$ satisfying:
 - $N_G(C) \setminus C = X$;
 - the left child of the root is labeled with $A = C \cup X$, which does not contain any clique-minimal separator;
 - and the right subtree of the root is an atom tree of $G \setminus C$.

In order to prevent any confusion, the reader has to notice that an atom tree is *not* a tree-decomposition (as defined in Section 4.1). In fact, an atom tree can be seen as the trace of some execution of the algorithm of [34, 45] to compute the clique-decomposition. Indeed, it is proved in [34] that in an atom tree, the leaves are labeled with the atoms of the graph, and each atom labels exactly one leaf of the tree. Given a *fixed* atom tree, this yields a natural ordering of the atoms by increasing depth. We follow this ordering so that we can apply our basic step of substitution sequentially, using at each step $i \in \{1, \dots, l\}$ the same clique-minimal separator X_i as in the atom tree to separate the connected component containing A_i from the remaining vertices to explore.

The whole process is illustrated with Figure 8. The numbers reported in Tab. 8i illustrate the interest of our pre-processing method for the computation of the hyperbolicity. Indeed, the graph G of Fig. 8a has 28 nodes and so 20 475 4-tuples, while the sum of the numbers of 4-tuples in the graphs G_i^* (Figs. 8c–8h) is 1 800. We thus significantly reduce the search space. Moreover, a simple cutting rule allows us to reduce the number of 4-tuples to consider to 1 575. To do so, we first order the graphs G_i^* by decreasing diameters, then iteratively compute the hyperbolicity of these graphs in this order, and stop exploration as soon as the diameter of a graph G_j^* is less than or equal to twice the largest value of δ computed so far.

5.2 Implementation and complexity analysis

5.2.1 Precomputation step and updates

We first focus on some computational tasks that have to be repeatedly executed at each step of our substitution method. In this section, we provide a high-level description for an implementation of these tasks.

Computation of the distances We heavily rely on the distance-matrix of the graph in our implementation, to compute the distances between some vertex (possibly newly added by our construction) and the vertices of some clique-minimal separator. Since we apply our basic substitution step as many times as there are atoms (*e.g.* Section 5.1.2), it is essential that we can update the distance-matrix of the graph quickly, so that such distances remain computable in constant-time.

Lemma 14. *Let G be a connected graph. We can embed in quadratic-time the distance-matrix of G into a data-structure, supporting:*

- constant-time access to the distance between a vertex and a vertex contained into a clique-minimal separator;
- constant-time updates when a simplicial vertex is added by our substitution method.

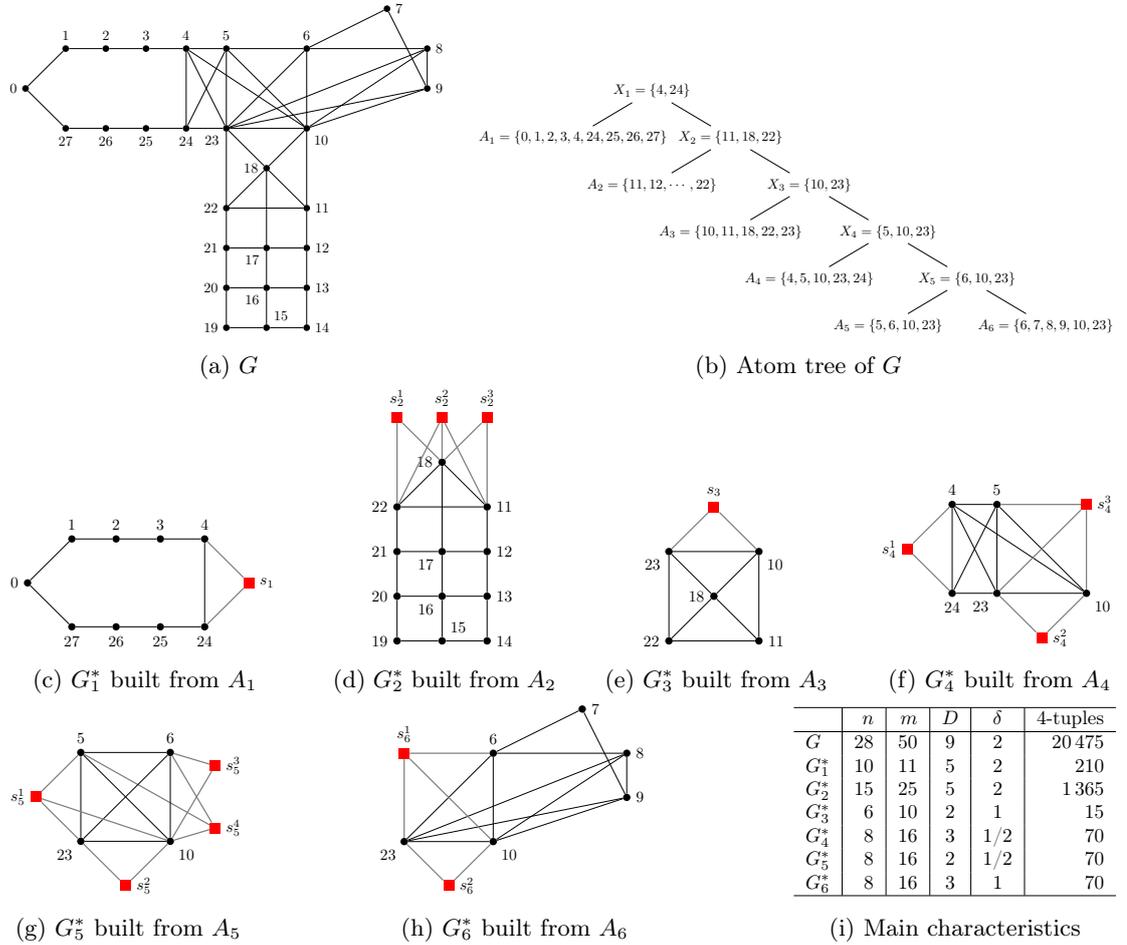


Figure 8: A connected graph G (Fig. 8a), an atom tree of the graph (Fig. 8b), the substitute of the atoms of G (Figs. 8c–8h), and the characteristics of these graphs (Tab. 8i).

Proof. The gist of such a structure is precisely Lemma 4. Let X be an $(A|B)$ -clique separator of G , and s be a simplicial vertex added in the substitute graph of $G[B]$. Let $a \in A \setminus X$ satisfying $N(s) = \{x \in X : d_G(a, x) = d_G(a, X)\}$. Then we have for every $b \in B$, $d(b, s) = d_G(a, b) + (1 - d_G(a, X))$.

It thus follows that once the substitution operation has been completed, we only need to relate every new simplicial vertex added by our construction to some vertex of $V(G)$ with an offset, so that we can compute the distances in the substitute graphs. The offset can be computed in constant-time by picking a neighbor of the simplicial vertex s , and accessing in constant-time to the distance between this neighbor and a vertex substituted by s . \square

Note that the data-structure of Lemma 14 does not support the computation of distances between two vertices added by our construction. We can safely ignore this drawback, as we never require to compute such distances in our method.

Determining the vertex sets of the substitute graphs A similar problem is to compute efficiently, at step i of our algorithm, the connected component containing the atom A_i when we disconnect our current (dynamical) graph using the clique-minimal separator X_i . Note that, due to the (simplicial) vertices and edges possibly added at previous steps, the classical algorithm for this problem cannot be ensured to run in $O(m)$ -time. Instead, we propose a method to update this component dynamically, starting from A_i . To do that, let us consider the following partition of the vertex set $V(G)$:

$$V_1 = A_1 \setminus X_1, \dots, V_i = A_i \setminus X_i, \dots, V_{l-1} = A_{l-1} \setminus X_{l-1}, V_l = A_l.$$

Lemma 15. *Let G be a connected graph, T be an atom tree of G , and A_1, \dots, A_l be its atoms. We denote by X_i the clique-minimal separator of G labeling the parent node of A_i in T , and by V_i^* the vertex set of the substitute graph of the atom A_i .*

Then, for every simplicial vertex s_i added at step i of our substitution method, we can decide the vertex set V_j^ , $j > i$, to which we have to add s_i in subsequent steps, in $O(n|X_i|)$ -time.*

Proof. It can be checked that at step i , the simplicial vertices added into $V_i^* \setminus A_i$ are those of previous steps whose neighborhood partly lies into $V_i = A_i \setminus X_i$. So, at every step of our substitution method, if a simplicial vertex is added by our construction, we consider the minimum index j satisfying V_j contains a neighbour of the vertex, and we dynamically update the vertex set V_j^* containing A_j by adding this new vertex into it. Since only $O(n)$ vertices are added at step i , and that their neighborhood is contained into X_i , the $O(n|X_i|)$ -time complexity follows. \square

5.2.2 Applying simplification rules

We now focus on some algorithmic aspects of the substitution operation (Section 5.1).

In this section, we will assume that the distance-matrix of the graph is given. Given the set $B \setminus X$, it is straightforward to compute in $O(n|X|)$ -time all of the subsets $X_b = \{x \in X : d_G(b, x) = d_G(b, X)\}$, for every $b \in B \setminus X$. A naive implementation would then consist in adding a simplicial vertex for every b and by doing so, we would lose all the benefit of the separation in terms of size of the graphs. However, we defined rules in order to avoid this worst-case in some situations, which potentially decrease the number of simplicial vertices to add in the substitute graphs. The goal of this section is to give hints on an efficient way to implement these rules.

Partition refinement techniques Indeed, it may happen that $X_b = X_{b'}$ for some pair $b, b' \in B$, and in such a case we wish to add only one simplicial vertex in the substitute graph G_1^* .

To do that efficiently, we will use the well-known *partition refinement* techniques (e.g. see [27, 39]). Given a partition \mathcal{P} of a set V , and a subset $S \subseteq V$ called the *pivot*, the partition refinement of \mathcal{P} w.r.t. S consists in replacing every group V_i of \mathcal{P} by the non-empty groups amongst $V_i \cap S$ and $V_i \cap \bar{S}$. This can be achieved in $O(|S|)$ -time, up to the precomputation of an appropriate data-structure in linear $O(|V|)$ -time.

We deduce from this standard technique the following result:

Lemma 16. *Let G be a connected graph given by its distance-matrix, and X be a subset of $V(G)$. We define the relation \equiv_X over the set $V(G) \setminus X$ as $u \equiv_X v$ if, and only if,*

$$\{x \in X : d_G(u, x) = d_G(u, X)\} = \{x \in X : d_G(v, x) = d_G(v, X)\}.$$

The equivalence classes of the relation \equiv_X can be computed in $O(n|X|)$ -time.

Proof. Consider the following graph $G_X = (V, \{\{u, x\} : u \notin X, x \in X \text{ and } d_G(u, x) = d_G(u, X)\})$. Starting from $\mathcal{P} = \{V \setminus X\}$, we refine the partition with the sets $N_{G_X}(x)$, for every $x \in X$. The total cost is $O(\sum_{x \in X} |N_{G_X}(x)|) = O(n|X|)$. \square

5.2.3 Complexity analysis

Finally, to determine the time complexity of our substitution method, we will use the following result:

Lemma 17 ([6]). *Let G be a connected graph, and A_1, \dots, A_l be its atoms. Then $\sum_i |A_i| \leq n + m$.*

Corollary 18. *The substitute of the atoms of a connected graph G can be computed in $O(nm)$ -time.*

Proof. Let T be an atom tree of G , and A_1, \dots, A_l be the atoms of the graph. For every i , let X_i be the clique-minimal separator of G labeling the father node of leaf A_i in T . By Definition 13, $X_i \subseteq A_i$.

We first precompute the distance-matrix of G in $O(nm)$ -time, then we embed it in quadratic-time into the data-structure of Lemma 14. Also, for every i , we initialize the vertex set V_i^* with the atom A_i . The set V_i^* is the vertex set of the substitute graph containing A_i at step i of our substitution method. We then apply each step of our substitution method sequentially.

We can easily check that at step i , there are at most $O(n)$ vertices to consider. By Lemma 16, this allows us to compute the simplicial vertices to add at this step in $O(n|X_i|)$ -time. By Lemma 15, we can then update the sets V_j^* , $j > i$, with the same time complexity. Since at most $O(n)$ vertices are added at this step, we finally update the distances in $O(n)$ -time by Lemma 14.

Consequently, our modified clique-decomposition can be computed in $O(n \sum_i |X_i|)$ -time, that is in $O(n \sum_i |A_i|) = O(nm)$ -time by Lemma 17. \square

6 Hyperbolicity of outerplanar graphs

Last, we deal in this section with an algorithmic application of the substitution method of Section 5. We use it here to provide a *linear-time* algorithm for computing the hyperbolicity of a large class of graphs.

More precisely, we here constrain our study to outerplanar graphs. A *planar graph* is a graph drawable in the Euclidean plane so that edges may only intersect at their endpoints, and an *outerplanar graph* is a graph which keeps the planarity property whenever one adds a universal vertex to it. Equivalently, a graph is outerplanar if it is drawable in the Euclidean plane so that edges may only intersect at their endpoints, and all the vertices lie on a common face which is called the *outerface*. Such a drawing is furthermore called an outerplanar embedding, and it can be computed in linear-time [43]. Outerplanar graphs are stable by the minor operation, and a graph is outerplanar if, and only if, it is K_4 -minor-free and $K_{2,3}$ -minor-free. The reader may refer to [42] for other characterizations of this class of graphs.

An interesting subclass of outerplanar graphs is the class of cycles. Actually, for an outerplanar graph, the induced cycles are exactly its atoms. Hyperbolicity of cycles is completely characterized, and it can be computed using the length of the cycle:

Lemma 19 ([14, 46]). *Cycles of order $4p + \varepsilon \geq 3$, with $p \geq 0$ and $\varepsilon \in \{0, 1, 2, 3\}$, are $(p - 1/2)$ -hyperbolic when $\varepsilon = 1$, and p -hyperbolic otherwise.*

In the sequel, we will use Lemma 19 in order to prove the main result of this section (e.g. Theorem 26). Our proofs heavily rely on the notion of *weak dual* [2].

Definition 20. *Let G be a connected outerplanar graph. The weak dual of G is a tree T_G , whose nodes are labeled by the atoms of G , and such that every two nodes adjacent in T_G satisfy the atoms labeling them either share a single edge or a cut-point.*

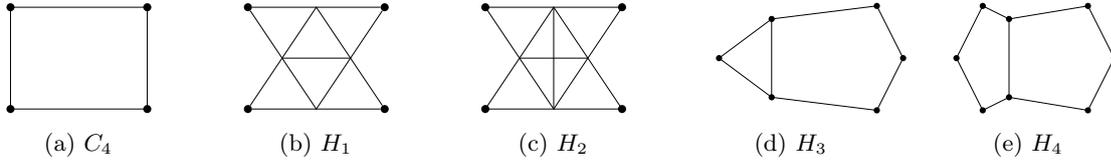


Figure 9: Characterization of 5-chordal $\frac{1}{2}$ -hyperbolic graphs, in terms of forbidden isometric subgraphs.

Note that a weak dual is nothing else than a tree-decomposition whose bags are the atoms of an outerplanar graph. If G is biconnected, then starting from an outerplanar embedding of the graph, we construct it by removing from the dual of the graph the universal vertex corresponding to the outerface (see Figure 10 for an example).

6.1 Outerplanar graphs with small hyperbolicity

As observed in Section 5, our substitution method for an exact computation of hyperbolicity requires the hyperbolicity of the graph to be at least 1. To overcome this drawback, we first characterize in this section outerplanar graphs that are $\frac{1}{2}$ -hyperbolic. Note that we only consider biconnected graphs, as the hyperbolicity of a graph is the maximum hyperbolicity taken over all its biconnected components, and the biconnected components of a graph are computable in linear-time [44].

Proposition 21. *A biconnected outerplanar graph is $\frac{1}{2}$ -hyperbolic if, and only if, either it is isomorphic to C_5 , or it is chordal and it does not contain the graph of Figure 9b as a subgraph. Furthermore, these conditions can be checked in linear-time.*

Proof. Let G be an outerplanar biconnected graph. We suppose G to be $\frac{1}{2}$ -hyperbolic. By Lemma 19, the graph C_5 is $\frac{1}{2}$ -hyperbolic. So, we will assume for the proof that G is not isomorphic to C_5 . We then remind that every induced cycle of G is isometric, as the induced cycles of G are exactly its atoms. As a result, we have by Lemma 19 that G only has induced cycles of length 3 or 5. Moreover, by [46], a 5-chordal graph is $\frac{1}{2}$ -hyperbolic if, and only if, it does not contain any graph of Figure 9 as an isometric subgraph¹.

By the hypothesis, G is C_4 -free and so, it does not contain the graph of Figure 9a as an isometric subgraph. Moreover, we claim that G is C_5 -free, as otherwise it would contain the graph of Figure 9d, or the graph of Figure 9e, as an isometric subgraph. Thus G has to be chordal. Also, we check that G cannot contain the graph of Figure 9c, as it is not outerplanar and being outerplanar is a hereditary property. Consequently, G is $\frac{1}{2}$ -hyperbolic if, and only if, it is chordal and it does not contain the graph of Figure 9b as an isometric subgraph.

Finally, recall that a graph is outerplanar if, and only if, it is K_4 -minor-free and $K_{2,3}$ -minor-free. Let H_1 be the graph of Figure 9b. We claim that every induced subgraph of G that is isomorphic to H_1 is isometric, as otherwise it would yield a $K_{2,3}$ -minor for G . In the same way, every subgraph of G that is isomorphic to H_1 is an induced subgraph of G , as otherwise it would yield a K_4 -minor for G . So, G is $\frac{1}{2}$ -hyperbolic if, and only if, it is chordal and it does not contain H_1 as a subgraph. Being chordal can be checked in linear-time [40], and a chordal outerplanar graph contains H_1 as a subgraph if, and only if, there are two adjacent vertices of degree 3 in its weak dual (see Figure 10). \square

¹The characterization of [46] is composed of six forbidden isometric subgraphs, but the sixth one is actually 6-chordal.

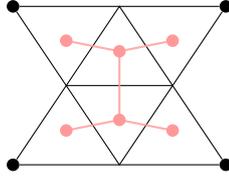


Figure 10: The forbidden subgraph of Figure 9b, and its characterization in the weak dual.

6.2 Substitute graphs of cycles

As we constrain ourselves to outerplanar graphs, recall that the atoms are exactly the induced cycles of the graph. Clearly, a clique-separator contained into a cycle is either a cut-point (in which case our substitution method never adds a simplicial vertex), or it is an edge-separator (in which case our substitution method might only add a single vertex, which has to be adjacent to both endpoints of the edge). Substitute graphs of cycles thus fall into the following subclass of outerplanar graphs:

Definition 22. *A sunshine graph is a biconnected outerplanar graph, containing a dominating induced cycle C satisfying every vertex not in C is a simplicial vertex of degree 2.*

Cycles are a subclass of such graphs, and two other examples of sunshine graphs are given in Figure 11. Moreover, we remark that an edge in a dominating cycle may be the neighborhood of at most one simplicial vertex out of the cycle, as otherwise it would yield a $K_{2,3}$ -minor for the graph. It can be noted in addition that except for the particular case of a diamond, there exists a unique dominating cycle in a sunshine graph, and every induced cycle which is not dominating is a triangle. Thus, if G is a sunshine graph and C is a dominating cycle of G , then we have by Theorem 11 that $\delta(C) \leq \delta(G) \leq \delta(C) + 1$. This difference can actually be decreased by $\frac{1}{2}$ as follows.

Lemma 23. *Let G be a sunshine graph, and C be a dominating cycle of G . Then we have:*

$$\delta(C) \leq \delta(G) \leq \delta(C) + \frac{1}{2}$$

Proof. By Theorem 11, we have $\delta(C) \leq \delta(G) \leq \delta(C) + 1$. So, our aim is to prove that no 4-tuple of G has a hyperbolicity greater than $\delta(C) + \frac{1}{2}$. By contradiction, let a, b, c, d be such that $\delta(a, b, c, d) = \delta(C) + 1$.

We arbitrarily orient the cycle C . For every $u \in \{a, b, c, d\} \setminus C$, we denote by $e_u = \{x_u, y_u\}$ the edge of C induced by its neighbours, where x_u denotes the head of the edge w.r.t. the orientation. Observe that for every $u, v \in \{a, b, c, d\} \setminus C$, we have $d(u, v) = 2 + \min\{d(x_u, y_v), d(x_v, y_u)\} = 1 + d(x_u, x_v) = 1 + d(y_u, y_v)$.

We then claim that there is exactly one vertex amongst a, b, c, d which belongs to the cycle C . Indeed, not all of a, b, c, d belong to C as the 4-tuple has a hyperbolicity greater than $\delta(C)$. Furthermore, at least three of them are not in C , as otherwise we would have $\delta(a, b, c, d) \leq \delta(C) + \frac{1}{2}$ by Lemmas 5 and 9. On the other hand, if none of them is in C , then we can check that $\delta(a, b, c, d) = \delta(x_a, x_b, x_c, x_d) \leq \delta(C)$, a contradiction. So, the claim is proved.

Finally, assume w.l.o.g. that $a \in C$. In such a case, we can check that $\delta(a, b, c, d) \leq \delta(a, x_b, x_c, x_d) + \frac{1}{2} \leq \delta(C) + \frac{1}{2}$. A contradiction. \square

Once the upper-bound for hyperbolicity is refined as above, the hyperbolicity of a sunshine graph can be computed in *linear-time*, using the following characterization.

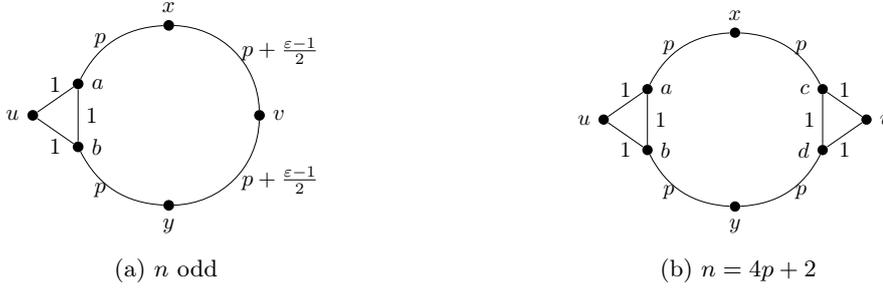


Figure 11: Substitute graphs of the atoms of an outerplanar graph.

Lemma 24. *Let G be a sunshine graph, and C be a dominating cycle for G of length $4p + \varepsilon \geq 3$, with $p \geq 0$ and $\varepsilon \in \{0, 1, 2, 3\}$. Assuming $G \setminus C$ is nonempty we have:*

- if ε is odd, then $\delta(G) = \delta(C) + \frac{1}{2}$;
- if $\varepsilon = 2$, then $\delta(G) = \delta(C) + \frac{1}{2}$ if, and only if, there exist two simplicial vertices out of C which are diametrically opposed;
- finally, if $\varepsilon = 0$, then $\delta(G) = \delta(C)$.

Proof. Recall that by the previous Lemma 24, we have $\delta(G) \leq \delta(C) + \frac{1}{2}$. Thus we only focus on finding 4-tuples u, v, x, y of hyperbolicity (at least) this value, and we choose one, if any, maximizing $|C \cap \{u, x, v, y\}|$. In the sequel, write $S_1 = d(u, v) + d(x, y)$, $S_2 = d(u, x) + d(v, y)$ and $S_3 = d(u, y) + d(v, x)$. We will assume in addition that $S_1 \geq S_2 \geq S_3$.

Case ε odd Equivalently, we have $\varepsilon \in \{1, 3\}$. In such a case, we have $\delta(C) = p + \frac{\min\{0, \varepsilon - 2\}}{2}$ by Lemma 19. Figure 11a exhibits a 4-tuple u, v, x, y satisfying:

$$\begin{aligned} S_1 &= \left(2p + \frac{\varepsilon + 1}{2}\right) + (2p + \min\{1, \varepsilon - 1\}) = 4p + \frac{\varepsilon + 1}{2} + \min\{1, \varepsilon - 1\} \\ S_2 &= (p + 1) + \left(p + \frac{\varepsilon - 1}{2}\right) = 2p + \frac{\varepsilon + 1}{2} \\ S_3 &= S_2 \end{aligned}$$

Hence, this 4-tuple has hyperbolicity $p + \frac{\min\{1, \varepsilon - 1\}}{2} = \delta(C) + \frac{1}{2}$.

Case $\varepsilon = 2$ In such a case, we have $\delta(C) = p$ by Lemma 19. We assume w.l.o.g. that $u \notin C$, and we claim that it implies $v \notin C$. Indeed, by the metric property of Lemma 8, and noticing that $S_1 \geq S_2 \geq S_3$, the vertex v has to be at equal distance l of both neighbours of u , as otherwise u could be replaced with one of its two neighbours, contradicting the maximality of $|C \cap \{u, x, v, y\}|$. Hence $v \notin C$ is impossible, as it would yield the length of C is $2l + 1 = 4p + 2$. It thus follows that $v \notin C$, and the length of C is in fact $2(l - 1) + 2 = 2l$, yielding $l = 2p + 1$.

Conversely, assume that there exist two simplicial vertices u, v that are diametrically opposed in G . We choose the 4-tuple u, x, v, y as in Figure 11b, and it satisfies:

$$\begin{aligned} S_1 &= (2p + 2) + (2p + 1) = 4p + 3 \\ S_2 &= 2(p + 1) = 2p + 2 \\ S_3 &= S_2 \end{aligned}$$

So, we have $\delta(u, v, x, y) = p + \frac{1}{2} = \delta(C) + \frac{1}{2}$.

Case $\varepsilon = 0$ Another application of Lemma 19 yields $\delta(C) = p$. Assuming $u \notin C$, we deduce as for the previous case that $v \notin C$, and v is at equal distance $l = 2p$ from both neighbours of u . Thus, C is partitioned by the neighborhoods of u and v , in the same way as in Figure 11b, into two paths of length $l - 1 = 2p - 1$. Furthermore, since the diameter of C is $2p$, those paths are geodesics of the cycle. By the proof of Lemma 23, at least one vertex $z \in \{x, y\}$ amongst the 4-tuple has to be in C so that $\delta(u, v, x, y) > \delta(C)$. As a result, by picking the geodesic containing z we get $d(u, z) + d(v, z) = 2 + (l - 1) = 2p + 1$. So, we have $\min\{d(u, z), d(v, z)\} \leq p$, contradicting the fact that $\delta(u, v, x, y) > \delta(C)$, as we have by [41] that $\delta(u, v, x, y) \leq \min\{d(u, z), d(v, z)\}$. To sum up, we always have $\delta(G) = \delta(C)$ in such a case. \square

6.3 Applying the substitution method in linear-time

Recall that our substitution method consists in constructing a substitute graph for every atom of the graph, then computing the hyperbolicity of every substitute graph, and finally taking the maximum over these hyperbolicity values and 1. In the case of an outerplanar graph, we provided in the previous section a characterization yielding a linear-time computation of the hyperbolicity of the substitute graphs of the atoms (*e.g.* Lemma 24). What now remains to prove is that the substitute graphs can also be computed in linear-time.

Lemma 25. *Let G be an outerplanar biconnected graph. The substitute graphs of the atoms of G can be computed in linear-time.*

Proof. First, we construct in linear-time an outerplanar embedding for G , then we construct from it the weak dual T_G of G . Let C_1, \dots, C_l be the atoms of G . We root T_G on an atom C_1 , which is an induced cycle. Note that $(T_G; C_1)$ naturally yields an atom tree, as defined in Definition 13: starting from a rooted tree $(T; C)$ initialized with $(T_G; C_1)$, we iterate as long as T is not empty, and at each step we disconnect all the atoms being the leaves of T sequentially. Such an ordering is enough to prove the correctness of the following algorithm.

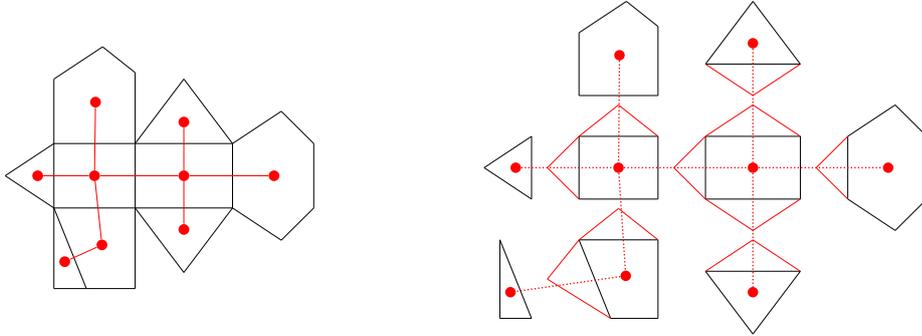


Figure 12: An application of the substitution method to an outerplanar graph.

- For every i , we initialize C_i^* with C_i .
- We start a depth-first search from the root, and for every visited atom C_i , once its subtree has been visited, we consider all its sons C_j , naming $e_{i,j}$ the edge-separator $C_i \cap C_j$. We add in C_i^* a simplicial vertex whose neighborhood is $e_{i,j}$ if, and only if, there is a vertex in C_j^* which is at equal distance to both endpoints of $e_{i,j}$. This is equivalent to have either the length of C_j is odd, or there is a simplicial vertex in $C_j^* \setminus C_j$ whose neighborhood is the edge diametrically opposed to $e_{i,j}$ in C_j .

- Finally, we start a breadth-first search from the root and for every visited atom $C_i \neq C_1$, we consider its parent atom, denoted by C_k , naming $e_{i,k}$ the edge-separator that it shares with it. As before, we add in C_i^* a simplicial vertex whose neighborhood is $e_{i,k}$ if, and only if, either the length of C_k is odd, or there is a simplicial vertex in $C_k^* \setminus C_k$ whose neighborhood is the edge diametrically opposed to $e_{i,k}$ in the atom C_k .

Note that for an atom, the depth-first search here is used to compute the simplicial vertices resulting from the disconnection of its sons, whereas the breadth-first search is used to compute the single vertex resulting from its own disconnection, if any. Using an atom tree as defined above, it can be checked that the resulting C_1^*, \dots, C_l^* are the substitute graphs of the atoms of G . \square

We finally conclude with the following theorem.

Theorem 26. *The hyperbolicity of a connected outerplanar graph is computable in linear-time.*

Proof. We can safely assume G to be biconnected by [44]. By [4, 28], G is 0-hyperbolic if, and only if, G is a clique. If it is not, then by Proposition 21, we can check whether it is $\frac{1}{2}$ -hyperbolic in linear-time.

From now on, assume $\delta(G) \geq 1$. By Lemma 25, we can compute the substitute graphs of the atoms of G in linear-time. We can thus conclude by Lemma 12 (*i.e.* the correctness of our substitution method), as these substitute graphs are sunshine graphs and their hyperbolicity is linear-time computable by Lemma 24. \square

7 Experimental evaluation

We report in this section on experiments performed with our substitution methodology on the graphs of five collaboration networks. This way, we aim to evaluate the computation time of the substitutes on some empirical graphs, and to better understand the factors impacting their size.

7.1 Datasets

We apply the algorithm presented in Section 5 to the collaboration networks of five different scientific communities [35], namely:

- **ca-AstroPh**, for the astrophysics community;
- **ca-CondMat**, for the condensed matter physics community;
- **ca-GrQc**, for the general relativity and quantum cosmology community;
- **ca-HepPh**, for the high energy physics-phenomenology community;
- and **ca-HepTh**, for the high energy physics-theory community.

In the **ca-*** graphs, nodes represent scientists and edges represent collaborations (*i.e.*, co-authoring a paper). These graphs are interesting to analyze the behavior of our algorithm and the size of their substitutes graphs because they have many cliques of various sizes. Indeed, a paper co-authored by k scientists induces a clique of size k in the graph. Furthermore, the number of co-authors per papers varies from one community to another. Therefore, we will observe different results in terms of the size of their substitute graphs, despite these graphs share many properties (see [35]).

7.2 Empirical results

We modified the clique-decomposition algorithm of [7] to implement the substitution method presented in Section 5. We used it to compute, for each graph, the substitute of each atom of the decomposition.

Decomposition into biconnected components We observed that all of the five graphs are composed of one largest biconnected component, that we call LBC. The component includes from 50% to 84,85% of all the vertices. This can be observed from the cumulative distribution of the size of the biconnected components in Fig 13a. The cumulative number of components is given as a percentage of the total number of biconnected components, and the size of the components as a percentage of the total number of vertices in the graph. We noticed that all the biconnected components but the LBC are small: only covering at most 1% of the vertices.

Clearly, the smallest biconnected components can be safely ignored for the computation of hyperbolicity, provided that a) their number of vertices is less than 4, or that b) their diameter is less than two times the hyperbolicity of the LBC, which is always the case for these graphs (see [14]). Thus, we now focus on the clique-decomposition of the LBC, and on its resulting substitute graphs.

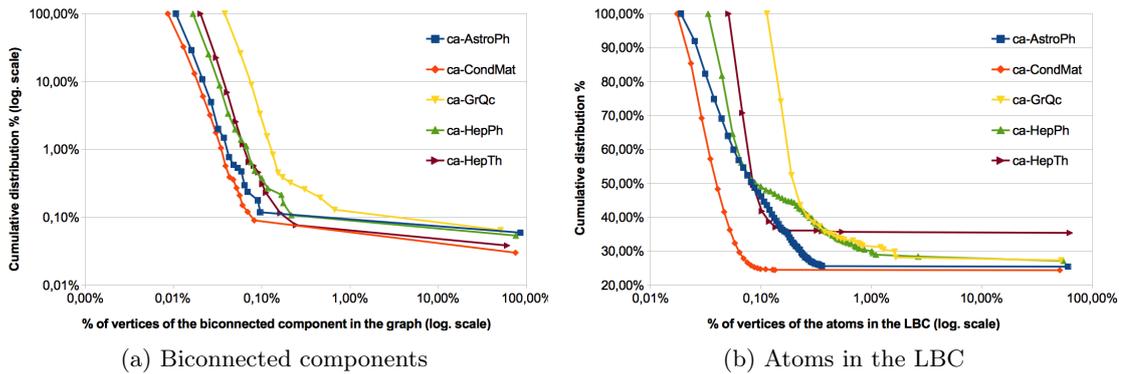


Figure 13: Cumulative distributions of the size of the biconnected components (Fig. 13a) and of the atoms in the LBC of each graph (Fig. 13b).

Clique-decomposition We plotted in Fig 13b the cumulative distribution of the size of the atoms of the LBC. The cumulative number of atoms is given as a percentage of the total number of atoms and the size of the atoms as a percentage of the total number of vertices in the LBC. Again, for all of the graphs, we observed one largest atom, that we call the LA. This atom includes from 50% to 60% of all the vertices, and all the other atoms only represent a small fraction of the overall vertices. In the worst case (ca-HepPh), all the atoms but the LA solely cover 2,65% of the vertices of the graph.

Moreover, like for the smallest biconnected components and as reported in [14], the substitute graphs of the smallest atoms can be safely ignored for the computation of hyperbolicity. As a result, the only component of the graphs to deal with for computing their hyperbolicity is the substitute graph of the LA. We will denote it LS in the sequel.

Size of the substitute graphs As explained in Section 5, the size of the LS depends on both the initial size of the LA and the number of added simplicial vertices. We have reported in Table 1 the original size n of each graph, the size n_B of its LBC, the size n_{LA} of the LA, and the size n_{LS} of the largest substitute. We have then computed the percentage R_{LA} of vertices that have been removed from the LBC to obtain the LA, that is $R_{LA} = \frac{n_B - n_{LA}}{n_B}$. We observe a significant reduction rate R_{LA} , varying from 37,40% to 49,22%. We have also computed the reduction rate R_{LS} of the LS with respect to the LBC, that is $R_{LS} = \frac{n_B - n_{LS}}{n_B}$. We observe that this reduction rate falls between 3,79% and 21,95%. This indicates that the substitution method adds many simplicial vertices to the LA when constructing the LS, despite the simplification rules presented in Section 5.2.2.

Instance name	n	n_B	n_{LA}	n_{LS}	R_{LA}	R_{LS}	Cost	Gain	Time (in sec.)
ca-CondMat	23 133	17 234	8 751	13 451	49,22%	21,95%	27,27%	-	672
ca-GrQc	5 242	2 651	1 386	2 343	47,72%	11,62%	36,10%	10,33%	5
ca-HepPh	12 008	9 025	4 925	8 287	45,43%	8,18%	37,25%	13,77%	167
ca-AstroPh	18 772	15 929	9 561	15 325	39,98%	3,79%	36,19%	18,16%	679
ca-HepTh	9 877	5 898	3 692	5 270	37,40%	10,65%	26,75%	11,30%	53

Table 1

We reported in Table 1 as *Cost* the percentage of vertices in the LBC representing the addition of new simplicial vertices. For the graph **ca-CondMat** - with the largest reduction rate R_{LS} -, the addition of simplicial vertices represents 27,27% of the size of its LBC, whereas for the **ca-AstroPh** graph - with the lowest reduction rate -, it goes up to 36,19%. This shows a difference of 18,16% between the R_{LS} of both extremal cases. The difference comes on the one hand, from the addition of 8,91% more simplicial vertices in the **ca-AstroPh** graph, and on the other hand from a difference of 9,25% between the reduction rates R_{LA} of their respective biggest atom LA. In the same way, when comparing the graphs **ca-HepTh**, **ca-HepPh** and **ca-GrQc**, respectively, with the graph **ca-CondMat**, we computed for R_{LA} a difference of 11,82%, 3,79% and 1,5%, respectively, plus a difference of 0,52%, 9,98% and 8,83%, respectively, for the addition of simplicial vertices. We thus concluded that the impact of n_{LA} and of the number of new simplicial vertices on the final size n_{LS} differs greatly depending on the graph.

7.3 Decomposition analysis

Having noticed the heterogeneous results of our empirical section, we are now analyzing in more details the properties causing the asymmetry between the various **ca-*** graphs.

Clique-decomposition We first analyzed the composition of the LA in terms of clique-separators. Let us denote by $\mathcal{X}_{LA} = \{X_1, \dots, X_l\}$ the clique-minimal separators that are contained into the LA, disconnecting the atoms $\mathcal{A}_{LA} = \{A_1, \dots, A_l\}$ from the LA, sequentially. We plotted in Fig 14a the cumulative distribution of the size of the clique-separators in the LA as a percentage of the total number of clique-separators. By doing so, we observed smaller clique-separators for the **ca-HepTh** and **ca-CondMat** graphs, with respectively a maximum size of 8 and 21, than for the three others graphs **ca-GrQc**, **ca-AstroPh** and **ca-HepPh**, having respectively clique-separators of maximum size 42, 53 and 192. Also, we reported in Table 2 that the ratio $R_{|\mathcal{X}_{LA}|}$ of the number of clique-minimal separators $|\mathcal{X}_{LA}|$ over the size n_{LA} of the LA, varies from 0.39 for **ca-AstroPh** to 0.54 for **ca-CondMat**. To sum up, there are more clique-separators in

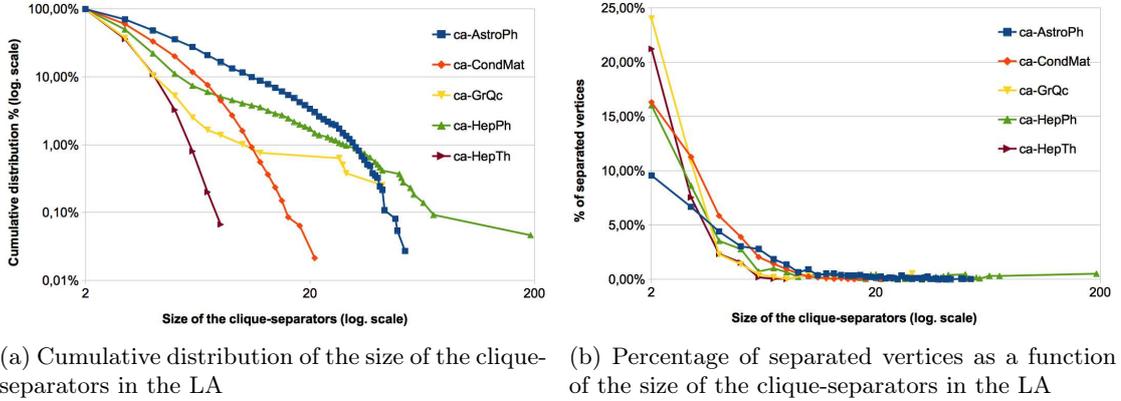


Figure 14

ca-CondMat than in ca-AstroPh, but there are larger clique-separators in ca-AstroPh than in ca-CondMat.

To complete our measurements, we related the size of clique-separators with the proportion of vertices that are disconnected by them from the LA. We reported in Table 2 as $\alpha_1 = n_B - n_{LA}$ the total number of vertices separated from the LA in the LBC, as $\alpha_2 = |V(\mathcal{A}_{LA} \setminus \mathcal{X}_{LA})|$ the number of vertices in the subset of atoms \mathcal{A}_{LA} which are not in the LA, and as $|\mathcal{X}_{LA}|$ the number of clique-minimal separators in the LA. Finally, we computed the fraction $\Delta_1 = \frac{\alpha_1 - \alpha_2}{n_B}$, quantifying the percentage of vertices that are neither contained into the LA, nor in any of the atoms in \mathcal{A}_{LA} . We reported as $\Delta_2 = R_{LA} - \Delta_1$ the fraction of vertices in some atom of \mathcal{A}_{LA} , hence those that are *directly* separated from the LA.

Our results put in evidence that most of the vertices are either contained into the LA, or into some other atom intersecting the LA. Other vertices comprise around 2,88% and 7,03% of the overall vertices. Moreover, as shown with Fig 14b, where we plotted the percentage of separated vertices as a function of the size of clique-minimal separators, smaller clique-separators of size ≤ 5 are responsible for a significant part (w.r.t. Δ_2) of the vertices disconnected from the LA in ca-CondMat (37,34% of vertices over 49,22%), whereas in ca-AstroPh they solely disconnect 23,67% over 39,98% of vertices. This difference is not balanced with clique-separators of larger size, even though these ones disconnect 13,43% of vertices in ca-AstroPh, while only 5,67% in ca-CondMat. A comparison of ca-CondMat with ca-HepPh yielded similar results. In contrast, for the graphs ca-GrQc and ca-HepTh, we noticed that 6,71% and 4,91% more vertices, respectively, than in CondMat, were disconnected by edge-separators. But the rest of the clique-minimal separators, of larger size, only disconnect 16,67% and 11,70% of the vertices, respectively, whereas 26,70% of them are separated in ca-CondMat. Therefore, most of the difference for the final size of the substitute graph LS comes from the number of vertices that are disconnected by clique-minimal separators of *small* size.

Substitute construction As illustrated with Figure 15, the number of additional vertices, resulting from the construction of the substitute graphs, depends on both the number and the size of the clique-minimal separators contained into the LA.

Instance name	α_1	α_2	$ \mathcal{X}_{LA} $	$R_{\mathcal{X}_{LA}}$	$\Delta_1\%$	$\Delta_2\%$
ca-CondMat	8 483	7 413	4 698	0,54	6,21%	43,01%
ca-GrQc	1 265	1 079	697	0,5	7,03%	40,69%
ca-HepPh	4 100	3 727	2 165	0,44	4,13%	41,3%
ca-AstroPh	6 368	5 910	3 709	0,39	2,88%	37,1%
ca-HepTh	2 206	1 942	1 503	0,41	4,47%	32,93%

Table 2

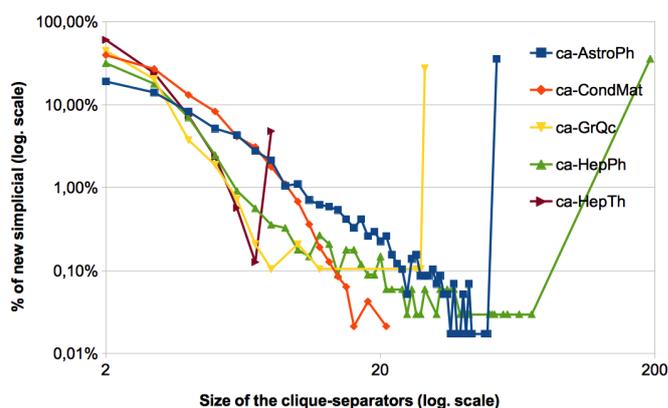


Figure 15: Number of added simplicial vertices

8 Conclusion

In this work, we proved a tight relation between the hyperbolicity of a graph and the maximum hyperbolicity taken over all its atoms. Our result subsumes the one in [11], as it directly implies that chordal graphs are 1-hyperbolic. Also, it implies that two graph classes extending chordal graphs, namely 2-chordal graphs [36, 37] and clique-separable graphs [25], have a bounded hyperbolicity.

In addition of our main result, we deduced from its proof a general substitution method, allowing us to modify the atoms at no extra-cost than the clique-decomposition; for graphs with hyperbolicity at least one, the maximum hyperbolicity taken over all the resulting graphs is exactly the hyperbolicity of the graphs, but the graphs to be considered may have a larger size than the atoms. Experiments suggest that the final size of the substitute graphs is mostly related with the number of clique-minimal separators of small size, and the disconnections resulting from them. We successfully applied this substitution method to outerplanar graphs, providing in this case a linear-time algorithm for computing the hyperbolicity. We let open whether the same can be done for other classes of graphs.

Finally, in the spirit of [2], it would be of interest to take advantage of the linear-time algorithm for outerplanar graphs, in order to yield an efficient computation of the hyperbolicity of planar graphs. Part of our future work will also consist in finding other graph decompositions which are applicable to the computation of this parameter.

References

- [1] M. Abu-Ata and F. F. Dragan. Metric tree-like structures in real-life networks: an empirical study. Technical Report arXiv:1402.3364, ArXiv, 2014.
- [2] B. Baker. Approximation algorithms for NP-complete problems on planar graphs. *Journal of the ACM (JACM)*, 41(1):153–180, 1994.
- [3] H.-J. Bandelt and V. Chepoi. Metric graph theory and geometry: a survey. *Contemporary Mathematics*, 453:49–86, 2008.
- [4] H.-J. Bandelt and H. Mulder. Distance-hereditary graphs. *Journal of Combinatorial Theory, Series B*, 41(2):182–208, 1986.
- [5] S. Bermudo, J. Rodríguez, J. Sigarreta, and J.-M. Vilaire. Gromov hyperbolic graphs. *Discrete Mathematics*, 313(15):1575–1585, 2013.
- [6] A. Berry, R. Pogorelcnik, and G. Simonet. Efficient clique decomposition of a graph into its atom graph. Technical Report RR-10-07, LIMOS, Aubière, France, Mar. 2010.
- [7] A. Berry, R. Pogorelcnik, and G. Simonet. An introduction to clique minimal separator decomposition. *Algorithms*, 3(2):197–215, 2010.
- [8] H. L. Bodlaender. Discovering treewidth. In *31st Conference on Current Trends in Theory and Practice of Computer Science (SOFSEM)*, volume 3381 of *Lecture Notes in Computer Science*, pages 1–16. Springer, 2005.
- [9] M. Boguñá, F. Papadopoulos, and D. V. Krioukov. Sustaining the Internet with hyperbolic mapping. *Nature Communications*, 1(62):1–18, Oct. 2010.
- [10] J. A. Bondy and U. Murty. *Graph theory with applications*, volume 290. Macmillan London, 1976.
- [11] G. Brinkmann, J. H. Koolen, and V. Moulton. On the hyperbolicity of chordal graphs. *Annals of Combinatorics*, 5(1):61–69, 2001.
- [12] J. Chakerian and S. Holmes. Computational tools for evaluating phylogenetic and hierarchical clustering trees. *Journal of Computational and Graphical Statistics*, 21(3):581–599, 2012.
- [13] V. Chepoi, F. F. Dragan, B. Estellon, M. Habib, and Y. Vaxès. Notes on diameters, centers, and approximating trees of δ -hyperbolic geodesic spaces and graphs. *Electronic Notes in Discrete Mathematics*, 31:231–234, 2008.
- [14] N. Cohen, D. Coudert, and A. Lancin. Exact and approximate algorithms for computing the hyperbolicity of large-scale graphs. Rapport de recherche RR-8074, Inria, Sept. 2012.
- [15] D. Coudert and G. Ducoffe. On the recognition of C_4 -free and $1/2$ -hyperbolic graphs. Research Report RR-8458, INRIA, Jan. 2014.
- [16] W. H. Cunningham. Decomposition of directed graphs. *SIAM Journal on Algebraic Discrete Methods*, 3(2):214–228, 1982.
- [17] P. de La Harpe and E. Ghys. *Sur les groupes hyperboliques d’après Mikhael Gromov*, volume 83. Progress in Mathematics, 1990.

-
- [18] M. Didi Biha, B. Kaba, M.-J. Meurs, and E. SanJuan. Graph decomposition approaches for terminology graphs. In *MICAI 2007: Advances in Artificial Intelligence*, volume 4827 of *Lecture Notes in Computer Science*, pages 883–893. Springer, 2007.
- [19] R. Diestel. *Graph theory, graduate texts in mathematics, vol. 173*. Springer, Heidelberg, 1997.
- [20] Y. Dourisboure and C. Gavaille. Tree-decompositions with bags of small diameter. *Discrete Mathematics*, 307(16):2008–2029, 2007.
- [21] F. Dragan. Tree-like structures in graphs: A metric point of view. In *39th International Workshop on Graph-Theoretic Concepts in Computer Science (WG)*, volume 8165 of *Lecture Notes in Computer Science*, pages 1–4. Springer, 2013.
- [22] A. Dress, K. Huber, J. Koolen, V. Moulton, and A. Spillner. *Basic Phylogenetic Combinatorics*. Cambridge University Press, Cambridge, UK, Dec. 2011.
- [23] H. Fournier, A. Ismail, and A. Vigneron. Computing the gromov hyperbolicity of a discrete metric space. Technical Report arXiv:1210.3323, ArXiv, Oct. 2012.
- [24] T. Gallai. Transitiv orientierbare graphen. *Acta Mathematica Hungarica*, 18(1):25–66, 1967.
- [25] F. Gavril. Algorithms on clique separable graphs. *Discrete Mathematics*, 19(2):159–165, 1977.
- [26] M. Gromov. Hyperbolic groups. In S. Gersten, editor, *Essays in Group Theory*, volume 8 of *Mathematical Sciences Research Institute Publications*, pages 75–263. Springer, New York, 1987.
- [27] M. Habib, C. Paul, and L. Viennot. Partition refinement techniques: An interesting algorithmic tool kit. *International Journal of Foundations of Computer Science*, 10(02):147–170, 1999.
- [28] E. Howorka. On metric properties of certain clique graphs. *Journal of Combinatorial Theory, Series B*, 27(1):67–74, 1979.
- [29] E. A. Jonckheere and P. Lohsoonthorn. Geometry of network security. In *American Control Conference*, volume 2, pages 976–981, Boston, MA, USA, 2004. IEEE.
- [30] B. Kaba, N. Pinet, G. Lelandais, A. Sigayret, and A. Berry. Clustering gene expression data using graph separators. *In silico biology*, 7(4):433–452, 2007.
- [31] W. S. Kennedy, O. Narayan, and I. Saniee. On the hyperbolicity of large-scale networks. Technical Report arXiv:1307.0031, ArXiv, 2013.
- [32] J. H. Koolen and V. Moulton. Hyperbolic bridged graphs. *European Journal of Combinatorics*, 23(6):683–699, 2002.
- [33] F. Le Gall. Faster algorithms for rectangular matrix multiplication. In *IEEE 53rd Annual Symposium on Foundations of Computer Science*, pages 514–523, New Brunswick, NJ, USA, 2012. IEEE.
- [34] H.-G. Leimer. Optimal decomposition by clique separators. *Discrete Mathematics*, 113(1):99–123, 1993.

-
- [35] J. Leskovec, J. Kleinberg, and C. Faloutsos. Graph evolution: Densification and shrinking diameters. *ACM Transactions on Knowledge Discovery from Data*, 1(1):1–41, Mar. 2007.
- [36] S. McCullough. 2-chordal graphs. In *Contributions to Operator Theory and its Applications*, pages 143–192. Springer, 1988.
- [37] S. McCullough. Minimal separators of 2-chordal graphs. *Linear algebra and its applications*, 184:187–199, 1993.
- [38] K. Olesen and A. Madsen. Maximal prime subgraph decomposition of bayesian networks. *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, 32(1):21–31, 2002.
- [39] R. Paige and R. Tarjan. Three partition refinement algorithms. *SIAM Journal on Computing*, 16(6):973–989, 1987.
- [40] D. Rose, R. Tarjan, and G. Lueker. Algorithmic aspects of vertex elimination on graphs. *SIAM Journal on computing*, 5(2):266–283, 1976.
- [41] M. A. Soto Gómez. *Quelques propriétés topologiques des graphes et applications à internet et aux réseaux*. PhD thesis, Univ. Paris Diderot (Paris 7), 2011.
- [42] M. Sysło. Characterizations of outerplanar graphs. *Discrete Mathematics*, 26(1):47–53, 1979.
- [43] R. Tamassia and I. Tollis. Planar grid embedding in linear time. *IEEE Transactions on Circuits and Systems*, 36(9):1230–1234, 1989.
- [44] R. E. Tarjan. Depth-first search and linear graph algorithms. *SIAM Journal on Computing*, 1(2):146–160, 1972.
- [45] R. E. Tarjan. Decomposition by clique separators. *Discrete Mathematics*, 55(2):221–232, 1985.
- [46] Y. Wu and C. Zhang. Hyperbolicity and chordality of a graph. *The Electronic Journal of Combinatorics*, 18(1):P43, 2011.



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