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# A construction of small regular bipartite graphs of girth 8

Camino Balbuena<sup>1†</sup>

<sup>1</sup>Departament de Matemàtica Aplicada III, Universitat Politècnica de Catalunya, Campus Nord, Jordi Girona 1 i 3, 08034 Barcelona, Spain

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Let  $q$  be a prime a power and  $k$  an integer such that  $3 \leq k \leq q$ . In this paper we present a method using Latin squares to construct adjacency matrices of  $k$ -regular bipartite graphs of girth 8 on  $2(kq^2 - q)$  vertices. Some of these graphs have the smallest number of vertices among the known regular graphs with girth 8.

**Keywords:** Incidence matrices, Latin squares, cages, girth.

## 1 Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow the books by Godsil and Royle [16] and by Lint and Wilson [21] for terminology and definitions.

Let  $G = (V(G), E(G))$  be a graph with vertex set  $V = V(G)$  and edge set  $E = E(G)$ . The *girth* of a graph  $G$  is the number  $g = g(G)$  of edges in a smallest cycle. The *degree* of a vertex  $v \in V$  is the number of vertices adjacent to  $v$ . A graph is called *regular* if all the vertices have the same degree. A *cage* is a  $k$ -regular graph with girth  $g$  having the smallest possible number of vertices. Simply counting the numbers of vertices in the distance partition with respect to a vertex yields a lower bound  $n_0(k, g)$  on the number of vertices  $n(k, g)$  in a cage, with the precise form of the bound depending on whether  $g$  is even or odd.

$$n_0(k, g) = \begin{cases} 1 + k + k(k-1) + \dots + k(k-1)^{(g-3)/2} & \text{if } g \text{ is odd;} \\ 2(1 + (k-1) + \dots + (k-1)^{g/2-1}) & \text{if } g \text{ is even.} \end{cases} \quad (1)$$

As defined by Biggs [7], the *excess* of a  $k$ -regular graph  $G$  is the difference  $|V(G)| - n_0(k, g)$ . A  $(k, g)$ -cage with even girth  $g$  and  $n_0(k, g)$  vertices is said to be a *generalized polygon graph*. Generalized polygon graphs exist if and only if  $g \in \{4, 6, 8, 12\}$  [7]. The question of the construction of graphs with small excess is a difficult one. When  $g = 6$ , the existence of a graph with  $n_0(k, 6) = 2(k^2 - k + 1)$

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vertices called *generalized triangle*, is equivalent to the existence of a projective plane of order  $k - 1$ , that is, a symmetric  $((n_0(k, 6))/2, k, 1)$ -design. It is known that these designs exist whenever  $k - 1$  is a prime power, but the existence question for other values remains unsettled. *Generalized quadrangles* when  $g = 8$ , and *generalized hexagons* when  $g = 12$  are also known to exist for all prime power values of  $k - 1$  [5, 16, 21].

Cages have been studied intensely since they were introduced by Tutte [27] in 1947. Erdős and Sachs [10] proved the existence of a graph for any value of the regularity  $k$  and the girth  $g$ , thus most of work carried out has been focused on constructing a smallest such graph [1, 2, 4, 6, 9, 11, 13, 15, 22, 23, 24, 28, 29]. Biggs is the author of an impressive report on distinct methods for constructing cubic cages [8]. For some time, Royle [26] kept a web-site in which all the cages known so far appear. More details about constructions on cages can be found in the survey by Wong [29], the survey by Holton and Sheehan [17] or on the more recent dynamic cage survey by Exoo and Jajcay [12].

It is conjectured that cages with even girth are bipartite [25, 29]. A graph is *bipartite* if its vertex set  $V$  can be partitioned into two partite sets,  $V_1$  and  $V_2$ , such that any edge has one end in  $V_1$  and the other in  $V_2$ . If the vertices are ordered in such a way that the vertices of  $V_1$  come first, then the adjacency matrix of a bipartite graph can be written in the form

$$A = \begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix}. \quad (2)$$

An *incidence graph* is a bipartite graph in which the elements of one part  $V_1$  are declared as *lines* and the elements of the other part  $V_2$  are declared as *points*. The terminology for incidence graphs is geometric. A point and a line are said to be *incident* if they are adjacent, thus the submatrix  $N$  of (2) is called an *incidence matrix* of the bipartite graph. If the number of points and the number of lines coincide, then  $N$  is clearly a square matrix. An incidence matrix  $N$  defines a *partial plane* when

- any line has at least two points, and
- two points are incident with at most one line.

Consequently, two lines of a partial plane have at most one point in common. The corresponding bipartite graph is called the *incidence graph of the partial plane*, which clearly has even girth  $g \geq 6$ . Thus for simplicity we shall say that *the partial plane has girth  $g/2$*  if and only if the corresponding incidence graph has girth  $g$ .

Let  $q$  be a prime power and  $k$  an integer such that  $3 \leq k \leq q$ . In [3], incidence matrices of  $(k, 6)$ -bipartite graphs of order  $2(kq - 1)$  were given. In this paper we present a method to construct the incidence matrices of  $k$ -regular bipartite graphs of girth  $g = 8$  on  $2q(kq - 1)$  vertices.

## 2 Position matrices

Let  $S$  denote a set of symbols and let  $A$  be a matrix whose elements are subsets of  $S$ . Given  $x \in S$  let  $P_x(A)$  be a  $(0, 1)$ -matrix of the same dimension as  $A$  that satisfies

$$(P_x(A))_{ij} = 1 \text{ if and only if } x \in A_{ij}.$$

Thus,  $P_x(A)$  is called the *position matrix of the symbol  $x$*  in  $A$ . Suppose that  $S = \{0, x_1, \dots, x_n\}$ . The position matrices of all the symbols in  $A$  different from 0 give rise to the following  $(0, 1)$ -matrix  $\mathcal{P}(A)$

called *position matrix* of  $A$ :

$$[\mathcal{P}(A)] = [P_{x_1}(A) \cdots P_{x_n}(A)].$$

Let  $\{A^1, A^2, \dots, A^r\}$  be a family of matrices of the same number of columns whose elements are subsets of  $S$ . Then the  $(0, 1)$ -matrix spanned by the position matrices of all of them

$$\begin{pmatrix} \mathcal{P}(A^1) \\ \mathcal{P}(A^2) \\ \vdots \\ \mathcal{P}(A^r) \end{pmatrix} = \begin{pmatrix} P_{x_1}(A^1) & \cdots & P_{x_n}(A^1) \\ P_{x_1}(A^2) & \cdots & P_{x_n}(A^2) \\ \vdots & & \vdots \\ P_{x_1}(A^r) & \cdots & P_{x_n}(A^r) \end{pmatrix}, \quad (3)$$

is said to be *the position matrix of the family*  $\mathcal{F} = \{A^1, A^2, \dots, A^r\}$ . The following example shows two matrices of order  $2 \times 2$  whose elements are subsets of  $S = \{0, a, b\}$  and the position matrix of them. From now on, if there is no confusion the 1-sets will be indicated as integers.

	MATRICES		SYMBOLS	
			$a$	$b$
$A^1$	$a$	$a$	$\mathbf{1}$	$\mathbf{1}$
	$b$	$b$	$\mathbf{0}$	$\mathbf{0}$
$A^2$	$\{a, b\}$	$0$	$\mathbf{1}$	$\mathbf{0}$
	$0$	$\{a, b\}$	$\mathbf{0}$	$\mathbf{1}$

(4)

As already mentioned in the Introduction, our main aim is to obtain incidence matrices of bipartite  $k$ -regular graphs of girth 8 with small excess. Such incidence matrices may be seen also as incidence matrices of partial planes which will be obtained by identifying row  $i$  of  $P_z(A^\alpha)$  as *line*  $i(\alpha)$ , and column  $j$  of  $P_z(A^\alpha)$  as *point*  $j(z)$ , for any matrix  $A^\alpha \in \mathcal{F}$  and  $z \in S - 0$ . To achieve our goal we propose the following definitions.

**Definition 2.1** *Let  $g \geq 4$  be an even number. A family of matrices  $\mathcal{F} = \{A^1, A^2, \dots, A^r\}$  of the same number of columns whose elements are subsets of a set of symbols  $S$  is said to have girth  $g$  if the position matrix of  $\mathcal{F}$  is the incidence matrix of a bipartite graph of girth  $g$ .*

Each matrix of (4) has girth  $g = \infty$ , and the two matrices  $A^1, A^2$  form a family of girth 8.

Let us recall that a *Latin square* of order  $q$  is a  $q \times q$  matrix with entries from a set of  $q$  symbols such that each symbol occurs exactly once in each row and exactly once in each column. A Latin square has clearly girth  $g = \infty$  because the position matrices of its elements are permutation matrices yielding the incidence matrix of a partial plane consisting in a set of parallel lines (since they have no common point).

In [3] we introduced the notion of quasi row-disjoint matrices as follows.

**Definition 2.2** [3] *Let  $A^1$  and  $A^2$  be two matrices of the same number of columns whose elements are subsets of a set of symbols  $S$  such that  $0 \in S$ . A pair  $(x, y)$  with  $x, y \in S$  belongs to the cartesian product of any two rows  $(A^1)_i \times (A^2)_h$  if and only if  $(x, y) \in (A^1)_{ij} \times (A^2)_{hj}$  for some  $j$ . Then  $A^1$  and  $A^2$  are said to be quasi row-disjoint if and only if the cartesian product of any two rows  $(A^1)_i, (A^2)_h$  contains at most one pair  $(x, x) \in (A^1)_i \times (A^2)_h$  with  $x \neq 0$ .*

The pair of matrices of the example (4) are quasi row-disjoint matrices. Moreover, in [3] we have stated the following theorem which roughly speaking says that a family of  $r$  matrices is quasi row-disjoint if and only if its girth is at least 6.

**Theorem 2.1** [3] *Let  $A^1$  and  $A^2$  be two matrices each one of girth at least 6 of the same number of columns and whose elements are subsets of a set of symbols  $S$  such that  $0 \in S$ . Then  $A^1$  and  $A^2$  are quasi row-disjoint if and only if the family  $\{A^1, A^2\}$  has girth at least 6.*

In the next theorem we give a sufficient condition for a family of matrices to have girth at least 8.

**Theorem 2.2** *Let  $\mathcal{F} = \{A^1, \dots, A^r\}$  be a set of  $r \geq 2$  quasi row-disjoint matrices of the same number of columns whose elements are subsets of a set of symbols  $S$  such that  $0 \in S$ . Let  $(A^u)_i, (A^v)_{i'}, (A^w)_{i''}$  denote any three mutually distinct rows of matrices  $A^u, A^v, A^w \in \mathcal{F}$ . Then the girth of  $\mathcal{F}$  is at least 8 if the sets  $(A^u)_i \times (A^v)_{i'}, (A^v)_{i'} \times (A^w)_{i''}, (A^u)_i \times (A^w)_{i''}$  contains at most two distinct pairs  $(x, x), (y, y)$  with  $x, y \neq 0, x, y \in S$ .*

**Proof:** Suppose  $(x, x) \in (A^u)_i \times (A^v)_{i'}, x \neq 0$ . Therefore the position matrix of  $\mathcal{F}$  has the following entries equal to 1:

$$P_x(A^u)(i, j) = P_x(A^v)(i', j) = 1, \quad (5)$$

where  $P_x(A^u)$  and  $P_x(A^v)$  are the position matrix of the element  $x$  in  $A^u$  and  $A^v$  respectively. Recall that for any given matrix  $A^\alpha \in \mathcal{F}$  and  $z \in S - 0$ , row  $i$  of  $P_z(A^\alpha)$  is line  $i(\alpha)$ , and column  $j$  of  $P_z(A^\alpha)$  is point  $j(z)$ . Consequently, (5) means that lines  $i(u)$  and  $i'(v)$  have point  $j(x)$  as a common point.

Analogously,  $(y, y) \in (A^v)_{i'} \times (A^w)_{i''}, y \neq 0$ , is equivalent to:

$$P_y(A^v)(i', j') = P_y(A^w)(i'', j') = 1,$$

or in other words, lines  $i'(v)$  and  $i''(w)$  have the point  $j'(y)$  in common, with  $j'(y) \neq j(x)$  because  $y \neq x$ . Thus if there exists  $z \neq 0, z \neq x, y$ , such that  $(z, z) \in (A^u)_i \times (A^w)_{i''}$ , then lines  $i(u)$  and  $i''(w)$  have the point  $j''(z)$  in common,  $j'' \neq j, j'$ , yielding that the partial plane defined by the position matrix of  $\mathcal{F}$  contains the triangle  $j(x)j'(y)j''(z)$ . In other words, the position matrix of  $\mathcal{F}$  is the incidence matrix of a bipartite graph of girth less than 8.  $\square$

Our immediate goal is to derive a method for constructing a family of matrices with girth 8, because the position matrix of this family will be the incidence matrix of a bipartite graph of girth 8.

### 3 Method

Throughout this work let  $[[n]]$  denote the set of non negative integers  $\{0, 1, \dots, n\}$  and  $(n) = [[n]] \setminus \{0\}$ . Let  $I_n$  be the identity matrix and denote by  $(t \times F)I_n$  the matrix obtained from  $I_n$  by replacing each one with a subset  $\{t\} \times F$  of  $\{t\} \times [[n]]$  for some  $t \in [[n]]$  and  $F \subseteq [[n]]$ . In the following theorem we demonstrate a method for obtaining a family of matrices with girth 8 using Latin squares.

**Theorem 3.1** *Let  $q$  be a prime power, and let  $\mathbb{F}_q$  be the Galois field of order  $q$ . For each  $u, t \in \mathbb{F}_q$ , define the  $q \times q$  matrix  $L^{u,t}$  by*

$$L^{u,t}(i, j) = i + uj + ut, \quad i, j \in \mathbb{F}_q.$$

*Then the following assertions hold:*

(i) For all  $u, v \in \mathbb{F}_q \setminus \{0\}$ ,  $u \neq v$ , and  $t, t' \in \mathbb{F}_q$  the matrices  $L^{u,t}$  and  $L^{v,t'}$  are quasi row-disjoint Latin squares with entries from  $[[q-1]]$ .

(ii) For any given  $u, t \in \mathbb{F}_q$ ,  $u \neq 0$ , the matrix  $L^{u,t} \times L^{u+u^{-1},t}$  has  $q^2$  distinct entries. Moreover, the position matrix of the family

$$\{L^{u,t} \times L^{u+u^{-1},t} : t \in \mathbb{F}_q\}$$

is a  $(0,1)$ -matrix of order  $q^2 \times q^3$  considering symbol  $(0,0)$  different from 0, and it is the incidence matrix of a partial plane consisting in  $q^2$  parallel lines with  $q$  points on each.

(iii) The family  $\{L^{u,t} \times L^{u+u^{-1},t} : u, t \in \mathbb{F}_q, u \neq 0\}$  has girth 8 and its position matrix has order  $(q^3 - q^2) \times q^3$  and  $q$  ones in each row and  $q-1$  ones in each column.

(iv) The position matrix of the family

$$\{L^{u,t} \times L^{u+u^{-1},t} : u, t \in \mathbb{F}_q, u \neq 0\} \cup \{(t \times [[q-1]])I_q : t \in \mathbb{F}_q\}$$

is the incidence matrix of a  $q$ -regular bipartite graph of girth 8 with  $q^3$  vertices in each partite set.

**Proof:** (i) Clearly  $L^{u,t}$  and  $L^{v,t'}$  are  $q \times q$  Latin squares on  $[[q-1]]$ . Let us show that they are quasi row-disjoint. Otherwise, there exists  $i, i', j, j' \in \mathbb{F}_q$  with  $j \neq j'$  such that

$$\begin{aligned} L^{u,t}(i, j) &= L^{v,t'}(i', j), \\ L^{u,t}(i, j') &= L^{v,t'}(i', j'). \end{aligned}$$

Equivalently,

$$\begin{aligned} i + uj + ut &= i' + vj + vt', \\ i + uj' + ut &= i' + vj' + vt'. \end{aligned}$$

Therefore  $u(j - j') = v(j - j')$  implying that  $u = v$  or  $j = j'$ , a contradiction in either case.

(ii) Note that  $L^{u+u^{-1},t}$  is also a Latin square if  $u + u^{-1} \neq 0$ , otherwise  $L^{0,t}(i, j) = i$ . In either case it is very easy to check that  $L^{u,t}$  and  $L^{u+u^{-1},t}$  are orthogonal, which implies that  $L^{u,t} \times L^{u+u^{-1},t}$  has  $q^2$  distinct entries. Hence, the position matrix of the family  $\{L^{u,t} \times L^{u+u^{-1},t} : t \in \mathbb{F}_q\}$  is a  $(0,1)$ -matrix of order  $q^2 \times q^3$  by considering entry  $(0,0)$  different from 0.

Let us show now that for every  $i, i', j, t, t' \in \mathbb{F}_q$  such that  $t \neq t'$  we have

$$(L^{u,t} \times L^{u+u^{-1},t})(i, j) \neq (L^{u,t'} \times L^{u+u^{-1},t'})(i', j).$$

Otherwise we would have

$$\begin{aligned} i + uj + ut &= i' + uj + ut', \\ i + (u + u^{-1})j + (u + u^{-1})t &= i' + (u + u^{-1})j + (u + u^{-1})t', \end{aligned}$$

implying  $t = t'$ , which is a contradiction. As a consequence, the position matrix of the set  $\{L^{u,t} \times L^{u+u^{-1},t} : t \in \mathbb{F}_q\}$  has one unique entry equal to 1 in each column. Considering the rows of this  $(0,1)$ -matrix as lines and the columns as points, this is equivalent to say that each point belongs to a unique line. Thus, this  $(0,1)$ -matrix is the incidence matrix of  $q^2$  parallel lines with  $q$  points on each.

(iii) First, let us show that the position matrix of the family  $\mathcal{M} = \{L^{u,t} \times L^{u+u^{-1},t} : u, t \in \mathbb{F}_q, u \neq 0\}$  has  $q$  entries equal to 1 in each row and  $q - 1$  entries equal to 1 in each column. By (ii) the position matrix of each set  $\{L^{u,t} \times L^{u+u^{-1},t} : t \in \mathbb{F}_q\}$  contributes with a unique 1 in each column, yielding that each column of the position matrix of  $\mathcal{M}$  has  $q - 1$  entries equal to 1. Since each matrix  $L^{u,t} \times L^{u+u^{-1},t}$  has  $q^2$  distinct entries, then the  $q$  position matrices of symbols starting with the same  $x$  for any  $x \in \mathbb{F}_q$  contribute with one unique 1 in each row. So the position matrix of  $\mathcal{M}$  has  $q$  entries equal to 1 in each row. Therefore, we conclude that the position matrix of  $\mathcal{M}$  has order  $(q^3 - q^2) \times q^3$  and  $q$  entries equal to 1 in each row and  $q - 1$  entries equal to 1 in each column.

Next, let us show that the girth of  $\mathcal{M}$  is at least 6. Otherwise, there exists  $u, v \in \mathbb{F}_q \setminus \{0\}$ ,  $u \neq v$  and  $i, i', j, j' \in \mathbb{F}_q$  with  $j \neq j'$  for which

$$\begin{aligned} (L^{u,t} \times L^{u+u^{-1},t})(i, j) &= (L^{v,t'} \times L^{v+v^{-1},t'})(i', j), \\ (L^{u,t} \times L^{u+u^{-1},t})(i, j') &= (L^{v,t'} \times L^{v+v^{-1},t'})(i', j'). \end{aligned}$$

This implies that  $L^{u,t}$  and  $L^{v,t'}$  are not quasi row-disjoint, a contradiction with item (i). Thus the family  $\{L^{u,t} \times L^{u+u^{-1},t} : u, t \in \mathbb{F}_q, u \neq 0\}$  has girth at least 6. Next let us show that the girth is 8 applying Theorem 2.2. By way of contradiction assume that for three elements  $u, v, w$  of  $\mathbb{F}_q \setminus \{0\}$  there exist three pairwise distinct columns  $j, j', j''$  for which

$$\begin{aligned} (L^{u,t} \times L^{u+u^{-1},t})(i, j) &= (L^{v,t'} \times L^{v+v^{-1},t'})(i', j) \\ (L^{v,t'} \times L^{v+v^{-1},t'})(i', j') &= (L^{w,t''} \times L^{w+w^{-1},t''})(i'', j') \\ (L^{w,t''} \times L^{w+w^{-1},t''})(i'', j'') &= (L^{u,t} \times L^{u+u^{-1},t})(i, j''). \end{aligned} \tag{6}$$

Note that  $u, v, w$  must be three distinct elements because by (ii) each matrix  $L^{u,t} \times L^{u+u^{-1},t}$  has  $q^2$  distinct entries. Then from the equality between the first coordinates we have:

$$\begin{aligned} (u - v)j &= i' - i + vt' - ut; \\ (v - w)j' &= i'' - i' + wt'' - vt'; \\ (w - u)j'' &= i - i'' + ut - wt''. \end{aligned}$$

Hence

$$(v - u)j + (w - v)j' = (w - u)j''. \tag{7}$$

Moreover, from equalities between the second coordinates in (6) and taking into account (7) we obtain

$$(v^{-1} - u^{-1})j + (w^{-1} - v^{-1})j' = (w^{-1} - u^{-1})j''.$$

Multiplying this equality by  $uvw$  we get

$$w(u - v)j + u(v - w)j' = v(u - w)j''. \tag{8}$$

Multiplying (7) by  $w$  we also obtain

$$w(v - u)j + w(w - v)j' = w(w - u)j''. \tag{9}$$

Thus adding both equalities (8) and (9) we have

$$(u - w)(v - w)j' = (v - w)(u - w)j''.$$

Taking into account that  $u, v, w$  are mutually distinct we get that  $j' = j''$  which is a contradiction. Therefore  $\mathcal{M} = \{L^{u,t} \times L^{u+u^{-1},t} : u, t \in \mathbb{F}_q, u \neq 0\}$  has girth 8 as claimed.

(iv) By (iii) and applying Theorem 2.2, we only need to prove that for all  $u, v \in \mathbb{F}_q \setminus \{0\}$  with  $u \neq v$ , any three matrices  $L^{u,t} \times L^{u+u^{-1},t}$ ,  $L^{v,t'} \times L^{v+v^{-1},t'}$  and  $(t'' \times [[q-1]])I_q$  have girth 8. Otherwise we would have

$$\begin{aligned} L^{u,t} \times L^{u+u^{-1},t}(i, j) &= L^{v,t'} \times L^{v+v^{-1},t'}(i', j) \\ L^{v,t'} \times L^{v+v^{-1},t'}(i', j') &\in (t'' \times [[q-1]])I_q(j', j') \\ L^{u,t} \times L^{u+u^{-1},t}(i, j') &\in (t'' \times [[q-1]])I_q(j', j') \end{aligned}$$

Then  $L^{u,t}(i, j) = L^{v,t'}(i', j)$  and  $L^{u,t}(i, j') = L^{v,t'}(i', j') = t''$ , meaning that  $L^{u,t}$  and  $L^{v,t'}$  are not quasi row-disjoint, contradicting item (i). Further, by (iii), we know that the position matrix of  $\{L^{u,t} \times L^{u+u^{-1},t} : u, t \in \mathbb{F}_q, u \neq 0\}$  has  $q$  ones in each row and  $q-1$  ones in each column. Since the rows of the position matrix of  $(t \times [[q-1]])I_q$  contributes with one additional one, then the result follows.  $\square$

To illustrate the method of Theorem 3.1, both the matrices provided by this theorem and their position matrix for the first case  $q = 2$  are shown in Table 1. From now on, if there is no confusion an entry  $(x, y)$  will be denoted as  $xy$ . Thus this (0,1)-matrix is the incidence matrix of a 2-regular graph of girth 8, which consists of two cycles of girth 8. In Table 2 the matrices for  $q = 3$  are also depicted. The corresponding position matrix is the incidence matrix of a 3-regular graph of girth 8 on 27 vertices in each partite set. Table 3 contains the matrices for  $q = 4$ . Their position matrix is the incidence matrix of a 4-regular bipartite graph on 64 vertices in each partite set.

MATRICES	SYMBOLS			
	00	01	10	11
$L^{1,0} \times L^{0,0}$	00    10 11    01	<b>1</b> 0   0 0 0 0   0 <b>1</b> 0 0 <b>1</b> 0		
$L^{1,1} \times L^{0,1}$	10    00 01    11	0 <b>1</b> 0 0 <b>1</b> 0   0 0 0 0 <b>1</b> 0   0 0   0 <b>1</b>		
$(0 \times [[1]])I_2$	{00, 01}    0 0    {00, 01}	<b>1</b> 0 <b>1</b> 0   0 0   0 0 0 <b>1</b> 0 <b>1</b> 0 0   0 0		
$(1 \times [[1]])I_2$	{10, 11}    0 0    {10, 11}	0 0   0 0 <b>1</b> 0 <b>1</b> 0 0 0   0 0   0 <b>1</b> 0 <b>1</b>		

**Tab. 1:** Case  $q = 2$ .

Let us call *array of  $r$  symbols and  $n$  columns* the matrix of order  $r \times n$

$$O_{r,n} = \begin{pmatrix} 1 & \cdots & 1 \\ 2 & \cdots & 2 \\ \vdots & \vdots & \vdots \\ r & \cdots & r \end{pmatrix}.$$

When  $r = n$  the array is denoted by  $O_n$ . It is easy to see that the position matrix of  $O_{r,n}$  is the incidence matrix of a partial plane consisting in  $r$  parallel lines, each one having  $n$  points. Using the position matrices of these arrays denoted by  $\mathcal{P}(O_{r,n})$  and applying Theorem 3.1, we now present the method for constructing the desired  $(k, 8)$ -bipartite graphs.



	$L^{1,t} \times L^{2,t}$	$L^{2,t} \times L^{1,t}$	$(t \times [[2]])I_3$		
t=0	00 12 21	00 21 12	{00, 01, 02}	0	0
	11 20 02	11 02 20	0	{00, 01, 02}	0
	22 01 10	22 10 01	0	0	{00, 01, 02}
t=1	12 21 00	21 12 00	{10, 11, 12}	0	0
	20 02 11	02 20 11	0	{10, 11, 12}	0
	01 10 22	10 01 22	0	0	{10, 11, 12}
t=2	21 00 12	12 00 21	{20, 21, 22}	0	0
	02 11 20	20 11 02	0	{20, 21, 22}	0
	10 22 01	01 22 10	0	0	{20, 21, 22}

**Tab. 2:** Matrices for the case  $q = 3$ .

	$L^{1,t} \times L^{0,t}$	$L^{2,t} \times L^{1,t}$	$L^{3,t} \times L^{1,t}$	$(t \times [[3]])I_4$			
t=0	00 10 20 30	00 21 32 13	00 31 12 23	{00, 01, 02, 03}	0	0	0
	11 01 31 21	11 30 23 02	11 20 03 32	0	{00, 01, 02, 03}	0	0
	22 32 02 12	22 03 10 31	22 13 30 01	0	0	{00, 01, 02, 03}	0
	33 23 13 03	33 12 01 20	33 02 21 10	0	0	0	{00, 01, 02, 03}
t=1	10 00 30 20	21 00 13 32	31 00 23 12	{10, 11, 12, 13}	0	0	0
	01 11 21 31	30 11 02 23	20 11 32 03	0	{10, 11, 12, 13}	0	0
	32 22 12 02	03 22 31 10	13 22 01 30	0	0	{10, 11, 12, 13}	0
	23 33 03 13	12 33 20 1	02 33 10 21	0	0	0	{10, 11, 12, 13}
t=2	20 30 00 10	32 13 00 21	12 23 00 31	{20, 21, 22, 23}	0	0	0
	31 21 11 01	23 02 11 30	03 32 11 20	0	{20, 21, 22, 23}	0	0
	02 12 22 32	10 31 22 03	30 01 22 13	0	0	{20, 21, 22, 23}	0
	13 03 33 23	01 20 33 12	21 10 33 02	0	0	0	{20, 21, 22, 23}
t=3	30 20 10 00	13 32 21 00	12 23 00 31	{30, 31, 32, 33}	0	0	0
	21 31 01 11	02 23 30 11	03 32 11 20	0	{30, 31, 32, 33}	0	0
	12 02 32 22	31 10 03 22	30 01 22 13	0	0	{30, 31, 32, 33}	0
	03 13 23 33	20 01 12 33	21 10 33 02	0	0	0	{30, 31, 32, 33}

**Tab. 3:** Matrices for the case  $q = 4$ .

**Theorem 3.2** Let  $q$  be a power prime, and let  $\mathbb{F}_q$  be the Galois field of order  $q$ . For each  $u, t \in \mathbb{F}_q$ ,  $u \neq 0$ , let  $L^{u,t}$  be the matrix  $L^{u,t}(i, j) = i + uj + ut$ ,  $i, j \in \mathbb{F}_q$ , and let  $M$  be the position matrix of the family

$$\{L^{u,t} \times L^{u+u^{-1},t} : u, t \in \mathbb{F}_q, u \neq 0\} \cup \{(t \times [[q-1]])I_q : t \in \mathbb{F}_q\}.$$

Then the following assertions hold:

(i) The  $(0, 1)$ -matrix

$$\boxed{M \mid \mathcal{P}(O_{q^2, q})^\top} \quad (10)$$

is the incidence matrix of a bipartite graph of girth 8 with  $q^3 + q^2$  vertices in one partite set having degree  $q + 1$ , and  $q^3$  vertices in the other partite set having degree  $q$ .

- (ii) Let  $(L^{u,t} \times L^{u+u^{-1},t})_0$  denote the matrix obtained from  $L^{u,t} \times L^{u+u^{-1},t}$  by replacing each entry  $0x$  with  $0$  for all  $x \in \mathbb{F}_q$ . Let  $M_0$  be the position matrix of the family

$$\{(L^{u,t} \times L^{u+u^{-1},t})_0 : u, t \in \mathbb{F}_q, u \neq 0\} \cup \{(t \times [[q-1]])I_q : t \in \mathbb{F}_q, t \neq 0\}.$$

Then

$$M_0 \begin{array}{|c|} \hline \mathcal{P}(O_{q^2-q,q})^\top \\ \hline [0] \\ \hline \end{array} \quad (11)$$

is the incidence matrix of a  $q$ -regular bipartite graph of girth 8 with  $q^3 - q$  vertices in each partite set.

- (iii) Let  $k$  be an integer such that  $3 \leq k \leq q - 1$  and let  $(L^{u,t} \times L^{u+u^{-1},t})_{q-k}$  denote the matrix obtained from  $L^{u,t} \times L^{u+u^{-1},t}$  by replacing with  $0$  the entries  $0y$  and  $(x, x+s)$  for all  $x, y, s \in \mathbb{F}_q$  for  $s = 0, \dots, q-1-k$ . Let  $M_{q-k}$  be the position matrix of

$$\{(L^{u,t} \times L^{u+u^{-1},t})_{q-k} : u = 1, \dots, k-1\} \cup \{(t \times ([q-1] \setminus \{t, t+1, \dots, t+q-1-k\}))I_q : t \neq 0\}.$$

Suppose that the  $q$  columns of  $O_{kq-q,q}$  are indexed by  $j \in \mathbb{F}_q$ . Let  $O_{kq-q,q}^*$  be the matrix obtained from  $O_{kq-q,q}$  by changing for  $0$  the entries  $(i, -u^2s)$  for all  $u = 1, \dots, k-1, i = (u-1)q+1, \dots, uq$  and  $s = 0, \dots, q-1-k$ . Then

$$M_{q-k} \begin{array}{|c|} \hline \mathcal{P}(O_{kq-q,q}^*)^\top \\ \hline [0] \\ \hline \end{array} \quad (12)$$

is the incidence matrix of a  $k$ -regular bipartite graph of girth 8 with  $kq^2 - q$  vertices in each partite set.

**Proof:** (i) From Theorem 3.1, it follows that  $M$  is the incidence matrix of a bipartite graph of girth 8 with  $q^3$  lines and  $q^3$  points. Thus we need to prove that the  $q^2$  columns of  $\mathcal{P}(O_{q^2,q})^\top$  can be added to  $M$  without decreasing the girth 8. It is readily seen that after adding these columns, the girth is at least 6, because by Theorem 3.1, for each  $u \in \mathbb{F}_q \setminus \{0\}$  the set  $\{L^{u,t} \times L^{u+u^{-1},t} : t \in \mathbb{F}_q\}$  consists of  $q^2$  parallel lines. Thus suppose that (10) contains as a sub-matrix the incidence matrix of a cycle of length 6. Then there exists  $u, v \in \mathbb{F}_q \setminus \{0\}$  and  $i, i', i'', j, j' \in \mathbb{F}_q$  with  $u \neq v, i \neq i'$  and  $j \neq j'$  such that

$$\begin{aligned} (L^{u,t} \times L^{u+u^{-1},t})(i, j) &= (L^{v,t'} \times L^{v+v^{-1},t'})(i'', j), \\ (L^{u,t} \times L^{u+u^{-1},t})(i', j') &= (L^{v,t'} \times L^{v+v^{-1},t'})(i'', j'). \end{aligned}$$

From the equalities between their coordinates we obtain

$$\begin{aligned} i - i' + u(j - j') &= v(j - j'), \\ i - i' + (u + u^{-1})(j - j') &= (v + v^{-1})(j - j'). \end{aligned}$$

Hence  $u^{-1}(j - j') = v^{-1}(j - j')$ , implying  $u = v$  or  $j = j'$ , a contradiction in either case. Further if  $(L^{u,t} \times L^{u+u^{-1},t})(i, j) \in (t' \times [[q-1]]I_q)(i'', j)$ , and  $(L^{u,t} \times L^{u+u^{-1},t})(i', j') \in (t' \times [[q-1]]I_q)(i'', j')$ , then then  $i'' = j = j'$  which is also a contradiction.

MATRICES			SYMBOLS						
			10	11	12	20	21	22	
0	12	21	0 0 0	0 0 0	0 1 0	0 0 0	0 0 1	0 0 0	1 0 0 0 0 0
11	20	0	0 0 0	1 0 0	0 0 0	0 1 0	0 0 0	0 0 0	1 0 0 0 0 0
22	0	10	0 0 1	0 0 0	0 0 0	0 0 0	0 0 0	1 0 0	1 0 0 0 0 0
12	21	0	0 0 0	0 0 0	1 0 0	0 0 0	0 1 0	0 0 0	0 1 0 0 0 0
20	0	11	0 0 0	0 0 1	0 0 0	1 0 0	0 0 0	0 0 0	0 1 0 0 0 0
0	10	22	0 1 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 1	0 1 0 0 0 0
21	0	12	0 0 0	0 0 0	0 0 1	0 0 0	1 0 0	0 0 0	0 0 1 0 0 0
0	11	20	0 0 0	0 1 0	0 0 0	0 0 1	0 0 0	0 0 0	0 0 1 0 0 0
10	22	0	1 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 1 0	0 0 1 0 0 0
0	21	12	0 0 0	0 0 0	0 0 1	0 0 0	0 1 0	0 0 0	0 0 0 1 0 0
11	0	20	0 0 0	1 0 0	0 0 0	0 0 1	0 0 0	0 0 0	0 0 0 1 0 0
22	10	0	0 1 0	0 0 0	0 0 0	0 0 0	0 0 0	1 0 0	0 0 0 1 0 0
21	12	0	0 0 0	0 0 0	0 1 0	0 0 0	1 0 0	0 0 0	0 0 0 0 1 0
0	20	11	0 0 0	0 0 1	0 0 0	0 1 0	0 0 0	0 0 0	0 0 0 0 1 0
10	0	22	1 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 1	0 0 0 0 1 0
12	0	21	0 0 0	0 0 0	1 0 0	0 0 0	0 0 1	0 0 0	0 0 0 0 0 1
20	11	0	0 0 0	0 1 0	0 0 0	1 0 0	0 0 0	0 0 0	0 0 0 0 0 1
0	22	10	0 0 1	0 0 0	0 0 0	0 0 0	0 0 0	0 1 0	0 0 0 0 0 1
{10, 11, 12}	0	0	1 0 0	1 0 0	1 0 0	0 0 0	0 0 0	0 0 0	0 0 0 0 0 0
0	{10, 11, 12}	0	0 1 0	0 1 0	0 1 0	0 0 0	0 0 0	0 0 0	0 0 0 0 0 0
0	0	{10, 11, 12}	0 0 1	0 0 1	0 0 1	0 0 0	0 0 0	0 0 0	0 0 0 0 0 0
{20, 21, 22}	0	0	0 0 0	0 0 0	0 0 0	1 0 0	1 0 0	1 0 0	0 0 0 0 0 0
0	{20, 21, 22}	0	0 0 0	0 0 0	0 0 0	0 1 0	0 1 0	0 1 0	0 0 0 0 0 0
0	0	{20, 21, 22}	0 0 0	0 0 0	0 0 0	0 0 1	0 0 1	0 0 1	0 0 0 0 0 0

**Tab. 4:** Case  $q = 3$  for (ii) of Theorem 3.2. Incidence matrix of a (3,8)-graph on 48 vertices.

(ii) This item follows directly from the fact that  $M_0$  is a sub-matrix of  $M$  obtained by deleting the first  $q^2$  columns, which correspond to the position of the symbols starting by 0. Hence  $M_0$  has  $q^3 - q^2$  columns. Moreover, the total number of rows of  $M_0$  is the number of matrices  $L^{u,t} \times L^{u+u^{-1},t}$ , (that is,  $(q-1)q$ ) plus the number of matrices  $(t \times ([q-1]))I_q$ ,  $t \neq 0$ , (a total of  $q-1$ ) multiplied by  $q$ , that is,  $q(q^2 - q + q - 1) = q^3 - q$ . Thus  $M_0$  is a matrix of order  $(q^3 - q) \times (q^3 - q^2)$ . Since  $\mathcal{P}(O_{q^2-q,q})^\top$  contributes with  $q^2 - q$  more columns, then (11) is a square matrix of order  $q^3 - q$ . Reasoning as in (iii) of Theorem 3.1, we obtain that (11) is the incidence matrix of a bipartite graph of girth 8, which has  $q^3 - q$  columns and  $q^3 - q$  rows both having  $q$  ones, so this item is valid. By way of example, both the matrices provided by this item (ii) and their position matrix for the case  $q = 3$  are shown in Table 4. Thus this (0,1)-matrix is the incidence matrix of a 3-regular graph of girth 8 on 24 vertices in each partite set.

(iii) Note that (12) is a sub-matrix of (10), then it is the incidence matrix of a bipartite graph of girth 8. Moreover,  $M_{q-k}$  is obtained from  $M$  by deleting the first  $q^2$  columns which corresponds to the position matrix of symbols starting by 0, and by deleting also  $(q-k)(q-1)q$  columns corresponding to the symbols  $(x+s, x+2s)$  for all  $s = 0, 1, \dots, q-1-k$  which have been changed for 0. Then the total number of columns of  $M_{q-k}$  is

$$q^3 - q^2 - (q-k)(q-1)q = kq(q-1).$$

The total number of rows of  $M_{q-k}$  is given by the number of matrices  $(L^{u,t} \times L^{u+u^{-1},t})_{q-k}$ ,  $u = 1, \dots, k-1$ ,  $t \in \mathbb{F}_q$ , plus the number of matrices  $(t \times ([q-1] \setminus \{t, t+1, \dots, t+q-1-k\}))I_q : t \neq 0, t \neq q-k$ , (a total of  $q-1$ ) multiplied by  $q$ , that is

$$(k-1)q^2 + (q-1)q = kq^2 - q.$$

Thus  $M_{q-k}$  is a matrix of order  $(kq^2 - q) \times (kq^2 - kq)$ . Since  $\mathcal{P}(O_{kq-q,q}^*)^\top$  contributes with  $kq - q$  more columns, then (12) is a square matrix of order  $kq^2 - q$ .

To finish the proof of this item, we only need to show that (12) has  $k$  ones in each row and  $k$  ones in each column. To see this, let us show that given a fixed  $s = 0, 1, \dots, q - 1 - k$ , the entries  $(y, y + s)$  for all  $y \in \mathbb{F}_q$ , are in the same column of each matrix  $L^{u,t} \times L^{u+u^{-1},t}$ . Suppose  $L^{u,t} \times L^{u+u^{-1},t}(i, j) = (y, y + s)$ , that is

$$i + uj + ut = y, \text{ and } i + (u + u^{-1})j + (u + u^{-1})t = y + s.$$

Then  $u^{-1}(j + t) = s$ , which implies  $j = us - t$ . Thus our claim is true since the symbols  $(y, y + s)$  are placed in the same column  $us - t$  of the matrix  $L^{u,t} \times L^{u+u^{-1},t}$ . Therefore, after changing for 0 the symbols  $0x$  and  $(y, y + s)$  for all  $x, y \in \mathbb{F}_q$  and for all  $s = 0, \dots, q - 1 - k$  we obtain a new matrix having  $k$  entries different from 0 in the rows  $i$  such that  $i + u^2s = 0$ , and in the remaining rows  $k - 1$  entries different from 0. Since in the rows  $i$  of  $\mathcal{P}(O_{qk-q,q}^*)^\top$  such that  $i + u^2s \neq 0$  there is a 1 by hypothesis, then  $\mathcal{P}(O_{qk-q,q}^*)^\top$  contributes with one additional entry equal to 1 in the same rows as those having  $k - 1$  entries different from 0. Hence (12) is a square matrix of order  $kq^2 - q$  having  $k$  ones in each row and clearly in each column.

By way of example, both the matrices provided by this item (iii) and their position matrix for the case  $q = 4$  and  $k = 3$  are shown in Table 5; and the corresponding matrices for  $q = 5$  and  $k = 3$  are shown in Table 6. Thus, these (0,1)-matrices are the incidence matrix of a 3-regular graph of girth 8 on 44 vertices in each partite set for  $q = 4$  and 70 vertices in each partite set for  $q = 5$ .  $\square$

## 4 Conclusion

For  $q$  a prime power and  $3 \leq k \leq q$  we have presented a method providing the incidence matrices of  $k$ -regular bipartite graphs of girth 8 with  $kq^2 - q$  vertices in each partite set. Thus if  $n(k, 8)$  denotes the order of a  $(k, 8)$ -cage, it follows from (1) that

$$2k(k^2 - 2k + 2) \leq n(k, 8) \leq 2q(kq - 1).$$

Hence the  $q$ -regular bipartite graphs constructed in this work have an excess of  $4q^2 - 6q$ . And the  $(q - 1)$ -regular bipartite graphs have an excess of  $8q^2 - 20q + 10$ .

As regards to known upper bounds on  $n(k, g)$ , Lazebnik, Ustimenko and Woldar [19] gave the following result: Let  $k \geq 2$  and  $g \geq 5$  be integers, and let  $q$  denote the smallest odd prime power for which  $k \leq q$ . Then,

$$n(k, g) \leq 2kq^{\frac{3}{4}g-a}, \quad (13)$$

where  $a = 4, 11/4, 7/2, 13/4$  for  $g \equiv 0, 1, 2, 3 \pmod{4}$ , respectively. According to (13),  $n(k, 8) \leq 2kq^2$ , therefore the graphs provided by our method also improve this result for  $g = 8$ . A construction giving this upper bound (13) for  $g = 8$  appeared for the first time in [18] and was used later in [14] and probably, the simplest exposition of it is in Section 2.4 of [20]. In [15],  $(q, 8)$ -graphs with  $2q(q^2 - 2)$  vertices were constructed using geometrical techniques. But for regularities  $k < q$  the graphs constructed in this paper have the smallest number of vertices among the known regular graphs with girth 8.



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