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# Ore and Erdős type conditions for long cycles in balanced bipartite graphs

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We conjecture Ore and Erdős type criteria for a balanced bipartite graph of order  $2n$  to contain a long cycle  $C_{2n-2k}$ , where  $0 \leq k < n/2$ . For  $k = 0$ , these are the classical hamiltonicity criteria of Moon and Moser. The main two results of the paper assert that our conjectures hold for  $k = 1$  as well.

**Keywords:** bipartite graph, cycle, long cycle, hamiltonicity, degree sum

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## 1 Introduction

One of the classical problems of graph theory is the study of sufficient conditions for a graph to contain a Hamilton cycle. In this paper we are primarily interested in two types of such conditions. Namely, the ones that put constraints on degree sums of pairs of non-adjacent vertices, and those that combine bounds on the size of a graph with bounds on its minimal degree. The first approach is due to Ore (see Section 2 for notation):

**Theorem 1.1** (Ore, [12]). *Let  $G$  be a graph of order  $n \geq 3$ , in which*

$$d_G(x) + d_G(y) \geq n$$

*for every pair of non-adjacent vertices  $x$  and  $y$ . Then  $G$  contains a Hamilton cycle.*

It follows immediately from Ore's theorem that the minimal size of a graph of order  $n \geq 3$  that guarantees hamiltonicity is  $\binom{n-1}{2} + 2$ . Erdős generalized this condition by adding a bound on the minimal degree of a graph:

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**Theorem 1.2** (Erdős, [9]). *Let  $G$  be a graph of order  $n \geq 3$  and minimal degree  $\delta(G) \geq r$ , where  $1 \leq r < n/2$ . Then  $G$  contains a Hamilton cycle, provided*

$$\|G\| > \max \left\{ \binom{n-r}{2} + r^2, \binom{n - \lfloor \frac{n-1}{2} \rfloor}{2} + \left\lfloor \frac{n-1}{2} \right\rfloor^2 \right\}.$$

The above conditions can, of course, be significantly strengthened in case of a balanced bipartite graph. The following two theorems are bipartite counterparts of Ore and Erdős criteria, respectively.

**Theorem 1.3** (Moon and Moser, [11]). *Let  $G$  be a bipartite graph of order  $2n$ , with colour classes  $X$  and  $Y$ , where  $|X| = |Y| = n \geq 2$ . Suppose that  $d_G(x) + d_G(y) \geq n + 1$  for every pair of non-adjacent vertices  $x \in X$  and  $y \in Y$ . Then  $G$  contains a Hamilton cycle.*

**Theorem 1.4** (Moon and Moser, [11]). *Let  $G$  be a bipartite graph of order  $2n$ , with colour classes  $X$  and  $Y$ ,  $|X| = |Y| = n \geq 2$ , and minimal degree  $\delta(G) \geq r$ ,  $1 \leq r \leq n/2$ . Then  $G$  contains a Hamilton cycle, provided  $\|G\| > n(n-r) + r^2$ .*

Our goal is to generalize the above criteria to long cycles, that is, cycles of length  $2n - 2k$ , where  $0 \leq k < n/2$ . We state the following two conjectures, that include Theorems 1.3 and 1.4 as special cases ( $k = 0$ ).

**Conjecture A.** *Let  $G$  be a 2-connected balanced bipartite graph of order  $2n$ , with colour classes  $X$  and  $Y$ ,  $|X| = |Y| = n \geq 5$ , and let  $k < n/2$  be a non-negative integer. If*

$$d_G(x) + d_G(y) \geq n - k + 1$$

*for every pair of non-adjacent  $x \in X$  and  $y \in Y$ , then  $G$  contains a cycle of length  $2n - 2k$ .*

**Conjecture B.** *Let  $G$  be a balanced bipartite graph of order  $2n$  and minimal degree  $\delta(G) \geq r \geq 1$ , where  $n \geq 2k + 2r$  and  $k \in \mathbb{Z}$ . If*

$$\|G\| > n(n - k - r) + r(k + r)$$

*then  $G$  contains a cycle of length  $2n - 2k$ .*

The main two results of this paper, Theorems A and B (Section 3), assert that our conjectures hold true for  $k = 1$ . We believe the conjectures to be significantly harder in case  $k \geq 2$ .

It should be mentioned here that analogous generalizations to long cycles of Ore's and Erdős's theorems have been studied in ordinary graphs. Woodall [14, Thm. 11] gives a complete list of Erdős type conditions for a graph of order  $n$  to contain a cycle of length  $n - k$  for any  $0 \leq k \leq \frac{n-3}{2}$ . The Ore type criterion is conjectured in [1], and follows from a result of Linial [10] in case  $k \leq 1$ .

**Remark 1.5.** Both the degree sum condition of Conjecture A and the bound on the size of Conjecture B are sharp, as can be seen in Example 1.6 below. It is also necessary to assume 2-connectedness in Conjecture A (Example 1.7). Finally, a quick look at  $C_6$  and  $C_8$  shows that Conjecture A would fail for  $n < 5$ .

**Example 1.6.** Let  $G_1$  be a balanced bipartite graph, with colour classes  $X$  and  $Y$ ,  $|X| = |Y| = n$ , where  $X = A \cup B$ ,  $Y = C \cup D$ ,  $|A| = k + r$ ,  $|B| = n - k - r$ ,  $|C| = r$ , and  $|D| = n - r$ . Moreover, assume that  $N_{G_1}(x) = C$  for all  $x \in A$ , and  $N_{G_1}(x) = Y$  for all  $x \in B$ . Then  $d_{G_1}(x) + d_{G_1}(y) = n - k$  for every pair  $x \in A$  and  $y \in D$ , and, in general,  $d_{G_1}(x) + d_{G_1}(y) \geq n - k$  for every pair of  $x \in X$  and  $y \in Y$ . If  $n \geq 2k + 2r$ , then  $\delta(G_1) = r \geq 1$  and  $\|G_1\| = n(n - k - r) + r(k + r)$ , but  $G_1$  does not contain a cycle of length  $2n - 2k$ .

**Example 1.7.** Let  $G_2 = (X, Y; E)$  be a balanced bipartite graph obtained from the disjoint union of  $H_1 = K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$  and  $H_2 = K_{\lceil n/2 \rceil, \lceil n/2 \rceil}$  by adding a single edge joining a vertex of  $H_1$  with a vertex of  $H_2$ . Then  $d_{G_2}(x) + d_{G_2}(y) \geq n$  for every pair of non-adjacent vertices  $x \in X$  and  $y \in Y$ , nonetheless  $G_2$  contains no cycle of length  $2n - 2$ . In fact,  $G_2$  contains no long cycle whatsoever.

The next section contains the inventory of basic definitions and results used throughout the paper. In Section 3 we state our main results, Theorems A and B, and their consequences. In particular, by combining Theorems A and B, we obtain a complete Erdős type characterisation of balanced bipartite graphs that do not contain cycles of length  $2n - 2$  (Theorem 3.6). The last two sections are devoted to proofs of the two main results.

## 2 Notation and tools

All graphs considered are undirected, have no loops and no multiple edges. Given a graph  $G$ , we denote by  $\|G\|$  the size (i.e., number of edges) of  $G$ , and by  $V(G)$  the vertex set of  $G$ . A bipartite graph is often denoted by  $G = (X, Y; E)$ , where  $X$  and  $Y$  are the two colour classes of  $G$ , and  $E = E(G)$  is the edge set of  $G$ . When  $|X| = |Y|$ , we say that  $G$  is *balanced*. Given a vertex  $x \in V(G)$ ,  $N_G(x)$  denotes the set of vertices adjacent to  $x$  in  $G$ ,  $d_G(x)$  the degree of  $x$  in  $G$  (i.e.,  $d_G(x) = |N_G(x)|$ ), and  $\delta(G)$  the minimal vertex degree in  $G$ . If  $L \subset V(G)$  is a vertex subset of  $G$ , then  $G - L$  denotes the subgraph of  $G$  induced by  $V(G) \setminus L$ , and  $N_G(L)$  is the set of neighbours of all the vertices in  $L$ . Given distinct vertices  $x$  and  $y$  of  $G$ , an  $x - y$  path is a path in  $G$  with endvertices  $x$  and  $y$ . We denote by  $C_l$  a cycle of length  $l$ , and by  $K_{n,n}$  a complete balanced bipartite graph of order  $2n$ . Finally, recall that a graph is called *2-connected* if the removal of any single vertex does not disconnect  $G$ .

In this section we have gathered results used in the proofs of Theorems A and B. First of all, we recall two hamiltonicity criteria obtained by Moon and Moser [11].

**Theorem 2.1** (Moon and Moser, [11]). *Let  $G$  be a balanced bipartite graph of order  $2n \geq 4$ , with  $\delta(G) \geq \frac{n+1}{2}$ . Then  $G$  contains a Hamilton cycle.*

**Theorem 2.2** (Moon and Moser, [11]). *Let  $G = (X, Y; E)$  be a balanced bipartite graph of order  $2n$ , and let  $S_m = \{x \in X : d_G(x) \leq m\}$ ,  $T_m = \{y \in Y : d_G(y) \leq m\}$  for  $m \in \mathbb{Z}$ . If, for every  $1 \leq m \leq n/2$ , the sets  $S_m$  and  $T_m$  are of cardinalities less than  $m$ , then  $G$  is hamiltonian.*

We shall need the following strengthening of Theorem 1.4.

**Theorem 2.3** (Wojda and Woźniak, [13]). *Let  $G(n, r)$  denote a bipartite graph with colour classes  $X = P \cup Q$  and  $Y = R \cup S$  such that  $|P| = |R| = r$ ,  $|Q| = |S| = n - r$ ,  $N_{G(n,r)}(x) = R$  for all  $x \in P$ , and  $N_{G(n,r)}(x) = Y$  for all  $x \in Q$ . Let  $G$  be a balanced bipartite graph of order  $2n \geq 4$ , minimal degree  $\delta(G) \geq r \geq 1$ , and size  $\|G\| \geq n(n - r) + r^2$ . Then  $G$  contains a Hamilton cycle, else  $r \leq n/2$  and  $G$  is isomorphic to  $G(n, r)$ .*

A bipartite graph of order  $2n$  is called *bipancyclic* if it contains cycles of lengths  $2k$  for all  $2 \leq k \leq n$ .

**Theorem 2.4** (Bagga and Varma, [5]). *Let  $G = (X, Y; E)$  be a balanced bipartite graph of order  $2n \geq 8$ . If  $d_G(x) + d_G(y) \geq n + 1$  for every pair of non-adjacent vertices  $x \in X$  and  $y \in Y$ , then  $G$  is bipancyclic.*

**Theorem 2.5** (Entringer and Schmeichel, [8]). *Let  $G$  be a hamiltonian bipartite graph of order  $2n \geq 8$ . If  $\|G\| > n^2/2$ , then  $G$  is bipancyclic.*

We will also need to know the cycle structure of an  $n/2$ -regular hamiltonian bipartite graph  $G$  of order  $2n$ . Notice that then  $\|G\| = n^2/2$ , so the above theorem does not apply. We then have:

**Theorem 2.6** (J. Adamus, [2]). *Let  $G$  be an  $n/2$ -regular hamiltonian bipartite graph of order  $2n$ . Then  $G$  contains a cycle  $C$  of length  $2n - 2$ . Moreover, if  $C$  can be chosen to omit a pair of adjacent vertices, then  $G$  is bipancyclic.*

Given a balanced bipartite graph  $G = (X, Y; E)$ , one defines a  $k$ -biclosure  $BCl_k(G)$  of  $G$  as the graph obtained from  $G$  by successively joining pairs of non-adjacent vertices  $x \in X$  and  $y \in Y$ , with degree sum of at least  $k$ , until no such pair remains. Closely related to this construction is the notion of  $k$ -bistability: A property  $\mathcal{P}$  defined on all balanced bipartite graphs of order  $2n$  is called  $k$ -bistable when, whenever  $G + xy$  has the property  $\mathcal{P}$  and  $d_G(x) + d_G(y) \geq k$ , then  $G$  itself has the property  $\mathcal{P}$ .

**Theorem 2.7** (Bondy and Chvátal, [7]). *A balanced bipartite graph  $G$  of order  $2n$  is hamiltonian if and only if its  $(n + 1)$ -biclosure  $BCl_{n+1}(G)$  is so.*

**Theorem 2.8** (Amar, Favaron, Mago and Ordaz, [4]). *The property of containing a cycle of length  $2n - 2$  is  $(n + 2)$ -bistable on balanced bipartite graphs of order  $2n$ .*

### 3 Long cycles in balanced bipartite graphs

Suppose we want to know whether a balanced bipartite graph  $G = (X, Y; E)$  has the property of containing a long cycle  $C_{2n-2k}$  for some  $0 \leq k < n/2$ . Given Theorem 1.3 of Moon and Moser, a natural question arises: Can one impose such a property by decreasing the bound on the degree sum of non-adjacent vertices by  $k$ ? We believe the answer to this question be positive (Conjecture A). As shown in Example 1.6, any lower bound on the degree sum of non-adjacent vertices  $x \in X$  and  $y \in Y$  which ensures  $C_{2n-2k} \subset G$  is at least  $n - k + 1$ . On the other hand, decreasing the bound below  $n + 1$  imposes additional assumptions on the graph. Interestingly enough, without the 2-connectedness constraint the graph could contain no long cycles at all (see Example 1.7). The following result gives a positive answer to the above question in case  $k = 1$ .

**Theorem A.** *Let  $G = (X, Y; E)$  be a 2-connected balanced bipartite graph of order  $2n \geq 4$ , such that  $d_G(x) + d_G(y) \geq n$  for every pair of non-adjacent vertices  $x \in X$  and  $y \in Y$ . Then  $G$  contains an even cycle of length at least  $2n - 2$ .*

We postpone the proof of the theorem to Section 4. Right now we will show that Theorem A implies Conjecture A for  $k = 1$ .

**Corollary 3.1.** *Conjecture A holds for  $k = 1$ .*

**Proof:** Let  $G = (X, Y; E)$  be a balanced bipartite graph of order  $2n$  that satisfies the assumptions of Conjecture A. By Theorem A above,  $G$  contains an even cycle of length at least  $2n - 2$ , so without loss of generality one may assume that  $G$  is hamiltonian.

Let  $x \in X$ , say, be a vertex of minimal degree  $\delta(G)$  in  $G$ . Then  $Y$  contains precisely  $n - \delta(G)$  vertices non-adjacent to  $x$ , each of degree at least  $n - \delta(G)$  (as  $d_G(x) + d_G(y) \geq n$  for  $xy \notin E$ ). Counting the edges incident with  $Y$ , we get

$$\|G\| \geq (n - \delta(G)) \cdot (n - \delta(G)) + \delta(G) \cdot \delta(G).$$

Observe that  $(n - \delta(G))^2 + \delta(G)^2 > n^2/2$  iff  $\delta(G) \neq n/2$ . Hence  $\|G\| > n^2/2$ , provided  $\delta(G) \neq n/2$ , and thus  $G$  contains  $C_{2n-2}$ , by Theorem 2.5. If, in turn,  $\delta(G) = n/2$ , then the result follows from Theorem 2.6.  $\square$

Let us now turn to Erdős type criteria. In [3], the second author conjectured the following sufficient condition for a balanced bipartite graph to contain a long cycle  $C_{2n-2k}$  (proved in [3] under considerably stronger assumptions).

**Conjecture 3.2** (L. Adamus, [3]). *Let  $G$  be a balanced bipartite graph of order  $2n$ , where  $n \geq 2k + 2$ ,  $k \in \mathbb{Z}$ . If  $\|G\| > n(n - k - 1) + k + 1$ , then  $G$  contains a cycle of length  $2n - 2k$ .*

Notice that both assumptions of the conjecture are weakest possible, as shown by the following two examples.

**Example 3.3.** Consider a graph  $G_1$  of Example 1.6, with  $r = 1$ . This graph has precisely  $n(n - k - 1) + k + 1$  edges, and it contains no cycle of length greater than  $2n - 2k - 2$ .

**Example 3.4.** Let  $G_3 = (X, Y; E)$  be a balanced bipartite graph, with colour classes of the form  $X = A \cup B$ ,  $Y = C \cup D$ , where  $|A| = |D| = k + 1$ ,  $|B| = |C| = n - k - 1$ . Fix a vertex  $y_0$  in  $C$ , and let  $N_{G_3}(x) = C$  for all  $x \in A$ , and  $N_{G_3}(x) = D \cup \{y_0\}$  for all  $x \in B$ . Then  $\|G_3\| > n(n - k - 1) + k + 1$  for  $k + 3 \leq n \leq 2k + 1$ , yet  $G_3$  contains no cycle of length greater than  $2n - 2k - 2$ . Hence the necessity of the assumption  $n \geq 2k + 2$ .

Interestingly, a similar graph was recently shown in [6] to be a counterexample to Győri's conjecture on  $C_{2l}$ -free bipartite graphs.

In light of Example 3.3 above, we ask: By how much can we decrease the lower bound on the size of a given graph  $G$  ensuring the existence of a cycle of length  $2m - 2k$ , knowing that the minimal degree of  $G$  is greater than 1? We address this question in Conjecture B. Certain special cases of Conjecture B are known true:  $k = 0$  is Theorem 1.4,  $k = r = 1$  is done in [3]. The following theorem (proved in Section 5 below) shows that the conjecture also holds for  $k = 1$  and arbitrary  $r$ .

**Theorem B.** *Let  $G = (X, Y; E)$  be a balanced bipartite graph of order  $2n$  and minimal degree  $\delta(G) \geq r \geq 1$ , where  $n \geq 4$  and  $n \geq 2r + 1$ . Let*

$$g(n, r) = n(n - 1 - r) + r(1 + r) + 1.$$

*Then  $G$  contains a cycle of length  $2n - 2$ , provided  $\|G\| \geq g(n, r)$ .*

Notice that Theorems 2.1 and 1.4 can be put together as follows:

**Theorem 3.5.** *Let  $G$  be a balanced bipartite graph of order  $2n \geq 4$ , with minimal degree  $\delta(G) \geq r$ . Then  $G$  contains a Hamilton cycle, provided*

- (1)  $n \leq 2r - 1$  or
- (2)  $n \geq 2r$  and  $\|G\| > n(n - r) + r^2$ .

Along the same lines, we combine Theorem 2.4 and Theorems A and B to prove the following criterion for cycles of length  $2n - 2$ .

**Theorem 3.6.** *Let  $G = (X, Y; E)$  be a balanced bipartite graph of order  $2n \geq 8$ , with minimal degree  $\delta(G) \geq r \geq 1$ . Then  $G$  contains a cycle of length  $2n - 2$ , provided*

- (1)  $n \leq 2r - 1$  or
- (2)  $n = 2r$  and  $\|G\| \geq 2r^2 + r + 1$  or
- (3)  $n \geq 2r + 1$  and  $\|G\| \geq n(n - 1 - r) + r(1 + r) + 1$ .

**Remark 3.7.** The lower bounds of conditions (2) and (3) are sharp: For an extremal graph for (2), consider the graph  $G_3$  from Example 3.4 with  $k + 1 = r$ ; for (3), consider  $G_1$  from Example 1.6 with  $k = 1$ .

**Proof of Theorem 3.6:**

- (1) Since  $n \leq 2r - 1$  iff  $r \geq (n + 1)/2$ , then the degree sum is greater than or equal to  $n + 1$  for every pair of vertices in  $G$  (in particular, for non-adjacent ones). By Theorem 2.4,  $G$  is then bipancyclic.
- (2) The bound on the size of  $G$  together with  $\delta(G) \geq r = n/2$  force 2-connectedness. Also, the degree sum is at least  $2r = n$  for every pair of vertices in  $G$ . Hence, by Corollary 3.1,  $G$  contains  $C_{2n-2}$ .
- (3) This is Theorem B. □

## 4 Proof of Theorem A

As 2-connectedness of a graph  $G$  implies  $\delta(G) \geq 2$ , the assertion of the theorem holds true for  $n \leq 3$ , by Theorem 2.1. Suppose then there exists  $n \geq 4$  for which the assertion fails. Let  $G = (X, Y; E)$  be a maximal 2-connected balanced bipartite graph of order  $2n$ , in which  $d_G(x) + d_G(y) \geq n$  for all non-adjacent  $x \in X, y \in Y$ , without a cycle of length at least  $2n - 2$ . By maximality of  $G$ ,  $G + xy$  contains a cycle of length at least  $2n - 2$ , and hence  $G$  contains an  $x - y$  path of length  $2n - 3$  or  $2n - 1$  for every pair of non-adjacent  $x \in X, y \in Y$ .

We shall show first that  $G$  contains a Hamilton path. Suppose not. Let  $x \in X, y \in Y$  be non-adjacent vertices and let  $P$  be an  $x - y$  path in  $G$  of length  $2n - 3$ ; say,  $P = u_1 v_1 u_2 v_2 \dots u_{n-1} v_{n-1}$ , where  $X = \{u_1, \dots, u_n\}, Y = \{v_1, \dots, v_n\}, u_1 = x$  and  $v_{n-1} = y$ . Put  $I_P = \{1 \leq i \leq n - 1 \mid u_1 v_i \in E\}$  and  $J_P = \{1 \leq i \leq n - 1 \mid u_i v_{n-1} \in E\}$ . Then  $I_P \cap J_P = \emptyset$ , for if  $i_0 \in I_P \cap J_P$ , then  $G$  contains a cycle  $u_1 v_{i_0} u_{i_0+1} \dots v_{n-1} u_{i_0} v_{i_0-1} \dots v_1 u_1$  of length  $2n - 2$ ; a contradiction.

As  $|I_P| = d_{G[V(P)]}(x)$  and  $|J_P| = d_{G[V(P)]}(y)$ , we obtain

$$d_{G[V(P)]}(x) + d_{G[V(P)]}(y) = |I_P| + |J_P| = |I_P \cup J_P| \leq n - 1,$$

where  $G[V(P)]$  denotes the subgraph of  $G$  induced by the vertex set of  $P$ . This shows that at least one of the vertices  $u_1$  and  $v_{n-1}$  has a neighbour among the remaining vertices  $u_n, v_n$  of  $G - P$ ; say,  $v_{n-1}u_n \in E$ . Notice that then  $u_n v_n \notin E$ , for otherwise  $u_1 \dots v_{n-1} u_n v_n u_1$  would be a Hamilton path. Similarly,  $u_1 v_n \notin E$ . Hence, in particular,  $I_P$  contains indices of all the neighbours of  $u_1$  in  $G$ , so  $|I_P| = d_G(u_1)$ . Let now  $K_P = \{1 \leq i \leq n - 1 \mid u_i v_n \in E\}$ . Then  $|K_P| = d_G(v_n)$ , and as  $d_G(u_1) + d_G(v_n) \geq n$ , it follows that there exists  $i_0 \in I_P \cap K_P$ . Then  $v_n u_{i_0} v_{i_0-1} \dots u_1 v_{i_0} u_{i_0+1} \dots v_{n-1} u_n$  is a Hamilton path in  $G$ ; a contradiction.

Let now  $x \in X$  and  $y \in Y$  be a pair of non-adjacent vertices such that  $G$  contains a Hamilton  $x - y$  path  $P$ ; say,  $P = u_1 v_1 \dots u_n v_n$ , where  $X = \{u_1, \dots, u_n\}$ ,  $Y = \{v_1, \dots, v_n\}$ ,  $x = u_1$  and  $y = v_n$ . Put  $I_G = \{1 \leq i \leq n \mid u_i v_i \in E\}$  and  $J_G = \{1 \leq i \leq n \mid u_i v_n \in E\}$ . Then  $|I_G| = d_G(x)$ ,  $|J_G| = d_G(y)$  and  $I_G \cap J_G = \emptyset$ , for if  $i_0 \in I_G \cap J_G$ , then  $u_1 v_{i_0} u_{i_0+1} \dots v_n u_{i_0} v_{i_0-1} \dots v_1 u_1$  is a Hamilton cycle in  $G$ . Hence

$$n \geq |I_G \cup J_G| = |I_G| + |J_G| = d_G(x) + d_G(y) \geq n,$$

so that, for every  $1 \leq i \leq n$ ,

$$\text{either } u_i \in N_G(y) \text{ or else } v_i \in N_G(x). \quad (\star)$$

Let  $d = d_G(y)$ . Denote by  $x_1, \dots, x_d$  those of the vertices  $u_1, \dots, u_n$  that are adjacent to  $y$ , ordered according to the orientation of  $P$  (from  $x$  to  $y$ ). Let  $y_1, \dots, y_d$  be the vertices of  $Y$  that lie on  $P$  next to the respective  $x_1, \dots, x_d$ ; then  $y_d = y$ .

Observe that if  $x_1 = u_i$  with  $i < n - d + 1$ , then there exists  $1 \leq j \leq d - 1$  such that  $y_j = v_l$ , where  $u_{l+1} \notin N_G(y)$ . Then  $v_{l+1} \in N_G(x)$  and we obtain a cycle  $u_1 v_{l+1} u_{l+2} \dots v_n u_l v_{l-1} \dots v_1 u_1$  of length  $2n - 2$  in  $G$ ; a contradiction.

Therefore  $x_1 = u_{n-d+1}$ , and hence  $N_G(y)$  coincides with the set  $\{u_{n-d+1}, \dots, u_n\}$ , call it  $U$ . Then  $\{y_1, \dots, y_d\}$  coincides with  $V := \{v_{n-d+1}, \dots, v_n\}$ , and by  $(\star)$ ,  $N_G(x) = Y \setminus V$ .

Suppose now that, for every  $v \in V$ ,  $N_G(v) \subset U$ . Then, for all  $u \in X \setminus U$  and  $v \in V$ ,  $u$  and  $v$  are non-adjacent, hence  $N_G(u) \subset Y \setminus V$ . Consequently,  $d_G(u_i) \leq n - d$  ( $i \leq n - d$ ), and  $d_G(v_j) \leq d$  ( $j \geq n - d + 1$ ). But  $u_i$  and  $v_j$  being non-adjacent, we also have  $d_G(u_i) + d_G(v_j) \geq n$ , which implies that  $d_G(u_i) = n - d$  and  $d_G(v_j) = d$ , and hence

$$N_G(u_i) = Y \setminus V \text{ and } N_G(v_j) = U \text{ for all } i \leq n - d, j \geq n - d + 1.$$

Thus  $G$  contains a complete bipartite graph  $K_{d,d}$  spanned on the vertices of  $U$  and  $V$ , and a complete bipartite  $K_{n-d, n-d}$  spanned on  $X \setminus U$  and  $Y \setminus V$ .

Now,  $G$  being 2-connected, it must contain two independent edges  $u_{i_1} v_{j_1}$  and  $u_{i_2} v_{j_2}$  for some  $i_1, i_2 \geq n - d + 1$  and  $j_1, j_2 \leq n - d$ . One immediately verifies that such a graph contains a cycle of length  $2n - 2$ , again contradicting the choice of  $G$ .

We can therefore conclude that there exists a vertex  $v_j$ , with  $n - d + 1 \leq j \leq n - 1$ , adjacent to a  $u_i$ , where  $i \leq n - d$ . Then  $u_1 v_i \dots u_j v_n u_n \dots v_j u_i v_{i-1} \dots v_1 u_1$  is a Hamilton cycle in  $G$ . This contradiction completes the proof of the theorem.  $\square$



## 5 Proof of Theorem B

Throughout this section we will frequently refer to the exceptional graph  $G(n, r)$  of Theorem 2.3. Recall that by  $G(n, r)$  we denote a balanced bipartite graph of order  $2n$ , with colour classes  $X = P \cup Q$  and  $Y = R \cup S$ , where  $|P| = |R| = r$ ,  $|Q| = |S| = n - r$ ,  $N_{G(n, r)}(x) = R$  for all  $x \in P$ , and  $N_{G(n, r)}(x) = Y$  for all  $x \in Q$ .

Let, as before,  $g(n, r) = n(n - 1 - r) + r(1 + r) + 1$ . We shall first show the following lemma.

**Lemma 5.1.** *Let  $G = (X, Y; E)$  be a balanced bipartite graph of order  $2n$  and minimal degree  $\delta(G) \geq r \geq 1$ , where  $n \geq 4$  and  $n \geq 2r + 1$ . Let  $\|G\| \geq g(n, r)$ , and assume there exists a pair of vertices  $x \in X$  and  $y \in Y$  such that  $d_G(x) + d_G(y) \leq n$  and  $\delta(G - \{x, y\}) \geq r$ . Then  $G$  contains a cycle of length  $2n - 2$ .*

**Proof:** Suppose  $G$  contains no cycle of length  $2n - 2$ . Then  $G - \{x, y\}$  contains no such cycle either, and as  $\delta(G - \{x, y\}) \geq r$ , Theorem 2.3 implies that

$$\|G - \{x, y\}\| \leq (n - 1)(n - 1 - r) + r^2 = n^2 - 2n - nr + r^2 + r + 1.$$

On the other hand,

$$\|G - \{x, y\}\| \geq g(n, r) - (d_G(x) + d_G(y)) \geq n^2 - 2n - nr + r^2 + r + 1.$$

Hence  $d_G(x) + d_G(y) = n$ , the vertices  $x$  and  $y$  are non-adjacent,  $G - \{x, y\}$  equals  $G(n - 1, r)$ , and  $r \leq (n - 1)/2$ . Without loss of generality, we may assume that  $x$  belongs to the colour class of  $G$  containing  $P \cup Q$  of  $G(n - 1, r)$ .

Now, either  $d_G(x) \geq r + 1$  or  $d_G(x) = r$ . In the first case,  $x$  must have at least two neighbours in  $S$  or else at least one neighbour in both  $S$  and  $R$ . One easily verifies that then  $G$  contains a cycle of length  $2n - 2$ , omitting  $y$  and a single vertex of  $P$ ; a contradiction.

If, in turn,  $d_G(x) = r$ , then  $d_G(y) = n - r$  and  $y$  must have neighbours in both  $P$  and  $Q$ , since  $r \leq (n - 1)/2 < n/2$ . Consequently,  $G$  contains a cycle of length  $2n - 2$ , omitting  $x$  and a vertex of  $S$ , which again contradicts the choice of  $G$ .  $\square$

We are now in position to prove Theorem B.

For a proof by contradiction, consider a graph  $G$  satisfying the assumptions of Theorem B, that does not contain a cycle of length  $2n - 2$ . Observe first that  $\|G\| > n^2/2$ . Indeed, the difference  $g(n, r) - n^2/2$  is always positive. Hence, by Theorem 2.5,  $G$  is not hamiltonian. Consequently, Theorem 2.2 implies that there exists a positive integer  $m \leq n/2$  such that at least one of the sets  $S_m = \{x \in X : d_G(x) \leq m\}$ ,  $T_m = \{y \in Y : d_G(y) \leq m\}$  has cardinality greater than or equal to  $m$ .

Let  $l$  be the least such  $m$ . Without loss of generality, we may assume that  $l$  is realized in  $X$ ; i.e.,  $|\{x \in X : d_G(x) \leq l\}| \geq l$ . Order the vertices of  $X = \{x_1, \dots, x_n\}$  so that  $r \leq d_G(x_1) \leq \dots \leq d_G(x_n)$ . Then, by minimality of  $l$ , we have  $l = \min\{i : d_G(x_i) \leq i\}$ . Of course,  $r \leq l \leq n/2$ . Put  $L = \{x_1, \dots, x_l\}$ .

The rest of the proof proceeds in two cases, depending on  $l$  being equal to or greater than  $r$ .

**Case 1:**

$l = r$ . We will first show that all the vertices of  $Y$  have degrees greater than  $r$ . Suppose to the contrary that there exists  $y_1 \in Y$  with  $d_G(y_1) = r$ . Then

$$\|G - \{x_1, y_1\}\| \geq g(n, r) - 2r = n^2 - n - nr + r^2 - r + 1,$$

and  $\delta(G - \{x_1, y_1\}) \geq r - 1$ . On the other hand, by Theorem 2.3,

$$\|G - \{x_1, y_1\}\| \leq (n-1)(n-r) + (r-1)^2 = n^2 - n - nr + r^2 - r + 1.$$

Hence  $d_G(x_1) + d_G(y_1) = 2r$  so that  $x_1 y_1 \notin E$  and  $G - \{x_1, y_1\}$  equals  $G(n-1, r-1)$ . By comparison of degrees, one readily verifies that  $x_1$  belongs to that colour class of  $G$  that contains  $P \cup Q$  of  $G(n-1, r-1)$ ; in fact,  $L = \{x_1\} \cup P$ . Consider the sets  $R$  and  $S$  of the other colour class of  $G(n-1, r-1)$ . As  $|N_G(x_1)| = r > |R|$  and  $x_1 y_1 \notin E$ , it follows that either  $x_1$  has neighbours in both  $R$  and  $S$  or else it has at least two neighbours in  $S$ . In any case, as in the proof of Lemma 5.1, one easily finds a cycle of length  $2n - 2$  in  $G$ , omitting  $y_1$  and a vertex of  $P$ ; a contradiction. Thus  $d_G(y) \geq r + 1$  for every  $y \in Y$ .

Next observe that every vertex of  $Y$  has a neighbour in  $L$ . Suppose otherwise, and let  $y_1 \in Y$  be such that  $N_G(y_1) \subset X \setminus L$ . Notice that all vertices of  $X \setminus L$  have degrees greater than  $r$ , for otherwise  $g(n, r) \leq \|G\| \leq (r+1)r + (n-r-1)n = g(n, r) - 1$ . Consequently, by removing  $y_1$  and a vertex of  $L$ , say  $x_1$ , we do not decrease the minimal degree in the remainder of  $G$ . But, as  $N_G(y_1) \subset X \setminus L$ , we have  $d_G(y_1) \leq n - r$ , hence  $d_G(x_1) + d_G(y_1) \leq r + (n - r) = n$ , and by Lemma 5.1,  $G$  contains a cycle of length  $2n - 2$ ; a contradiction.

Consider the graph  $G - L$ . Notice that

$$\|G - L\| \geq g(n, r) - r^2 = n^2 - n - nr + r + 1.$$

Moreover, we claim that  $d_{G-L}(x) + d_{G-L}(y) \geq n$  for every pair of non-adjacent  $x \in X \setminus L$  and  $y \in Y$ . For if  $d_{G-L}(x) + d_{G-L}(y) \leq n - 1$  for a pair of non-adjacent  $x \in X \setminus L$  and  $y \in Y$ , then, by the above inequality,

$$\|(G - L) - \{x, y\}\| \geq n^2 - 2n - nr + r + 2 > (n - r - 1)(n - 1),$$

which contradicts  $(G - L) - \{x, y\}$  being a bipartite graph with colour classes of cardinality  $n - r - 1$  and  $n - 1$ .

Taking into account that every vertex in  $Y$  has a neighbour in  $L$ , we now obtain that

$$d_G(x) + d_G(y) \geq n + 1 \quad \text{for all non-adjacent } y \in Y \text{ and } x \in X \setminus L.$$

Let  $\tilde{G}$  be the bipartite graph obtained from  $G$  by joining all the non-adjacent vertices of  $Y$  and  $X \setminus L$ . As  $|X \setminus L| = n - r$  and every  $y \in Y$  has a neighbour in  $L$ , we get that  $d_{\tilde{G}}(y) \geq n - r + 1$  for all  $y \in Y$ . Hence  $d_{\tilde{G}}(x) + d_{\tilde{G}}(y) \geq n + 1$  for every pair of non-adjacent vertices  $x \in X$  and  $y \in Y$ . Therefore, joining all the non-adjacent vertices of  $X$  and  $Y$  in  $\tilde{G}$  with degree sum of at least  $n + 1$  yields a complete bipartite graph  $K_{n,n}$ . As  $\tilde{G}$  was obtained from  $G$  also by joining certain non-adjacent vertices of  $X$  and  $Y$  with degree sum of at least  $n + 1$ , this shows that the  $(n + 1)$ -biclosure of  $G$  equals  $K_{n,n}$ . Thus, by Theorem 2.7,  $G$  contains a Hamilton cycle, which, as we observed at the beginning of this proof, is impossible.

**Case 2:**

$l \geq r + 1$ . In this case  $n \geq 2r + 2$  (as  $l \leq n/2$ ) and  $r \geq 2$  (for otherwise  $l = r = 1$ , by minimality); hence  $|L| \geq 3$ . Moreover,  $d_G(x_{l-1}) = d_G(x_l) = l$ , by minimality of  $l$ .

Suppose first that  $d_G(x) + d_G(y) \geq n + 2$  for every pair of non-adjacent  $x \in X \setminus L$  and  $y \in Y$ . Let  $G'$  be the bipartite graph obtained from  $G$  by joining all the non-adjacent vertices of  $X \setminus L$  and  $Y$ . We claim that every  $y \in Y$  has a neighbour in  $L$  (in  $G'$ ). Suppose otherwise, and let  $y_1 \in Y$  be such that  $N_{G'}(y_1) \subset X \setminus L$ . Then  $d_{G'}(y_1) \leq n - l$ , hence  $d_{G'}(x_1) + d_{G'}(y_1) \leq n$ . Moreover,  $\delta(G' - \{x_1, y_1\}) \geq r$ , as all the vertices in  $X \setminus L$  have degrees of at least  $l \geq r + 1$ , and  $d_{G'}(y) \geq n - l \geq l \geq r + 1$  for all  $y \in Y$ . Then Lemma 5.1 implies that  $G'$  contains a cycle of length  $2n - 2$ , and hence, by Theorem 2.8, so does  $G$ ; a contradiction.

Notice that  $G'$  was obtained from  $G$  by joining only pairs of vertices with degree sum of at least  $n + 2$ . Also, as every vertex  $y \in Y$  has a neighbour in  $L$  (in  $G'$ ), we have  $d_{G'}(y) \geq n - l + 1$ . Recall that  $d_{G'}(x_l) = d_G(x_l) = l$  and  $d_{G'}(x_{l-1}) = d_G(x_{l-1}) = l$ . Hence

$$d_{G'}(x_l) + d_{G'}(y) \geq n + 1 \quad \text{and} \quad d_{G'}(x_{l-1}) + d_{G'}(y) \geq n + 1 \quad \text{for all } y \in Y.$$

Let  $G^{(2)}$  be the graph obtained from  $G'$  by joining  $x_l$  and  $x_{l-1}$  with all the vertices of  $Y$ . Then  $d_{G^{(2)}}(y) \geq n - l + 2$  for all  $y \in Y$ , and as  $d_{G^{(2)}}(x_{l-2}) = d_G(x_{l-2}) \geq l - 1$  (by minimality of  $l$ ), we get that

$$d_{G^{(2)}}(x_{l-2}) + d_{G^{(2)}}(y) \geq n + 1 \quad \text{for all } y \in Y.$$

Let now  $G^{(3)}$  be the graph obtained from  $G^{(2)}$  by joining  $x_{l-2}$  with all the non-adjacent vertices of  $Y$ . In general, let  $G^{(m)}$  ( $m \geq 3$ ) be obtained from  $G^{(m-1)}$  by joining  $x_{l-m+1}$  with all the non-adjacent vertices of  $Y$ . Then  $G^{(l)} = K_{n,n}$ , and  $G^{(m)}$  is obtained from  $G^{(m-1)}$  by joining only pairs of vertices with degree sum of at least  $n + 1$ . Thus  $G^{(l)} = BC l_{n+1}(G)$ , so that the  $(n + 1)$ -biclosure of  $G$  is a complete bipartite graph. Now Theorem 2.7 implies that  $G$  contains a Hamilton cycle, which again leads to contradiction.

To complete the proof, it remains to consider the case when there is a pair of non-adjacent  $x^0 \in X \setminus L$  and  $y^0 \in Y$  with  $d_G(x^0) + d_G(y^0) \leq n + 1$ . This however can only happen when  $n = 2r + 2$  or  $n = 2r + 3$ . For let us suppose that  $n \geq 2r + 4$ , and put  $f(l) = l^2 + (n - l - 1)(n - 1) + n + 2$ . We show  $\|G\| < f(l)$  and  $f(l) \leq g(n, r)$ , and thus obtain a contradiction with the assumption  $\|G\| \geq g(n, r)$ . If  $G$  contains a pair of non-adjacent vertices  $x \in X \setminus L$  and  $y \in Y$  with  $d_G(x) + d_G(y) \leq n + 1$ , then

$$\|G\| \leq |L| \cdot l + |X \setminus (L \cup \{x\})| \cdot |Y \setminus \{y\}| + d_G(x) + d_G(y) \leq f(l) - 1.$$

As the derivative of  $f$  equals  $f'(l) = -n + 2l + 1$ , it follows that  $f(l)$  is decreasing for  $l \leq (n - 1)/2$ , and hence maximal at  $l = r + 1$ . One immediately verifies that  $f(r + 1) \leq g(n, r)$  for  $n \geq 2r + 4$ . If, on the other hand,  $l > (n - 1)/2$ , then  $l = n/2$  (since  $l \leq n/2$ ), and it is again immediate to check that  $f(n/2) \leq g(n, r)$  for  $n \geq 2r + 4$ .

**Subcase 2.1:**

$n = 2r + 2$ . Then  $r + 1 \leq l \leq n/2$  yields  $l = r + 1$ , and we obtain

$$\|G - \{x^0, y^0\}\| \geq g(2r + 2, r) - (2r + 3) = 3r^2 + 3r. \quad (1)$$

On the other hand,

$$\|G - \{x^0, y^0\}\| \leq |L| \cdot l + |X \setminus (L \cup \{x^0\})| \cdot |Y \setminus \{y^0\}| = 3r^2 + 3r + 1. \quad (2)$$

Hence

$$3r^2 + 3r \leq \|G - \{x^0, y^0\}\| \leq 3r^2 + 3r + 1 \text{ and } 2r + 2 \leq d_G(x^0) + d_G(y^0) \leq 2r + 3.$$

Suppose first that  $\|G - \{x^0, y^0\}\| = 3r^2 + 3r + 1$ . Then, by (2),  $d_G(x) = l$  for all  $x \in L$ , and  $N_G(y^0) \cap L = \emptyset$ ; in particular,  $d_G(x_1) + d_G(y^0) \leq l + (n - l) = n$ . Moreover,  $N_G(y) \supset X \setminus (L \cup \{x^0\})$  for all  $y \in Y \setminus \{y^0\}$ , and  $d_G(x) \geq r + 1$  for all  $x \in X$ , so that  $\delta(G - \{x_1, y^0\}) \geq r$ , and by Lemma 5.1,  $G$  contains a cycle of length  $2n - 2$ ; a contradiction.

Therefore we may assume that  $\|G - \{x^0, y^0\}\| = 3r^2 + 3r$ . By (1),  $d_G(x^0) + d_G(y^0) = 2r + 3$ , and what's more,  $d_G(x) + d_G(y) \geq 2r + 3$  for all non-adjacent  $x \in X \setminus L$  and  $y \in Y$ . Indeed, if  $d_G(x^1) + d_G(y^1) \leq 2r + 2$  for some non-adjacent  $x^1 \in X \setminus L$  and  $y^1 \in Y$ , then by (1) and (2),  $\|G - \{x^1, y^1\}\| = 3r^2 + 3r + 1$ , which leads to contradiction, as above.

We will now show that  $|N_G(L)| > r + 1$ . Suppose otherwise, that is, suppose  $|N_G(L)| = l = r + 1$ . Then  $N_G(y^0) \cap L = \emptyset$ , for else  $N_G(L) \ni y^0$  implies

$$\|G - \{x^0, y^0\}\| \leq |L| \cdot (l - 1) + |X \setminus (L \cup \{x^0\})| \cdot |Y \setminus \{y^0\}| = 3r^2 + 2r,$$

which is impossible. Therefore  $d_G(y^0) = n - l - 1 = r$ ; in particular,  $d_G(x_1) + d_G(y^0) \leq l + r < n$ . Notice that, as  $G - \{x^0, y^0\}$  only has one edge less than the right-hand side of (2), every neighbour of  $y^0$  in  $G$  has degree at least  $n - 2 = 2r$ , and every neighbour of  $x_1$  has at least  $l - 1 = r$  other neighbours in  $L$  ( $x_1$  being the only vertex whose degree could be less than  $l$ ). Thus  $\delta(G - \{x_1, y^0\}) \geq r$ , and we get a contradiction, by Lemma 5.1. Thus  $|N_G(L)| > r + 1$ .

It is now not difficult to see that  $BCl_{n+1}(G) = K_{n,n}$ : Recall that we have verified that  $d_G(x) + d_G(y) \geq 2r + 3 = n + 1$  for all non-adjacent  $x \in X \setminus L$  and  $y \in Y$ . Let  $G'$  be the graph obtained from  $G$  by joining all the non-adjacent vertices of  $X \setminus L$  and  $Y$ . Next observe that, by minimality of  $l = r + 1$ ,  $d_{G'}(x_{r+1}) = d_G(x_{r+1}) = r + 1$ , and as  $|N_G(L)| > r + 1$ , at least one non-neighbour of  $x_{r+1}$ , say  $y'$ , has a neighbour among the other vertices of  $L$ . Hence  $|N_{G'}(y')| \geq |X \setminus L| + 1$ , so that  $d_{G'}(x_{r+1}) + d_{G'}(y') \geq (r + 1) + (r + 2) = n + 1$ . Let  $G^{(2)}$  be obtained from  $G'$  by joining  $x_{r+1}$  with  $y'$ , and hence increasing the degree of  $x_{r+1}$  to  $r + 2$ . Then  $d_{G^{(2)}}(x_{r+1}) + d_{G^{(2)}}(y) \geq n + 1$  for all  $y \in Y$ . Let  $G^{(3)}$  be obtained from  $G^{(2)}$  by joining  $x_{r+1}$  with all the non-adjacent vertices of  $Y$ . Now  $d_{G^{(3)}}(y) \geq r + 2$  for all  $y \in Y$ . By minimality of  $l$  again,  $d_{G^{(3)}}(x_r) = d_G(x_r) = r + 1$ , and hence  $d_{G^{(3)}}(x_r) + d_{G^{(3)}}(y) \geq 2r + 3$  for all  $y \in Y$ . Let  $G^{(4)}$  be obtained from  $G^{(3)}$  by joining  $x_r$  with all the non-adjacent vertices of  $Y$ . Then  $d_{G^{(4)}}(y) \geq r + 3$  for all  $y \in Y$ , and hence, as  $\delta(G^{(4)}) \geq \delta(G) \geq r$ ,  $d_{G^{(4)}}(x) + d_{G^{(4)}}(y) \geq 2r + 3$  for all non-adjacent  $x \in X$  and  $y \in Y$ . Joining all the non-adjacent pairs  $x \in X$ ,  $y \in Y$  of  $G^{(4)}$  with degree sum of at least  $n + 1$  we thus obtain  $K_{n,n}$ . Since at each stage we only joined pairs of vertices with degree sum of at least  $n + 1$ , this shows that  $K_{n,n} = BCl_{n+1}(G)$ . By Theorem 2.7,  $G$  contains a Hamilton cycle; a contradiction.

**Subcase 2.2:**

$n = 2r + 3$ . Again,  $r + 1 \leq l \leq n/2$  yields  $l = r + 1$ , and we have

$$\|G - \{x^0, y^0\}\| \geq g(2r + 3, r) - (2r + 4) = 3r^2 + 6r + 3,$$

and, on the other hand,

$$\|G - \{x^0, y^0\}\| \leq |L| \cdot l + |X \setminus (L \cup \{x^0\})| \cdot |Y \setminus \{y^0\}| = 3r^2 + 6r + 3.$$

Therefore both inequalities must, in fact, be equalities; in particular,  $d_G(x_1) = l$  and  $d_G(x) \geq r + 1$  for all  $x \in X$ ,  $N_G(y^0) \cap L = \emptyset$ , so that  $d_G(y^0) \leq n - l$ , and finally  $|N_G(y)| \geq |X \setminus (L \cup \{x^0\})| = r + 1$  for all  $y \in Y \setminus \{y^0\}$ . Thus, again,  $G$  with the vertices  $x_1, y^0$  satisfies the assumptions of Lemma 5.1, hence  $G$  contains a cycle of length  $2n - 2$ ; a contradiction. This completes the proof of Theorem B.  $\square$

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