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# Independent sets in $(P_6, \text{diamond})$ -free graphs

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We prove that on the class of  $(P_6, \text{diamond})$ -free graphs the Maximum-Weight Independent Set problem and the Minimum-Weight Independent Dominating Set problem can be solved in polynomial time.

**Keywords:** Maximum-Weight Independent Set; Minimum-Weight Independent Dominating Set; polynomial-time algorithm.

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## 1 Introduction

An *independent set* (or a *stable set*) in a graph  $G$  is a subset of pairwise nonadjacent vertices of  $G$ . An independent set of  $G$  is *maximal* if it is not properly contained in any other independent set of  $G$ .

The Maximum-Weight Independent Set (*WIS*) problem is the following: Given a graph  $G = (V, E)$  and a weight function  $w$  on  $V$ , determine an independent set of  $G$  of maximum weight. Let  $\alpha_w(G)$  denote the maximum weight of an independent set of  $G$ . The WIS problem reduces to the *IS* problem if all vertices  $v$  have the same weight  $w(v) = 1$ .

The WIS problem is NP-hard [34] and remains difficult for cubic graphs [27] and for planar graphs [26], while it can be efficiently solved for various graph classes which include perfect graphs [33] (and the class of perfect graphs includes the chordal graphs),  $K_{1,3}$ -free graphs [2, 37, 40, 42, 45], and  $2K_2$ -free graphs [21, 22, 38].

The Minimum-Weight Independent Dominating Set (*WID*) problem is the following: Given a graph  $G = (V, E)$  and a weight function  $w$  on  $V$ , determine a maximal independent set of  $G$  of minimum weight. Let  $\iota_w(G)$  denote the minimum weight of a maximal independent set of  $G$ . The WID problem reduces to the *ID* problem if all vertices  $v$  have the same weight  $w(v) = 1$ .

The WID problem is NP-hard [28] and remains difficult for chordal graphs [18] and for  $2P_3$ -free perfect graphs [46], while it can be efficiently solved for various graph classes which include permutation graphs [15], locally independent well-dominated graphs [47], and  $2K_2$ -free graphs [21, 22, 38].

Both WIS and WID remain difficult for triangle-free graphs [43]. Also, for both IS and ID, the class of  $P_5$ -free graphs is the unique minimal class, defined by forbidding a single connected subgraph, for which the computational complexity is an open question (see [1, 3, 7]).

On the other hand, several papers introduced structural properties on graphs containing no long induced paths (see e.g. [5, 6, 19, 39]), often applied to design efficient algorithms for solving various NP-hard

problems, including WIS or WID, in subclasses of such graphs: concerning subclasses of  $P_5$ -free graphs, see e.g. [4, 9, 10, 11, 13, 16, 17, 23, 29, 31, 36]; concerning subclasses of  $P_6$ -free graphs, see e.g. [9, 24, 30, 35, 41, 44].

Let us focus on two such graph subclasses which involve triangle-free graphs as well.

The class of  $(P_5, \text{diamond})$ -free graphs: a recent paper [8] shows that such graphs have bounded clique-width and that a corresponding clique-width expression can be constructed in  $\mathcal{O}(n + m)$  time, which implies that a large class of NP-hard problems including WIS and WID can be solved for such graphs in  $\mathcal{O}(n + m)$  time.

The class of  $(P_6, \text{triangle})$ -free graphs: a recent paper [14] shows that such graphs have bounded clique-width and that a corresponding clique-width expression can be constructed in  $\mathcal{O}(n^2)$  time, which implies that a large class of NP-hard problems including WIS and WID can be solved for such graphs in  $\mathcal{O}(n^2)$  time.

This paper introduces a proof that WIS and WID can be solved for  $(P_6, \text{diamond})$ -free graphs in  $\mathcal{O}(n^7)$  time.

## 2 Notation and preliminaries

For any missing notation or reference, let us refer to [12].

Throughout this paper let  $G = (V, E)$  be a finite undirected graph without self-loops and multiple edges and let  $|V| = n, |E| = m$ . For every  $u \in V$ , let  $N(u) = \{v \in V : uv \in E\}$  be the set of *neighbors* of  $u$ . Let  $U, W$  be two subsets of  $V$ . Let  $N(U) = \{v \in V \setminus U : \text{there exists } u \in U \text{ such that } uv \in E\}$ . Let  $N_W(U) = N(U) \cap W$ . Let us say that  $U$  has a *join* (a *co-join*, respectively) with  $W$ , if each vertex in  $U$  is adjacent (is nonadjacent) to each vertex in  $W$ . Let  $v \in V$ . Let us say that:  $v$  *contacts*  $U$  if  $v$  is adjacent to some vertex of  $U$ ;  $v$  is *universal* for  $U$  if  $v$  is adjacent to each vertex of  $U$ ;  $v$  is *partial* to  $U$  if  $v$  contacts  $U$  but is not universal for  $U$ . Then let us say that  $U$ , with  $\emptyset \subset U \subset V$ , is a *module* of  $G$  – often called *homogeneous set* in the literature – if no vertex of  $V \setminus U$  is partial to  $U$ .

Let  $G[U]$  denote the subgraph of  $G$  induced by the vertex subset  $U$ . For any graph  $F$ ,  $G$  is *F-free* if  $G$  contains no induced subgraph isomorphic to  $F$ .

A *component* of  $G$  is the vertex set of a maximal connected subgraph of  $G$ . A component of  $G$  is *trivial* if it is a singleton, and *nontrivial* otherwise.

Concerning WIS and WID, algorithmically an easy reduction works if the graph is disconnected: that is, if  $G$  has components  $V_1, \dots, V_k$ , then  $\alpha_w(G) = \alpha_w(G[V_1]) + \dots + \alpha_w(G[V_k])$  and  $\iota_w(G) = \iota_w(G[V_1]) + \dots + \iota_w(G[V_k])$ .

A path  $P_k$  has vertices  $v_1, v_2, \dots, v_k$  and edges  $v_j v_{j+1}$  for  $1 \leq j < k$ . A cycle  $C_k$  has vertices  $v_1, v_2, \dots, v_k$  and edges  $v_j v_{j+1}$  for  $1 \leq j \leq k$  (subscript addition taken modulo  $k$ ). A *triangle* is a graph of three vertices  $a, b, c$  and edges  $ab, ac, bc$ . A *diamond* is a graph of four vertices  $a, b, c, d$  and edges  $ab, ac, ad, bc, bd$ .

A *clique* of  $G$  is a set of pairwise adjacent vertices of  $G$ . Notice that each component of  $G$  is a clique if and only if  $G$  is  $P_3$ -free.

A graph is *chordal* if it contains no induced  $C_k$ ,  $k \geq 4$ .

For chordal graphs, WIS and ID can be efficiently solved (see [25] and [20], respectively), while WID remains NP-hard on them [18].

In [32] the authors proved that distance-hereditary graphs have bounded clique-width, and that a corresponding clique-width expression can be constructed in  $\mathcal{O}(n + m)$  time. Since chordal diamond-free

graphs are distance-hereditary, a direct consequence is:

**Theorem 1 ([32])** *Both the WIS and the WID problems can be solved for chordal diamond-free graphs in  $\mathcal{O}(n + m)$  time.*  $\square$

In [14] the authors proved that  $(P_6, \text{triangle})$ -free graphs have bounded clique-width, and that a corresponding clique-width expression can be constructed in  $\mathcal{O}(n^2)$  time. A direct consequence is:

**Theorem 2 ([14])** *Both the WIS and the WID problems can be solved for  $(P_6, \text{triangle})$ -free graphs in  $\mathcal{O}(n^2)$  time.*  $\square$

Obviously, the WIS (or WID) problem on a graph  $G$  with vertex weight function  $w$  can be reduced to the same problem on subgraphs  $G[V \setminus N(v)]$  for every  $v \in V$  in the following way:

$$\alpha_w(G) = \max\{\alpha_w(G[V \setminus N(v)]) \mid v \in V\}$$

$$\iota_w(G) = \min\{\iota_w(G[V \setminus N(v)]) \mid v \in V\}$$

Thus, whenever WIS (or WID) is solvable in time  $T$  for every subgraph  $G[V \setminus N(v)]$  of  $G$  with  $v \in V$ , then it is solvable for  $G$  in time  $nT$ , plus  $\mathcal{O}(n^3)$  additional steps to generate those subgraphs.

Let us conclude with an observation which will be often used later.

**Observation 1** *Let  $G = (V, E)$  be a graph, and  $U \subseteq V$  with  $|U| = k$ . If one can solve WIS (or WID) for each induced subgraph of  $G[V \setminus U]$  in time  $T$ , then one can solve WIS (or WID) for  $G$  in time  $2^k(T + n^2)$ .*

**Proof:** Let  $I(U)$  be the family of independent sets of  $G[U]$ . Then to solve WIS (or WID) for  $G$  one can consider WIS (or WID) for  $|I(U)|$  subgraphs of  $G$ , i.e., for  $G[V \setminus U]$  and for  $G[I \cup (V \setminus (N(I) \cup U))]$  for every  $I \in I(U)$ . Since  $|I(U)| \leq 2^k$ , the assertion follows.  $\square$

**Remark:** The results of the next section are introduced only for WIS, by meaning that they hold for WID as well (by interchanging WIS with WID, and  $\alpha$  with  $\iota$ ).

### 3 Independent sets in $(P_6, \text{diamond})$ -free graphs

Let us introduce a preliminary result.

**Definition 1** *A graph  $G = (V, E)$  is green if there exists a partition  $\{X, Y\}$  of  $V$  (with  $X$  or  $Y$  possibly empty) such that:*

- (i)  $G[X]$  and  $G[Y]$  are  $P_3$ -free;
- (ii) each component of  $G[Y]$  is a module of  $G$ ;
- (iii) each vertex in  $Y$  is adjacent to at most one vertex in each component of  $G[X]$ .

Notice that every  $P_3$ -free graph is green.

**Lemma 1** *One can solve WIS for every green  $P_6$ -free graph in  $\mathcal{O}(n^3)$  time.*

**Proof:** Let  $G = (V, E)$  be a green  $P_6$ -free graph. Assume without loss of generality that  $G$  is connected. Let  $\{X, Y\}$  be a partition of  $V$  according to Definition 1. In particular, to our aim, by ii one can assume that  $Y$  is an independent set. In fact, one can contract each component  $K$  of  $G[Y]$  into a singleton  $u$  with  $w(u) = \alpha_w(G[K])$ : that can be done in polynomial time since each component of  $G[Y]$  is a clique.

Let  $W$  be the family of nontrivial components of  $G[X]$ . In particular, to our aim (similarly to above), one can assume that in each component  $Q \in W$  at most one vertex is nonadjacent to any vertex in  $Y$ .

**Claim 3.1** *There exists  $y^* \in Y$  such that  $y^*$  contacts every element of  $W$ .*

**Proof:** For any  $y \in Y$ , let  $J(y) = \{Q \in W : y \text{ contacts } Q\}$ . Let  $y^* \in Y$  be such that  $|J(y^*)| \geq |J(y)|$  for every  $y \in Y$ . We state that this vertex  $y^*$  is a proper choice for Claim 3.1. Assume for a contradiction that there exists a component  $Q \in W$  such that  $y^*$  does not contact  $Q$ . Since  $G$  is connected, one can select  $y \in Y$  belonging to a shortest path from  $y^*$  to  $Q$ , such that  $y$  contacts  $Q$ . By definition of  $W$  and by iii,  $Q$  contains two (adjacent) vertices  $q_1$  and  $q_2$  such that  $y$  is adjacent to  $q_1$  and nonadjacent to  $q_2$ . Then, since  $G$  is  $P_6$ -free,  $y$  contacts all the elements of  $W$  which are contacted by  $y^*$ . This implies  $|J(y^*)| < |J(y)|$ , a contradiction.  $\square$

**Claim 3.2** *There exists at most one element of  $W$  of cardinality greater than 2.*

**Proof:** Assume for a contradiction that there exist two elements of  $W$ , say  $\tilde{Q}$  and  $Q$ , with  $|\tilde{Q}| \geq 3$  and  $|Q| \geq 3$ . Let  $y^* \in Y$  according to Claim 3.1. Let  $\tilde{q} \in \tilde{Q}$  and  $q \in Q$  be adjacent to  $y^*$ . Since  $|\tilde{Q}| \geq 3$ , there exists  $a \in Y$  adjacent to  $q_a \in \tilde{Q}$ , with  $q_a \neq \tilde{q}$ . Then  $a$  is adjacent to  $\tilde{q}$ , otherwise  $a, q_a, \tilde{q}, y^*, q$  and any vertex in  $Q$  nonadjacent to  $a$  induce a  $P_6$ . Then, since  $|Q| \geq 3$ , there exists  $b \in Y$  adjacent to  $q_b \in Q$ , with  $q_b \neq q$ . By symmetry, one has that  $b$  is adjacent to  $\tilde{q}$ . Since  $|Q| \geq 3$ , there exists  $q' \in Q$  such that  $q' \neq q$  and  $q'$  is nonadjacent to  $b$ . Then  $q', q_b, b, \tilde{q}, q_a, a$  induce a  $P_6$ , a contradiction.  $\square$

If  $|Q| \leq 2$  for every  $Q \in W$ , then by iii  $G$  is triangle-free. Otherwise, by Claim 3.2 there exists at most one element, say  $\tilde{Q}$ , of  $W$  of cardinality greater than 2. One can solve WIS in  $G$  by solving WIS in  $G[V \setminus \tilde{Q}]$  and in  $G[V \setminus N(\tilde{q})]$  for every  $\tilde{q} \in \tilde{Q}$ . Since such graphs are triangle-free, the lemma follows by Theorem 2.  $\square$

### 3.1 Deleting $C_6$ 's in $(P_6, \text{diamond})$ -free graphs

Throughout this subsection assume that  $G = (V, E)$  is a  $(P_6, \text{diamond})$ -free graph containing a 6-cycle  $C$ , say with vertices  $v_i$  and edges  $v_i v_{i+1}$ ,  $i \in \{1, \dots, 6\}$  (subscript addition taken modulo 6). Let  $N(C)$  be the set of vertices from  $V \setminus C$  which are adjacent to some vertex in  $C$ . For any subset  $S$  of  $C$ , let  $M_S$  be the set formed by those vertices in  $N(C)$  which are adjacent to each vertex in  $S$  and are nonadjacent to each vertex in  $C \setminus S$ . In particular, let us write  $M_1$  for  $S = \{v_1\}$ ,  $M_{1,2}$  for  $S = \{v_1, v_2\}$ , and so on. Then let  $Z(k)$  denote the set of vertices of  $N(C)$  having exactly  $k$  neighbors in  $C$ .

Since  $G$  is  $(P_6, \text{diamond})$ -free:  $Z(1) = Z(5) = Z(6) = \emptyset$ ; each vertex in  $Z(2)$  belongs to some of the sets  $M_{i,i+2}$  or  $M_{i,i+3}$ ,  $i \in \{1, \dots, 6\}$  (subscript addition taken modulo 6); each vertex in  $Z(3)$  belongs to some of the sets  $M_{i,i+2,i+4}$  or  $M_{i,i+2,i+3}$  or  $M_{i,i+3,i+4}$ ,  $i \in \{1, \dots, 6\}$  (subscript addition taken modulo 6).

**Lemma 2** *Every component of  $G[Z(0)]$  is green.*

**Proof:** Let  $K$  be a component of  $G[Z(0)]$ . Since  $G$  is connected, there exists  $v \in V \setminus K$  which contacts  $K$ . If  $v$  is universal for  $K$ , then  $G[K]$  is  $P_3$ -free (since  $G$  is diamond-free). Then let us assume that  $v$  is partial to  $K$ , and prove that  $G[K]$  is green. Let us write  $X = K \cap N(v)$ , and  $Y = K \setminus N(v)$ . Since  $G$  is diamond-free,  $G[X]$  is  $P_3$ -free. Let  $T$  be a component of  $G[Y]$ . Then  $T$  is a module of  $G[K]$ : otherwise, for any  $x \in X$  partial to  $T$ , one has that two adjacent vertices in  $T$  together with  $x$ ,  $v$  and two vertices in  $C$  would induce a  $P_6$  (since  $G$  is  $(P_6, \text{diamond})$ -free,  $v$  is the endpoint of an induced  $P_3$  involving two vertices in  $C$ ). Then  $T$  is a clique (since  $G$  is diamond-free), i.e.,  $G[Y]$  is  $P_3$ -free. Furthermore, each vertex in  $Y$  is adjacent to at most one vertex in each component of  $X$ , otherwise a diamond arises involving  $v$ . Then the lemma follows.  $\square$

Let us fix any vertex of  $C$ , say  $v_2$ .

Let us prove that WIS can be solved for  $G[V \setminus N(v_2)]$  in  $\mathcal{O}(n^6)$  time.

A partition of  $V \setminus N(v_2)$  is given by  $\{\{v_2, v_4, v_5, v_6\}, M_{1,3,4,6}, M_{1,3,5}, M_{1,3,4}, M_{1,4,5}, M_{3,5,6}, M_{3,6,1}, M_{4,6,1}, M_{6,3,4}, M_{1,3}, M_{1,4}, M_{1,5}, M_{3,5}, M_{3,6}, M_{4,6}, Z(0)\}$ .

Since  $G$  is diamond-free, the sets  $M_{1,3,4,6}, M_{1,3,4}, M_{1,4,5}, M_{3,5,6}, M_{3,6,1}, M_{4,6,1}, M_{6,3,4}$  have cardinality at most one. Then, by Observation 1, to our aim it is sufficient to prove that WIS can be solved for each induced subgraph of  $G[U]$  in polynomial time, where a partition of  $U$  is given by

$\{M_{1,3,5}, M_{1,3}, M_{1,4}, M_{1,5}, M_{3,5}, M_{3,6}, M_{4,6}, Z(0)\}$ .

Since  $G$  is diamond-free:  $M_{1,3,5}, M_{1,3}, M_{1,4}, M_{1,5}, M_{3,5}, M_{3,6}, M_{4,6}$  are independent sets. Since  $G$  is  $(P_6, \text{diamond})$ -free:  $M_{1,3,5} \cup M_{1,3} \cup M_{1,5} \cup M_{3,5}$  is an independent set;  $M_{1,3} \cup M_{1,5} \cup M_{3,5} \cup M_{4,6}$  has a co-join with  $Z(0)$ .

For any  $W \subseteq U$ , let us write  $W^* = W \cap Z(0)$ .

For any  $W \subseteq U$ , let us say that a component  $K$  of  $G[W^*]$  is of:

**type 1** if  $K$  is not a clique and there exists a vertex in  $W \setminus Z(0)$  partial to  $K$ ;

**type 2** if  $K$  is a clique and there exists a vertex in  $W \setminus Z(0)$  partial to  $K$ ;

**type 3** otherwise.

Let  $T_1(W), T_2(W), T_3(W)$  respectively denote the union of components of  $G[W^*]$  of type 1, 2, 3.

Let us fix a subset  $W \subseteq U$ .

Notice that  $M_{1,3,5}$  is an independent set, since  $G$  is diamond-free.

Let us consider the following binary relation ' $\leq_1$ ' on  $M_{1,3,5}$ : for any  $a, b \in M_{1,3,5}$ ,  $a \leq_1 b$  if  $N_{T_1(W)}(a) \subseteq N_{T_1(W)}(b)$ . It is immediate to verify that  $(M_{1,3,5}, \leq_1)$  is a partially ordered set.

**Lemma 3** *Let  $y \in M_{1,3,5}$  be maximal for  $(M_{1,3,5}, \leq_1)$ . Then  $G[T_1(W) \setminus N(y)]$  is  $P_3$ -free.*

**Proof:** First let us prove that  $y$  contacts every component  $Q$  of type 1 of  $G[W^*]$ . Assume for a contradiction that there exists a component  $Q_1$  of type 1 of  $G[W^*]$  such that  $y$  does not contact  $Q_1$ . By definition of component of type 1, there exists  $y_1 \in W \setminus Z(0)$  partial to  $Q_1$ . Since  $G$  is  $P_6$ -free (also recall that  $W \subseteq U$ ),  $y_1 \in M_{1,3,5}$ . By the maximality of  $y$  there exists a vertex  $\tilde{q}$  in some component  $\tilde{Q}$  of type 1 of  $G[W^*]$  such that  $\tilde{q}$  is adjacent to  $y$  and nonadjacent to  $y_1$ . Then  $\tilde{q}, y, v_1, y_1$  and two adjacent vertices of  $Q_1$  induce a  $P_6$ , a contradiction.

To conclude the proof of the lemma one has to show that if  $y$  contacts a component  $Q$  of type 1 of  $G[W^*]$ , then  $G[Q \setminus N(y)]$  is  $P_3$ -free. This can be shown by applying the last part of the argument of Lemma 2.  $\square$

Let us consider the following binary relation ‘ $\leq_2$ ’ on  $M_{1,3,5}$ : for any  $a, b \in M_{1,3,5}$ ,  $a \leq_2 b$  if  $N_{T_2(W)}(a) \subseteq N_{T_2(W)}(b)$ . It is immediate to verify that  $(M_{1,3,5}, \leq_2)$  is a partially ordered set.

**Lemma 4** *Let  $y \in M_{1,3,5}$  be maximal for  $(M_{1,3,5}, \leq_2)$ . If  $T_1(W) = \emptyset$ , then  $G[(W \setminus N(y))^*]$  admits at most one component of type 2.*

**Proof:** Let  $Q_1$  be a component of type 2 of  $G[(W \setminus N(y))^*]$ . By definition of component of type 2 there exists  $y_1 \in (W \setminus N(y)) \setminus Z(0)$  partial to  $Q_1$ . Since  $G$  is  $P_6$ -free (also recall that  $W \subseteq U$ ),  $y_1 \in M_{1,3,5}$ . Let  $q_1 \in Q_1$  be adjacent to  $y_1$ ; let  $q' \in Q_1$  be nonadjacent to  $y_1$ . Since  $T_1(W) = \emptyset$ , there exists a component of type 2 of  $G[W^*]$ , say  $\tilde{Q}_1$ , such that  $Q_1 \subseteq \tilde{Q}_1$ . In particular, there exists  $\tilde{q}_1 \in \tilde{Q}_1 \setminus Q_1$  adjacent to  $y$  and nonadjacent to  $y_1$ , otherwise, by the maximality of  $y$  there would exist a vertex  $t \in W \setminus \tilde{Q}_1$  adjacent to  $y$  and nonadjacent to  $y_1$ , i.e., vertices  $q', q_1, y_1, v_1, y, t$  would induce a  $P_6$ .

Let us prove that  $Q_1$  is the unique component of type 2 of  $G[(W \setminus N(y))^*]$ . Assume for a contradiction that there exists another component of type 2, say  $Q_2$ , of  $G[(W \setminus N(y))^*]$ . Notice that  $y_1$  is nonadjacent to any vertex  $q_2$  of  $Q_2$ , otherwise  $q', \tilde{q}_1, y, v_1, y_1$  and  $q_2$  would induce a  $P_6$ . Then let  $y_2 \in M_{1,3,5}$  be partial to  $Q_2$ . By symmetry,  $y_2$  is nonadjacent to any vertex of  $Q_1$ . Then a vertex of  $Q_2$ ,  $y_2, v_1, y_1, q_1, q'$  induce a  $P_6$ , a contradiction.  $\square$

Now, let us consider the following cases:

1.  $T_1(U) = T_2(U) = \emptyset$ .

Then each component  $K$  of  $G[Z(0)]$  is a module of  $G[U]$ . Then, to our aim, one can assume that  $K$  is a singleton. In fact, one can contract  $K$  into a singleton  $u$  with  $w(u) = \alpha_w(G[K])$ : that can be done in polynomial time by Lemmas 2 and 1. So in general, one can assume that  $Z(0)$  is an independent set. One can solve WIS in  $G[U]$  by solving WIS in  $G[U \setminus M_{1,4}]$  and in  $G[U \setminus N(y)]$ , for every  $y \in M_{1,4}$ .

That can be done in  $\mathcal{O}(n^3)$  time. In fact, by the assumptions and by the above properties, one can verify that  $G[U \setminus M_{1,4}]$  is triangle-free, and that  $G[U \setminus N(y)]$  is triangle-free for every vertex  $y \in M_{1,4}$  (in particular, no vertex of  $M_{3,6} \setminus N(y)$  is adjacent to a vertex of  $Z(0) \setminus N(y)$ , otherwise a  $P_6$  arises). Then the assertion follows by Theorem 2.

For the other two cases we note that the existence of a component  $Q$  of type 1 or type 2 in  $U \cap Z(0)$  implies the existence of a vertex  $a \in M_{1,3,5}$  which is partial to  $Q$ , similarly to the proof above. So,  $M_{1,3,5}$  is nonempty.

2.  $T_1(U) = \emptyset, T_2(U) \neq \emptyset$ .

Based on  $(M_{1,3,5}, \leq_2)$ , the vertices  $y_1, \dots, y_h$  of  $M_{1,3,5}$  can be totally ordered so that  $y_i$  is maximal for  $(\{y_i, \dots, y_h\}, \leq_2)$  for  $i = 1, \dots, h$ . Then one can solve WIS in  $G[U]$  by sequentially solving WIS in  $G[U \setminus N(y_1)]$ , in  $G[(U \setminus \{y_1, \dots, y_{i-1}\}) \setminus N(y_i)]$  for  $i = 2, \dots, h$ , and in  $G[U \setminus M_{1,3,5}]$ .

That can be done in  $\mathcal{O}(n^5)$  time. In fact, let us first consider  $G[U \setminus N(y_1)]$ . If  $G[(U \setminus N(y_1))^*]$  admits no component of type 2, then one can refer to Case 1. Otherwise, by Lemma 4,  $G[(U \setminus N(y_1))^*]$  admits a unique component of type 2, say  $Q$ . Then one can solve WIS in  $G[U \setminus N(y_1)]$  by solving WIS in  $G[(U \setminus N(y_1)) \setminus Q]$  and in  $G[(U \setminus N(y_1)) \setminus N(q)]$ , for every  $q \in Q$ : since for each of such graphs  $G[H]$  one has that  $G[H^*]$  has no component of type 2, one can refer to Case 1 and to Lemmas 1 and 2. Now, let us consider  $G[(U \setminus \{y_1, \dots, y_{i-1}\}) \setminus N(y_i)]$  for  $i = 2, \dots, h$ : by the mentioned total order, one can apply the argument applied for  $G[U \setminus N(y_1)]$  in order to show that WIS can be

solved for such graphs in polynomial time. Finally, let us consider  $G[U \setminus M_{1,3,5}]$ : since no vertex in  $U \setminus M_{1,3,5}$  is partial to any component of  $G[Z(0)]$  (otherwise a  $P_6$  arises), one can refer to Case 1.

3.  $T_1(U) \neq \emptyset$ .

Based on  $(M_{1,3,5}, \leq_1)$ , the vertices  $y_1, \dots, y_h$  of  $M_{1,3,5}$  can be totally ordered so that  $y_i$  is maximal for  $(\{y_i, \dots, y_h\}, \leq_1)$  for  $i = 1, \dots, h$ . Then one can solve WIS in  $G[U]$  by sequentially solving WIS in  $G[U \setminus N(y_1)]$ , in  $G[(U \setminus \{y_1, \dots, y_{i-1}\}) \setminus N(y_i)]$  for  $i = 2, \dots, h$ , and in  $G[U \setminus M_{1,3,5}]$ .

That can be done in  $\mathcal{O}(n^6)$  time. In fact, let us first consider  $G[U \setminus N(y_1)]$ . By Lemma 3,  $G[T_1(W) \setminus N(y_1)]$  is  $P_3$ -free. Then  $G[(U \setminus N(y_1))^*]$  admits no component of type 1. Then one can refer to Case 2 and to Lemmas 1 and 2. Now, let us consider  $G[(U \setminus \{y_1, \dots, y_{i-1}\}) \setminus N(y_i)]$  for  $i = 2, \dots, h$ : by the mentioned total order, one can apply the argument applied for  $G[U \setminus N(y_1)]$  in order to show that WIS can be solved for such graphs in polynomial time. Finally, let us consider  $G[U \setminus M_{1,3,5}]$ : since no vertex in  $U \setminus M_{1,3,5}$  is partial to any component of  $G[Z(0)]$  (otherwise a  $P_6$  arises), one can refer to Case 1.

Let us summarize the above argument as follows.

**Theorem 3** *Let  $G = (V, E)$  be a  $(P_6, \text{diamond})$ -free graph containing a 6-cycle  $C$ . Then one can solve WIS for  $G[V \setminus N(c)]$  in  $\mathcal{O}(n^6)$  time, for any vertex  $c$  of  $C$ .  $\square$*

### 3.2 Deleting $C_5$ 's in $(P_6, \text{diamond}, C_6)$ -free graphs

Throughout this subsection assume that  $G = (V, E)$  is a  $(P_6, \text{diamond}, C_6)$ -free graph containing a 5-cycle  $C$ , say with vertices  $v_i$  and edges  $v_i v_{i+1}$ ,  $i \in \{1, \dots, 5\}$  (subscript addition taken modulo 5). Let  $N(C)$  be the set of vertices from  $V \setminus C$  which are adjacent to some vertex in  $C$ . For any subset  $S$  of  $C$ , let  $M_S$  be the set formed by those vertices in  $N(C)$  which are adjacent to each vertex in  $S$  and are nonadjacent to each vertex in  $C \setminus S$ . In particular, let us write  $M_1$  for  $S = \{v_1\}$ ,  $M_{1,2}$  for  $S = \{v_1, v_2\}$ , and so on. Then let  $Z(k)$  denote the set of vertices of  $N(C)$  having exactly  $k$  neighbors in  $C$ .

Since  $G$  is  $(P_6, \text{diamond})$ -free:  $Z(4) = Z(5) = \emptyset$ ; each element of  $Z(3)$  belongs to some of the sets  $M_{i, i+2, i+3}$ ,  $i \in \{1, \dots, 5\}$  (subscript addition taken modulo 5).

Similarly to the previous subsection, one has the following fact.

**Lemma 5** *Every component of  $G[Z(0)]$  is green.  $\square$*

**Lemma 6** *There exists a vertex  $c$  of  $C$  such that one of the following statements holds:*

- (i)  $M_i \setminus N(c) = \emptyset$  for all  $i \in \{2, \dots, 5\}$ , and  $M_{i, i+1} \setminus N(c) = \emptyset$  for all  $i \in \{1, \dots, 5\}$  (subscript addition taken modulo 5);
- (ii)  $M_i \setminus N(c) = \emptyset$  for all  $i \in \{1, \dots, 5\}$ , and  $M_{i, i+1} \setminus N(c) = \emptyset$  for all  $i \in \{1, \dots, 4\}$  (subscript addition taken modulo 5).

**Proof:** Since  $G$  is  $(P_6, C_6)$ -free, for all  $i \in \{1, \dots, 5\}$  (subscript addition taken modulo 5) one has that: if  $M_i \neq \emptyset$ , then  $M_{i+2} = M_{i+3} = M_{i+1, i+2} = M_{i+3, i+4} = \emptyset$ ; if  $M_{i, i+1} \neq \emptyset$ , then  $M_{i-1, i} = M_{i+1, i+2} = \emptyset$ . This implies the lemma.  $\square$

Let us fix any vertex of  $C$ , say  $v_2$ , guaranteed by Lemma 6.



Let us prove that one can solve WIS for  $G[V \setminus N(v_2)]$  in  $\mathcal{O}(n^4)$  time.

A partition of  $V \setminus N(v_2)$  is given by  $\{\{v_2, v_4, v_5\}, M_{1,3,4}, M_{3,4,5}, M_{1,3}, M_{1,4}, M_{3,5}, M_1, Z(0)\}$ .

Since  $G$  is diamond-free, the sets  $M_{1,3,4}, M_{3,4,5}$  have cardinality at most one. Then, by Observation 1, to our aim it is sufficient to prove that WIS can be solved for each induced subgraph of  $G[U]$  in polynomial time, where a partition of  $U$  is given by  $\{M_{1,3}, M_{1,4}, M_{3,5}, M_1, Z(0)\}$ .

Since  $G$  is diamond-free,  $M_{1,3}, M_{1,4}, M_{3,5}$  are independent sets. Since  $G$  is  $P_6$ -free:  $M_1$  has a co-join with  $Z(0)$ ; each vertex in  $M_{1,3} \cup M_{1,4} \cup M_{3,5}$  is not partial to any component of  $G[Z(0)]$ , i.e., each component of  $G[Z(0)]$  is a module of  $G[U]$ . So by assertions similar to Lemmas 5 and 1, one can assume that  $Z(0)$  is an independent set.

Now, let us consider the following cases, which are exhaustive by symmetry.

**Case 1** statement i of Lemma 6 holds.

**Case 1.1**  $M_1 = \emptyset$ . One can solve WIS in  $G[U]$  by solving WIS in  $G[U \setminus M_{3,5}]$  and in  $G[U \setminus N(y)]$  for every  $y \in M_{3,5}$ . Since  $G$  is diamond-free, one can verify that such graphs are triangle-free. Then in this case one can solve WIS for  $G[U]$  in  $\mathcal{O}(n^3)$  time by Theorem 2.

**Case 1.2**  $M_1 \neq \emptyset$ .

First assume that  $M_1$  is a clique. One can solve WIS in  $G[U]$  by solving WIS in  $G[U \setminus M_1]$  and in  $G[U \setminus N(y)]$  for every  $y \in M_1$ . Then, by referring to Case 1.1, in this case one can solve WIS for  $G[U]$  in  $\mathcal{O}(n^4)$  time.

Then assume that  $M_1$  is not a clique, i.e.,  $G[M_1]$  is disconnected (since  $G[M_1]$  is  $P_3$ -free). Then  $M_{3,5}$  is partitioned into  $\{M_{3,5}^0, M'_{3,5}\}$ , where  $M'_{3,5} = \{x \in M_{3,5} : x \text{ is universal for } M_1\}$ , and  $M_{3,5}^0 = \{x \in M_{3,5} : x \text{ does not contact } M_1\}$  (in fact if  $y, z \in M_1$  are nonadjacent and a vertex  $x \in M_{3,5}$  is adjacent to  $y$  and nonadjacent to  $z$ , then  $v_4, v_3, x, y, v_1, z$  induce a  $P_6$ ). One can solve WIS in  $G[U]$  by solving WIS in  $G[U \setminus M'_{3,5}]$  and in  $G[U \setminus N(y)]$  for every  $y \in M'_{3,5}$ .

That can be done in  $\mathcal{O}(n^4)$  time. Concerning graphs  $G[U \setminus N(y)]$  for every  $y \in M'_{3,5}$ , one can refer to Case 1.1. Then let us consider  $G[U \setminus M'_{3,5}]$ . Notice that  $M_{3,5}^0$  has a co-join with  $Z(0)$ , otherwise a  $P_6$  arises involving a vertex of  $M_1$ . Then  $U \setminus M'_{3,5}$  is partitioned into  $\{X, Y\}$ , where  $X = M_1 \cup M_{1,3} \cup M_{1,4}$  (i.e.,  $G[X]$  is  $P_3$ -free) and  $Y = M_{3,5}^0 \cup Z(0)$  (i.e.,  $Y$  is an independent set). Then each component of  $G[U \setminus M'_{3,5}]$  is either  $P_3$ -free or green. Then the assertion follows by Lemma 1.

**Case 2** statement ii of Lemma 6 holds.

One can solve WIS in  $G[U]$  by solving WIS in  $G[U \setminus M_{4,5}]$  and in  $G[U \setminus N(y)]$  for every  $y \in M_{4,5}$ . Since  $G$  is diamond-free,  $M_{4,5}$  is a clique. Then, by referring to Case 1.1 (3.2), in this case one can solve WIS for  $G[U]$  in  $\mathcal{O}(n^4)$  time.

Let us summarize the above argument as follows.

**Theorem 4** Let  $G = (V, E)$  be a  $(P_6, \text{diamond}, C_6)$ -free graph containing a 5-cycle  $C$ .

Then there exists a vertex  $c$  of  $C$  (which can be easily found) such that one can solve WIS for  $G[V \setminus N(c)]$  in  $\mathcal{O}(n^4)$  time.  $\square$

### 3.3 Deleting $C_4$ 's in $(P_6, \text{diamond}, C_6, C_5)$ -free graphs

Throughout this subsection assume that  $G = (V, E)$  is a  $(P_6, \text{diamond}, C_6, C_5)$ -free graph containing a 4-cycle  $C$ , say with vertices  $v_i$  and edges  $v_i v_{i+1}$ ,  $i \in \{1, \dots, 4\}$  (subscript addition taken modulo 4). Let  $N(C)$  be the set of vertices from  $V \setminus C$  which are adjacent to some vertex in  $C$ . For any subset  $S$  of  $C$ , let  $M_S$  be the set formed by those vertices in  $N(C)$  which are adjacent to each vertex in  $S$  and are nonadjacent to each vertex in  $C \setminus S$ . In particular, let us write  $M_1$  for  $S = \{v_1\}$ ,  $M_{1,2}$  for  $S = \{v_1, v_2\}$ , and so on. Then let  $Z(k)$  denote the set of vertices of  $N(C)$  having exactly  $k$  neighbors in  $C$ .

Since  $G$  is  $(P_6, \text{diamond})$ -free:  $Z(3) = Z(4) = \emptyset$ .

Similarly to the previous subsection, one has the following fact.

**Lemma 7** *Every component of  $G[Z(0)]$  is green.* □

Let us fix any vertex of  $C$ , say  $v_2$ .

Let us prove that WIS can be solved for  $G[V \setminus N(v_2)]$  in  $\mathcal{O}(n^6)$  time.

A partition of  $V \setminus N(v_2)$  is given by  $\{\{v_2, v_4\}, M_{1,3}, M_{3,4}, M_{4,1}, M_1, M_3, M_4, Z(0)\}$ . Then, by Observation 1, to our aim it is sufficient to prove that WIS can be solved for each induced subgraph of  $G[U]$  in polynomial time, where a partition of  $U$  is given by  $\{M_{1,3}, M_{3,4}, M_{4,1}, M_1, M_3, M_4, Z(0)\}$ ,

Let us introduce some preliminary definitions and lemmas.

Let us write:

$$\begin{aligned} M_1^0 &= \{x \in M_1 : x \text{ does not contact } M_4\} \\ M_3^0 &= \{x \in M_3 : x \text{ does not contact } M_4\} \\ M_4^0 &= \{x \in M_4 : x \text{ does not contact } M_1 \cup M_3\} \\ X &= \{x \in M_1 : x \text{ contacts } M_4\} \cup \{x \in M_4 : x \text{ contacts } M_1\} \\ Y &= \{x \in M_3 : x \text{ contacts } M_4\} \cup \{x \in M_4 : x \text{ contacts } M_3\} \end{aligned}$$

Let us write:

$$\begin{aligned} Z_1 &= M_1^0 \cup \{z \in Z(0) : z \text{ contacts } M_1^0\} \\ Z_3 &= M_3^0 \cup \{z \in Z(0) : z \text{ contacts } M_3^0\} \\ Z_4 &= M_4^0 \cup \{z \in Z(0) : z \text{ contacts } M_4^0\} \\ Z_X &= X \cup \{z \in Z(0) : z \text{ contacts } X\} \\ Z_Y &= Y \cup \{z \in Z(0) : z \text{ contacts } Y\} \\ \tilde{Z} &= \{z \in Z(0) : z \text{ does not contact } M_1 \cup M_3 \cup M_4\}. \end{aligned}$$

**Lemma 8** *The following facts hold:*

- (i) *each pair of the sets  $Z_1, Z_3, Z_4, Z_X, Z_Y, \tilde{Z}$  has a co-join;*
- (ii) *each component of  $G[Z_1 \cup Z_3 \cup Z_4 \cup \tilde{Z}]$  is green;*
- (iii) *each component of  $G[Z_X \cup Z_Y]$  is either  $P_3$ -free or bipartite.*

**Proof:**

**Proof of i.** Since  $G$  is  $C_5$ -free,  $M_1$  has a co-join with  $M_3$ . Then, since  $G$  is  $C_6$ -free, no vertex in  $M_4$  can be adjacent to a vertex in  $M_1$  and to a vertex in  $M_3$  at the same time. It follows that each pair of the sets  $M_1^0, M_3^0, M_4^0, X, Y$  has a co-join. Furthermore, since  $G$  is  $(P_6, C_6)$ -free, one can verify that two vertices chosen in two different sets – among the mentioned sets – cannot contact a component of  $Z(0)$  at the same time. Finally, since  $G$  is  $P_6$ -free, if a vertex in  $y \in M_1 \cup M_3 \cup M_4$  contacts a component  $K$  of  $G[Z(0)]$ , then  $y$  is universal for  $K$ . Then i follows.

**Proof of ii.** It is enough to deal with  $G[Z_1]$ , as the other subsets can be treated similarly. If  $Z_1 = M_1^0$ , then  $G[Z_1]$  is  $P_3$ -free. Otherwise, since  $G$  is  $(P_6, \text{diamond})$ -free, one can verify (similarly to the argument of Lemma 2) that each component of  $G[Z_1]$  is green.

**Proof of iii.** It is enough to deal with  $G[Z_1]$ , as the other subsets can be treated similarly. If  $Z_X = X$ , then since  $G$  is  $(\text{diamond}, C_5)$ -free, each component of  $G[Z_X]$  is bipartite. Otherwise, since  $G$  is  $P_6$ -free, each vertex in  $Z(0)$  contacting a component  $K$  of  $G[X]$  dominates  $K$ . Then, since  $G$  is diamond-free, each component of  $G[Z_X]$  is a clique.  $\square$

**Lemma 9** *If  $M_1 \neq \emptyset$  and  $M_3 \neq \emptyset$ , then:*

- (i)  $Z(0)$  has a co-join with  $M_1 \cup M_3$ ;
- (ii)  $X = Y = \emptyset$ .

**Proof:**

**Proof of i.** It follows since  $G$  is  $(P_6, C_6, C_5)$ -free.

**Proof of ii.** By symmetry, let us only prove that  $X = \emptyset$ . Assume for a contradiction that  $X \neq \emptyset$ . Then let  $x_1 \in M_1$  be adjacent to  $x_4 \in M_4$ . Let  $x_3 \in M_3$ . By i of Lemma 8,  $x_1$  and  $x_4$  are nonadjacent to  $x_3$ . Then  $x_4, x_1, v_1, v_2, v_3, x_3$  induce a  $P_6$ .  $\square$

**Lemma 10** *If a vertex  $y \in M_{1,3}$  contacts a component  $K$  of  $G[Z_X \cup Z_Y]$ , then  $K \setminus N(y)$  is either a clique or an independent set.*

**Proof:** By symmetry, let us consider only  $G[Z_X]$ . Let  $K$  be a component of  $G[X]$ . If  $G[K]$  is  $P_3$ -free, then the assertion trivially follows. Then, by iii of Lemma 8, assume that  $G[K]$  is bipartite. Let  $y \in M_{1,3}$  contact  $K$ . Notice that  $y$  cannot be adjacent to two adjacent vertices of  $K$ , otherwise a diamond arises involving  $v_1$ . Then, to avoid a  $P_6$ ,  $y$  is adjacent to all the vertices of a side of the bipartite graph, i.e.,  $K \setminus N(y)$  is an independent set.  $\square$

Let us write  $Z = Z_1 \cup Z_3 \cup Z_4 \cup Z_X \cup Z_Y \cup \tilde{Z}$ . Then  $\{M_{1,3}, Z\}$  is a partition of  $U$ .

For any  $W \subseteq U$ , let us write  $W^* = W \cap Z$ .

For any  $W \subseteq U$ , let us say that a component  $K$  of  $G[W^*]$  is of:

**type 1** if  $K$  is not a clique and there exists a vertex in  $W \setminus Z$  partial to  $K$ ;

**type 2** if  $K$  is a clique and there exists a vertex in  $W \setminus Z$  partial to  $K$ ;

**type 3** otherwise.

Let  $T_1(W), T_2(W), T_3(W)$  respectively denote the union of components of  $G[W^*]$  of type 1, 2, 3.

Let us fix a subset  $W \subseteq U$ .

Notice that  $M_{1,3}$  is an independent set, since  $G$  is diamond-free.

Let us consider the following binary relation ' $\leq_1$ ' on  $M_{1,3}$ : for any  $a, b \in M_{1,3}$ ,  $a \leq_1 b$  if  $N_{T_1(W)}(a) \subseteq N_{T_1(W)}(b)$ . It is immediate to verify that  $(M_{1,3}, \leq_1)$  is a partially ordered set.

**Lemma 11** *Let  $y \in M_{1,3}$  be maximal for  $(M_{1,3}, \leq_1)$ . Then  $G[T_1(W) \setminus N(y)]$  is  $P_3$ -free.*

**Proof:** If either  $M_1 = \emptyset$  or  $M_3 = \emptyset$ , then one can apply an argument similar to that of Lemma 3, by considering also (i) of Lemma 8 and Lemma 10 – in detail, if  $M_1 = \emptyset$  (if  $M_3 = \emptyset$ ), then vertex  $v_1$  (vertex  $v_3$ ) is universal for  $M_{1,3}$  and does not contact  $Z$ .

If  $M_1 \neq \emptyset$  and  $M_3 \neq \emptyset$ , then one can apply an argument similar to that of Lemma 3, by considering Lemma 9 and the fact that no element of  $M_{1,3}$  can be partial to a component of  $G[M_1 \cup M_3]$  (since  $G$  is diamond-free).  $\square$

Let us consider the following binary relation ' $\leq_2$ ' on  $M_{1,3}$ : for any  $a, b \in M_{1,3}$ ,  $a \leq_2 b$  if  $N_{T_2(W)}(a) \subseteq N_{T_2(W)}(b)$ . It is immediate to verify that  $(M_{1,3}, \leq_2)$  is a partially ordered set.

**Lemma 12** *Let  $y \in M_{1,3}$  be maximal for  $(M_{1,3}, \leq_2)$ . If  $T_1(W) = \emptyset$ , then  $G[(W \setminus N(y))^*]$  admits at most one component of type 2.*

**Proof:** One can apply the argument in the proof of Lemma 11, by considering Lemma 4 instead of Lemma 3.  $\square$

Now, let us consider the following cases.

**Case 1**  $M_{3,4} = M_{4,1} = \emptyset$ .

**Case 1.1**  $T_1(U) = T_2(U) = \emptyset$ .

Then each component  $K$  of  $G[Z]$  is a module of  $G[U]$ . Then, to our aim, one can assume that  $K$  is a singleton. In fact, one can contract  $K$  into a singleton  $k$  with  $w(k) = \alpha_w(G[K])$ : that can be done in  $\mathcal{O}(n^3)$  time by ii–iii of Lemma 8 and Lemma 1.

So in general, one can assume that  $Z$  is an independent set. Then  $G[U]$  is bipartite. In this case one can solve WIS for  $G[U]$  in time  $\mathcal{O}(n^3)$  by Theorem 2.

**Case 1.2**  $T_1(U) = \emptyset, T_2(U) \neq \emptyset$ .

Based on  $(M_{1,3}, \leq_2)$ , the vertices  $y_1, \dots, y_h$  of  $M_{1,3}$  can be totally ordered so that  $y_i$  is maximal for  $(\{y_i, \dots, y_h\}, \leq_2)$  for  $i = 1, \dots, h$ . Then one can solve WIS in  $G[U]$  by sequentially solving WIS in  $G[U \setminus N(y_1)]$ , in  $G[(U \setminus \{y_1, \dots, y_{i-1}\}) \setminus N(y_i)]$  for  $i = 2, \dots, h$ , and in  $G[U \setminus M_{1,3}]$ .

That can be done in  $\mathcal{O}(n^5)$  time. In fact, let us first consider  $G[U \setminus N(y_1)]$ . If  $G[(U \setminus N(y_1))^*]$  admits no component of type 2, then one can refer to Case 1.1. Otherwise, by Lemma 12,  $G[(U \setminus N(y_1))^*]$  admits a unique component of type 2, say  $Q$ . Then one can solve WIS in  $G[U \setminus N(y_1)]$  by solving WIS in  $G[(U \setminus N(y_1)) \setminus Q]$  and in  $G[(U \setminus N(y_1)) \setminus N(q)]$ , for every  $q \in Q$ : since for each of such graphs  $G[H]$  one has that  $G[H^*]$  has no component of type 2, one can refer to Case 1.1, to ii–iii of Lemma 8 and to Lemmas 7 and 1. Now, let us consider  $G[(U \setminus \{y_1, \dots, y_{i-1}\}) \setminus N(y_i)]$  for  $i = 2, \dots, h$ : by the mentioned total order, one can apply the argument applied for  $G[U \setminus N(y_1)]$  in

order to show that WIS can be solved for such graphs in polynomial time. Finally, let us consider  $G[U \setminus M_{1,3}] = G[Z]$ : then one can refer to Case 1.1.

**Case 1.3**  $T_1(U) \neq \emptyset$ .

Based on  $(M_{1,3}, \leq_2)$ , the vertices  $y_1, \dots, y_h$  of  $M_{1,3}$  can be totally ordered so that  $y_i$  is maximal for  $(\{y_i, \dots, y_h\}, \leq_2)$  for  $i = 1, \dots, h$ . Then one can solve WIS in  $G[U]$  by sequentially solving WIS in  $G[U \setminus N(y_1)]$ , in  $G[(U \setminus \{y_1, \dots, y_{i-1}\}) \setminus N(y_i)]$  for  $i = 2, \dots, h$ , and in  $G[U \setminus M_{1,3}]$ .

That can be done in  $\mathcal{O}(n^6)$  time. In fact, let us first consider  $G[U \setminus N(y_1)]$ . By Lemma 11,  $G[T_1(W) \setminus N(y_1)]$  is  $P_3$ -free. Then  $G[(U \setminus N(y_1))^*]$  admits no component of type 1. Then one can refer to Case 1.2 and to Lemmas 7 and 1. Now, let us consider  $G[(U \setminus \{y_1, \dots, y_{i-1}\}) \setminus N(y_i)]$  for  $i = 2, \dots, h$ : by the total order, one can apply the argument applied for  $G[U \setminus N(y_1)]$  in order to show that WIS can be solved for such graphs in polynomial time. Finally, let us consider  $G[U \setminus M_{1,3}] = G[Z]$ : then one can refer to Case 1.1.

**Case 2**  $M_{3,4} \cup M_{4,1} \neq \emptyset$ .

Since  $G$  is diamond-free,  $M_{3,4}$  and  $M_{4,1}$  are cliques. Since  $G$  is  $(P_6, \text{diamond}, C_6, C_5)$ -free, the following facts hold:  $M_{3,4} \cup M_{4,1}$  has a co-join with  $N(C) \setminus (N(v_2) \cup M_{3,4} \cup M_{4,1})$ ; a vertex in  $M_{3,4} \cup M_{4,1}$  and a vertex in  $N(C) \setminus (N(v_2) \cup M_{3,4} \cup M_{4,1})$  cannot contact a component of  $G[Z(0)]$  at the same time;  $M_{3,4}$  has a co-join with  $M_{4,1}$ .

First assume that  $M_{3,4} \neq \emptyset$  and  $M_{4,1} \neq \emptyset$ . Then  $M_{3,4} \cup M_{4,1}$  has a co-join with  $Z(0)$  (otherwise a  $P_6$  or a  $C_6$  arises). In general, by the above facts,  $M_{3,4} \cup M_{4,1}$  has a co-join with  $U \setminus (M_{3,4} \cup M_{4,1})$ . Then, since  $M_{3,4}$  and  $M_{4,1}$  are cliques, one can directly refer to Case 1.

Then assume that  $M_{3,4} = \emptyset$ . One can solve WIS for  $G[U]$  by solving WIS in  $G[U \setminus M_{4,1}]$  and in  $G[U \setminus N(y)]$ , for every  $y \in M_{4,1}$ . Since  $M_{4,1}$  is a clique, one can directly refer to Case 1.

The case in which  $M_{4,1} = \emptyset$  can be similarly treated, by symmetry.

Let us summarize the above argument as follows.

**Theorem 5** *Let  $G = (V, E)$  be a  $(P_6, \text{diamond}, C_6, C_5)$ -free graph containing a 4-cycle  $C$ . Then one can solve WIS for  $G[V \setminus N(c)]$  in  $\mathcal{O}(n^6)$  time, for any vertex  $c$  of  $C$ .  $\square$*

### 3.4 A solution for WIS and WID in $(P_6, \text{diamond})$ -free graphs

In this subsection we formalize an efficient method for solving WIS (or WID) in  $(P_6, \text{diamond})$ -free graphs. To this end, let us first summarize the results of the previous subsections in the following theorem.

**Theorem 6** *Let  $G = (V, E)$  be a connected  $(P_6, \text{diamond})$ -free graph containing a  $C_6$  or a  $C_5$  or a  $C_4$ . Then there exists a vertex  $c$  (which can be easily found) such that one can solve WIS (or WID) for  $G[V \setminus N(c)]$  in  $\mathcal{O}(n^6)$  time.*

**Proof:** If  $G$  contains a  $C_6$ , then the assertion follows by Theorem 3. If  $G$  is  $C_6$ -free and contains a  $C_5$ , then the assertion follows by Theorem 4. If  $G$  is  $(C_6, C_5)$ -free and contains a  $C_4$ , then the assertion follows by Theorem 5.  $\square$

To prove that WIS (or WID) is solvable in polynomial time on the class of  $(P_6, \text{diamond})$ -free graphs, it suffices to find a polynomial upper bound  $p(n) = \mathcal{O}(n^7)$  on the number of steps sufficient for any

allowed input of order  $n$ . If  $G$  is chordal, then we are done by Theorem 1. Otherwise, there exists a sixth-degree polynomial  $q(n)$  with the property that in any  $(P_6, \text{diamond})$ -free non-chordal graph one can determine a vertex  $x$  such that WIS (or WID) can be solved on  $G_1 = G[V \setminus N(x)]$  in  $q(n)$  time. If  $G'_1 = G - x$  is not chordal, then again one can find a vertex  $x'$  such that the problem can be solved on  $G_2 = G[V(G'_1) \setminus N(x')]$  in  $q(n)$  time, and so on. In this way we obtain some graphs  $G_1, G_2, \dots, G_k$  with  $k < n$ , such that each  $G_i$  is either chordal or admits an efficient WIS (or WID) algorithm. Thus, the total running time is  $O(n(q(n) + r(n)))$  where  $r(n)$  is the time needed to check whether the current graph is chordal and if it is not, then to find a suitable vertex under the conditions of Theorem 6.

Now, by Theorems 1 and 6 one obtains:

**Theorem 7** *Both the WIS and the WID problems can be solved for  $(P_6, \text{diamond})$ -free graphs in  $\mathcal{O}(n^7)$  time.  $\square$*

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