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Stéphane Girard

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# STATISTICAL INFERENCE FOR WEIBULL TAIL-DISTRIBUTIONS

Stéphane Girard

SMS/LMC-IMAG, Université Grenoble 1, INRIA

## Outline

1. Weibull tail-distributions.
2. Kernel estimators of the Weibull tail-coefficient.
3. Bias-reduced estimator of the Weibull tail-coefficient.
4. Estimation of extreme quantiles.
5. Simulation study.
6. Nidd river data.

## 1. Weibull tail-distributions.

Let  $X_1, X_2, \dots, X_n$  be a sequence of independent and identically distributed random variables with cumulative distribution function  $F$  such that

$$\text{(A1)} \quad 1 - F(x) = \exp(-H(x)), \quad H^{-1}(t) = \inf\{x, H(x) \geq t\} = t^\theta \ell(t),$$

where

- $\theta > 0$  is the Weibull tail-coefficient,
- $\ell$  is a slowly varying function *i.e.*

$$\ell(\lambda x)/\ell(x) \rightarrow 1 \text{ as } x \rightarrow \infty \text{ for all } \lambda > 0.$$

The inverse failure rate function  $H^{-1}$  is said to be regularly varying at infinity with index  $\theta$  and this property is denoted by  $H^{-1} \in \mathcal{R}_\theta$ .

In the following, we also assume a second order condition on  $\ell$ :

**(A2)** There exist  $\rho \leq 0$  and  $b(x) \rightarrow 0$  such that uniformly locally on  $\lambda \geq 1$

$$\log \left( \frac{\ell(\lambda x)}{\ell(x)} \right) \sim b(x) K_\rho(\lambda), \text{ when } x \rightarrow \infty,$$

with

$$K_\rho(\lambda) = \int_1^\lambda u^{\rho-1} du.$$

It can be shown that necessarily  $|b| \in \mathcal{R}_\rho$ . The second order parameter  $\rho \leq 0$  tunes the rate of convergence of  $\ell(\lambda x)/\ell(x)$  to 1. The closer  $\rho$  is to 0, the slower is the convergence.

## Examples:

	$\theta$	$\ell(x)$	$b(x)$	$\rho$
Absolute Gaussian $ \mathcal{N} (\mu, \sigma^2)$	$1/2$	$2^{1/2}\sigma - \frac{\sigma}{2^{3/2}} \frac{\log x}{x} + O(1/x)$	$\frac{1}{4} \frac{\log x}{x}$	$-1$
Gamma $\Gamma(\alpha \neq 1, \beta)$	$1$	$\frac{1}{\beta} + \frac{\alpha - 1}{\beta} \frac{\log x}{x} + O(1/x)$	$(1 - \alpha) \frac{\log x}{x}$	$-1$
Weibull $\mathcal{W}(\alpha, \lambda)$	$1/\alpha$	$\lambda$	$0$	$-\infty$

**Related work:** Berred (1991), Broniatowski (1993), Beirlant, Broniatowski, Teugels, Vynckier, (1995), Beirlant, Bouquiaux, Werker (2006).

## 2. Kernel estimators of the Weibull tail-coefficient.

**Principle:** Our approach is based on the following approximation. Denoting by  $q(t)$  the quantile function

$$q(t) = F^{-1}(1 - t) = H^{-1}(\log(1/t)) = (\log(1/t))^\theta \ell(\log(1/t)),$$

we obtain for  $t$  and  $s$  small:

$$\begin{aligned} \log(q(t)) - \log(q(s)) &= \theta(\log_2(1/t) - \log_2(1/s)) + \log\left(\frac{\ell(\log(1/t))}{\ell(\log(1/s))}\right) \\ &\simeq \theta(\log_2(1/t) - \log_2(1/s)), \end{aligned} \tag{1}$$

where  $\log_2(x) = \log \log(x)$  and since  $\ell \in \mathcal{R}_0$ . Considering  $t = i/n$ ,  $s = k_n/n$  and replacing  $F$  by its empirical counterpart yield

$$\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n}) \simeq \theta(\log_2(n/i) - \log_2(n/k_n)),$$

for  $i = 1, \dots, k_n - 1$  and where  $k_n$  is an intermediate sequence.

Our method: Estimation via linear combination of upper order statistics.

$$\hat{\theta}_n(\alpha) = \sum_{i=1}^{k_n-1} \alpha_{i,n} (\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n})) \bigg/ \sum_{i=1}^{k_n-1} \alpha_{i,n} (\log_2(n/i) - \log_2(n/k_n)),$$

where

- $\alpha_{i,n} = W(i/k_n) + \varepsilon_{i,n}$ ,
- $\varepsilon_{i,n}$ ,  $i = 1, \dots, k_n - 1$  is a non-random sequence, and
- $W : [0, 1] \rightarrow \mathbb{R}$  is a smooth score function, verifying

**(A3)**  $W$  has a continuous derivative  $W'$  on  $(0, 1)$ ,

**(A4)** There exist  $M > 0$ ,  $0 \leq q < 1/2$  and  $p < 1$  such that, for all  $x \in (0, 1)$ :

$$|W(x)| \leq Mx^{-q} \text{ and } |W'(x)| \leq Mx^{-p-q}.$$



**Asymptotic normality:** Suppose **(A1)**–**(A4)** hold. Then

$$k_n^{1/2}(\hat{\theta}_n(\alpha) - \theta) \xrightarrow{d} \mathcal{N}(0, \sigma^2(\theta, W)),$$

for any sequence  $(k_n)$  satisfying  $k_n \rightarrow \infty$  and

$$k_n^{1/2} \max\{b(\log(n/k_n)), 1/\log(n/k_n), \|\varepsilon\|_{n,\infty}\} \rightarrow 0,$$

where we have defined:

$$\begin{aligned} \|\varepsilon\|_{n,\infty} &= \max_{i=1,\dots,k_n-1} |\varepsilon_{i,n}|, \\ \mu(W) &= \int_0^1 W(x) \log(1/x) dx, \\ \sigma^2(W) &= \int_0^1 \int_0^1 W(x)W(y) \frac{\min(x,y)(1 - \max(x,y))}{xy} dx dy, \\ \sigma(\theta, W) &= \theta \frac{\sigma(W)}{\mu(W)}. \end{aligned}$$

**Example 1.** Constant weights  $\alpha_{i,n} = 1$  for all  $i = 1, \dots, k_n - 1$  yield an existing estimator (Girard, 2004):

$$\hat{\theta}_n^G = \frac{\sum_{i=1}^{k_n-1} (\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n}))}{\sum_{i=1}^{k_n-1} (\log_2(n/i) - \log_2(n/k_n))}.$$

We found back the same limiting result: If **(A1)** and **(A2)** hold then

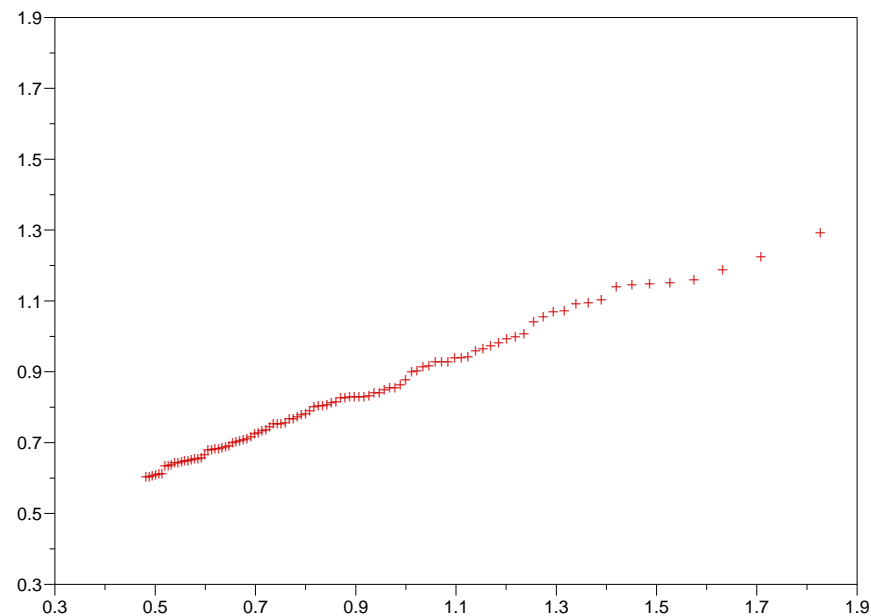
$$k_n^{1/2}(\hat{\theta}_n^G - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2),$$

for any sequence  $(k_n)$  satisfying  $k_n \rightarrow \infty$  and

$$k_n^{1/2} \max\{b(\log(n/k_n)), 1/\log(n/k_n)\} \rightarrow 0.$$

**Example 2:** A new estimator of the Weibull tail-coefficient based on a QQ-plot.

- Drawing the pairs  $(\log_2(n/i), \log(X_{n-i+1,n}))$  for  $i = 1, \dots, k_n$  gives a graph which is approximatively linear (with slope  $\theta$ ).
- Example :  $|\mathcal{N}|(0, 1)$  distribution,  $n = 500$ ,  $k_n = 100$ .



- $\hat{\theta}_n^Z$  is the least square estimator of  $\theta$  based on the  $k_n$  largest observations:

$$\hat{\theta}_n^Z = \frac{\sum_{i=1}^{k_n-1} (\log_2(n/i) - \tau_n) \log(X_{n-i+1,n})}{\sum_{i=1}^{k_n-1} (\log_2(n/i) - \tau_n) \log_2(n/i)},$$

where

$$\tau_n = \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log_2(n/i).$$

- Similar to the Zipf estimator for the extreme value index proposed by Kratz, Resnick (1996) and Schultze, Steinebach (1996).
- Particular case of  $\hat{\theta}_n(\alpha)$  with  $W(x) = -(\log(x) + 1)$ . Thus, under **(A1)** and **(A2)**,

$$k_n^{1/2}(\hat{\theta}_n^Z - \theta) \xrightarrow{d} \mathcal{N}(0, 2\theta^2),$$

for any sequence  $(k_n)$  satisfying  $k_n \rightarrow \infty$  and

$$k_n^{1/2} \max\{b(\log(n/k_n)), \log^2(k_n)/\log(n/k_n)\} \rightarrow 0.$$

### 3. Bias-reduced estimator of the Weibull tail-coefficient.

**Principle:** We focus on the case where the convergence in **(A2)** is slow, *i.e.*

**(A5)**  $x|b(x)| \rightarrow \infty$  as  $x \rightarrow \infty$ .

Let us note that this condition implies  $\rho \geq -1$ . Gamma and (absolute) Gaussian distribution fulfill **(A5)** whereas Weibull distribution do not. Condition **(A2)** can be used to precise approximation **(1)**:

$$\log(q(t)) - \log(q(s)) = \theta(\log_2(1/t) - \log_2(1/s)) + b(\log(1/s))K_\rho \left( \frac{\log(1/t)}{\log(1/s)} \right) (1 + o(1)). \quad (2)$$

## Exponential regression models:

- Define  $Z_i = i \log(n/i)(\log X_{n-i+1,n} - \log X_{n-i,n})$ ,  $i = 1, \dots, k_n$ . Then, under **(A1)**, **(A2)**, **(A5)**,

$$\sup_{1 \leq i \leq k_n} \left| Z_i - \left( \theta + b(\log(n/k_n)) \left( \frac{\log(n/i)}{\log(n/k_n)} \right)^\rho \right) f_i \right| = o_{\mathbb{P}}(b(\log(n/k_n))), \quad (3)$$

for any sequence  $(k_n)$  such that  $k_n \rightarrow \infty$  and  $\log k_n / \log n \rightarrow 0$ , and where  $(f_1, \dots, f_{k_n})$  is a vector of independent and standard exponentially distributed random variables.

- Similar to the ones proposed by Beirlant, Dierckx, Goegebeur, Matthys (1999), Feuerverger, Hall (1999) and Beirlant, Dierckx, Guillou, Starica (2002) in the case of Pareto-type distributions.
- One can plug the canonical choice  $\rho = -1$  in the regression model (3) without perturbing the approximation so that

$$\sup_{1 \leq i \leq k_n} \left| Z_i - \left( \theta + b(\log(n/k_n)) \frac{\log(n/k_n)}{\log(n/i)} \right) f_i \right| = o_{\mathbb{P}}(b(\log(n/k_n))). \quad (4)$$

**Application 1: Bias-reduced estimator.** Estimation of  $\theta$  and  $b(\log(n/k_n))$  by a Least-Square method after substituting  $\rho$  with the value  $-1$ :

$$\hat{\theta}_n^R = \bar{Z}_{k_n} - \hat{b}(\log(n/k_n))\bar{x}_{k_n}, \quad \hat{b}(\log(n/k_n)) = \frac{\sum_{i=1}^{k_n} (x_i - \bar{x}_{k_n})Z_i}{\sum_{i=1}^{k_n} (x_i - \bar{x}_{k_n})^2}, \quad (5)$$

where  $x_i = \log(n/k_n)/\log(n/i)$ ,  $\bar{x}_{k_n} = \frac{1}{k_n} \sum_{i=1}^{k_n} x_i$  and  $\bar{Z}_{k_n} = \frac{1}{k_n} \sum_{i=1}^{k_n} Z_i$ .

Asymptotic normality under **(A1)**, **(A2)**, **(A5)**:

$$\frac{k_n^{1/2}}{\log(n/k_n)} (\hat{\theta}_n^R - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2)$$

for any sequence  $(k_n)$  such that  $k_n \rightarrow \infty$  and

$$k_n^{1/2} b(\log(n/k_n)) / \log(n/k_n) \rightarrow \Lambda \neq 0.$$

**Application 2: Adaptive selection of  $k_n$ .** Adapted from Matthys, Beirlant (2003) in the context of extreme-value index estimation. Neglecting the bias correction in (5) yields the Maximum Likelihood estimator:

$$\hat{\theta}_n^{ML} = \bar{Z}_{k_n} = \frac{1}{k_n} \sum_{i=1}^{k_n} i \log(n/i) (\log X_{n-i+1,n} - \log X_{n-i,n}).$$

From (4), the Asymptotic Mean Squared Error (AMSE) associated to  $\hat{\theta}_n^{ML}$  is given by

$$AMSE(\hat{\theta}_n^{ML}) = \frac{\theta^2}{k_n} + \left( b(\log(n/k_n)) \frac{1}{k_n} \sum_{i=1}^{k_n} \frac{\log(n/k_n)}{\log(n/i)} \right)^2,$$

and can be estimated by

$$\widehat{AMSE}(\hat{\theta}_n^{ML}) = \frac{(\hat{\theta}_n^R)^2}{k_n} + \left( \hat{b}(\log(n/k_n)) \frac{1}{k_n} \sum_{i=1}^{k_n} \frac{\log(n/k_n)}{\log(n/i)} \right)^2.$$

The minimization of  $\widehat{AMSE}(\hat{\theta}_n^{ML})$  with respect to  $k_n$  gives rise to an adaptive selection procedure.



## 4. Estimation of extreme quantiles.

**Principle:** An extreme quantile  $x_{p_n}$  of order  $p_n < 1/n$  is defined by:

$$1 - F(x_{p_n}) = p_n.$$

Recall that, for small  $t$  and  $s$

$$\frac{q(t)}{q(s)} = \frac{H^{-1}(\log(1/t))}{H^{-1}(\log(1/s))} = \left( \frac{\log(1/t)}{\log(1/s)} \right)^\theta \frac{\ell(\log(1/t))}{\ell(\log(1/s))} \simeq \left( \frac{\log(1/t)}{\log(1/s)} \right)^\theta.$$

Considering  $t = p_n$ ,  $s = k_n/n$ , replacing  $F$  by its empirical counterpart and  $\theta$  by an estimator  $\hat{\theta}_n$  yield the following estimator

$$\hat{x}_{p_n}(\hat{\theta}_n) = X_{n-k_n+1,n} \left( \frac{\log(1/p_n)}{\log(n/k_n)} \right)^{\hat{\theta}_n} = X_{n-k_n+1,n} \exp \left( \hat{\theta}_n \log \tau_n \right),$$

where we have defined  $\tau_n = \log(1/p_n)/\log(n/k_n)$ .

**Asymptotic normality:** Based on the following result (Gardes, Girard, 2005). Suppose **(A1)** and **(A2)** hold. If moreover

$$1 \leq \liminf \tau_n \leq \limsup \tau_n < \infty,$$

and

$$k_n^{1/2}(\hat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

then

$$\frac{\log(n/k_n)k_n^{1/2}}{\log(k_n/(np_n))} \left( \frac{\hat{x}_{p_n}(\hat{\theta}_n)}{x_{p_n}} - 1 \right) \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

for any sequence  $(k_n)$  satisfying  $k_n \rightarrow \infty$ ,  $k_n/n \rightarrow 0$  and

$$k_n^{1/2}b(\log(n/k_n)) \rightarrow 0.$$

As a consequence, we obtain the asymptotic normality of  $\hat{x}_{p_n}(\hat{\theta}_n^G)$ ,  $\hat{x}_{p_n}(\hat{\theta}_n^Z)$  and  $\hat{x}_{p_n}(\hat{\theta}_n^{\text{BBTV}})$  where

$$\hat{\theta}_n^{\text{BBTV}} = \frac{\log(n/k_n)}{X_{n-k_n+1,n}} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} (X_{n-i+1,n} - X_{n-k_n+1,n})$$

is the estimator introduced by Beirlant, Broniatowski, Teugels, Vinckier (1995).

**Bias-reduced estimator:** Basing on the refined approximation (2) of  $\log q(t) - \log q(s)$ , it is natural to introduce

$$\hat{x}_{p_n}^R = X_{n-k_n+1,n} \exp \left( \hat{\theta}_n^R \log \tau_n + \hat{b}(\log(n/k_n)) K_{-1}(\tau_n) \right).$$

Asymptotic normality under **(A1)**, **(A2)**, **(A5)**:

$$\frac{k_n^{1/2}}{\log(n/k_n)} \left( \frac{\hat{x}_{p_n}^R}{x_{p_n}} - 1 \right) \xrightarrow{d} \mathcal{N}(\Lambda \mu(\tau), \theta^2 \sigma^2(\tau))$$

for any sequence  $(k_n)$  such that  $k_n \rightarrow \infty$ ,  $\tau_n \rightarrow \tau > 1$  and

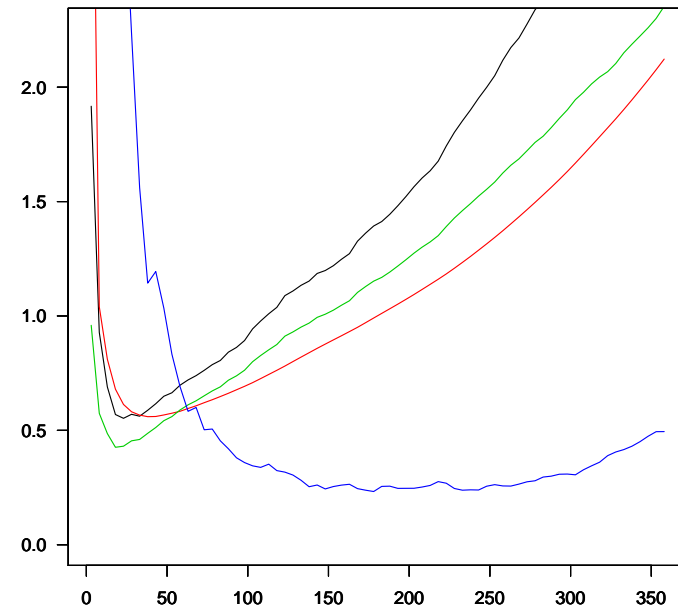
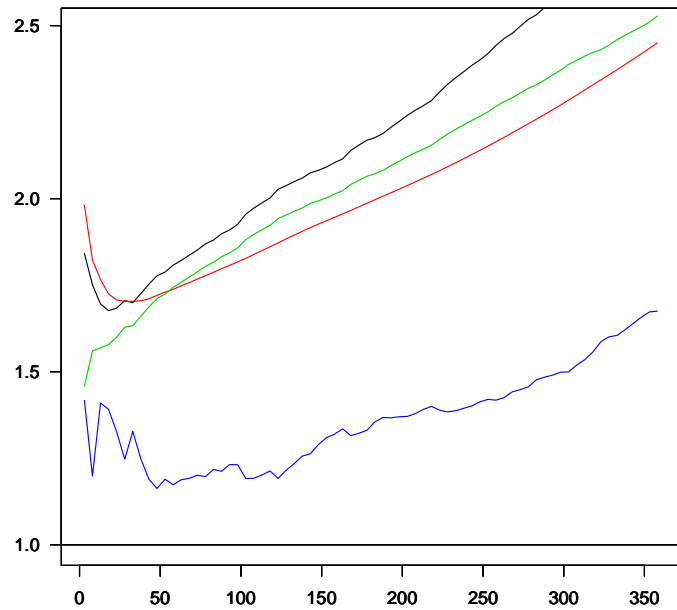
$$k_n^{1/2} b(\log(n/k_n)) / \log(n/k_n) \rightarrow \Lambda \neq 0,$$

where  $\mu(\tau) = (K_{-1}(\tau) - K_\rho(\tau))$  and  $\sigma^2(\tau) = (K_{-1}(\tau) - \log(\tau))^2$ .

## 5. Simulation study.

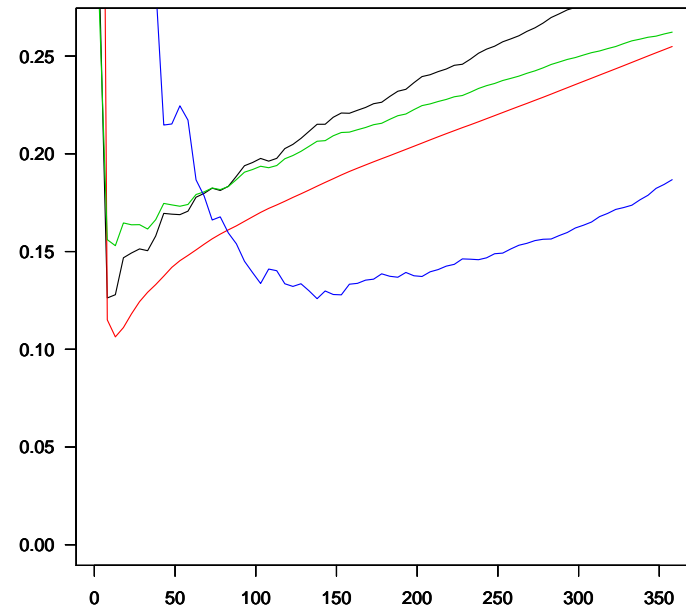
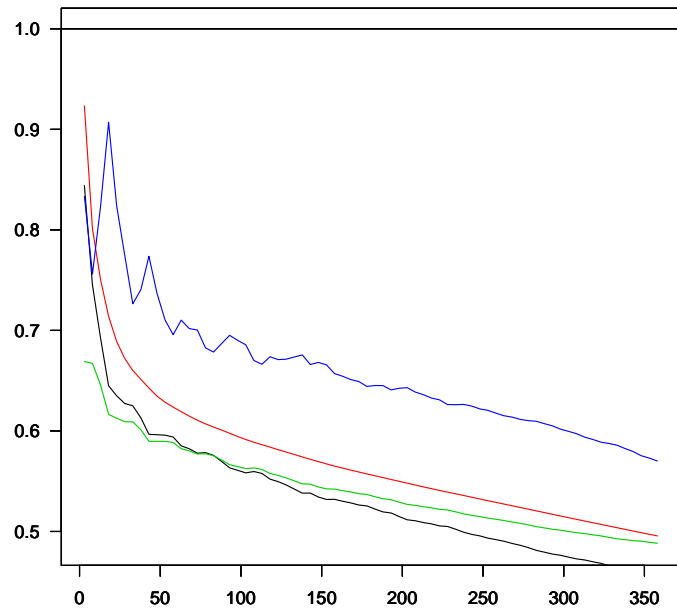
- Comparison of  $\hat{\theta}_n^G$  (in black),  $\hat{\theta}_n^Z$  (in red),  $\hat{\theta}_n^{ML}$  (in green) and  $\hat{\theta}_n^R$  (in blue) to the true  $\theta$  (black horizontal line).
- Simulated distributions : Absolute Gaussian  $|\mathcal{N}|(0, 1)$ , Gamma  $\Gamma(0.25, 1)$ ,  $\Gamma(4, 1)$ , and Weibull  $\mathcal{W}(4, 4)$ ,  $\mathcal{W}(0.25, 0.25)$ .
- Sample size  $n = 500$ ,  $k_n \in \{2, \dots, 360\}$ , 100 replications.
- Computation of the mean estimate (Hill plot, left pannel) and of the Mean Square Error (MSE, right pannel).

Gamma  $\Gamma(0.25, 1) \longrightarrow \theta = 1, b(x) > 0$



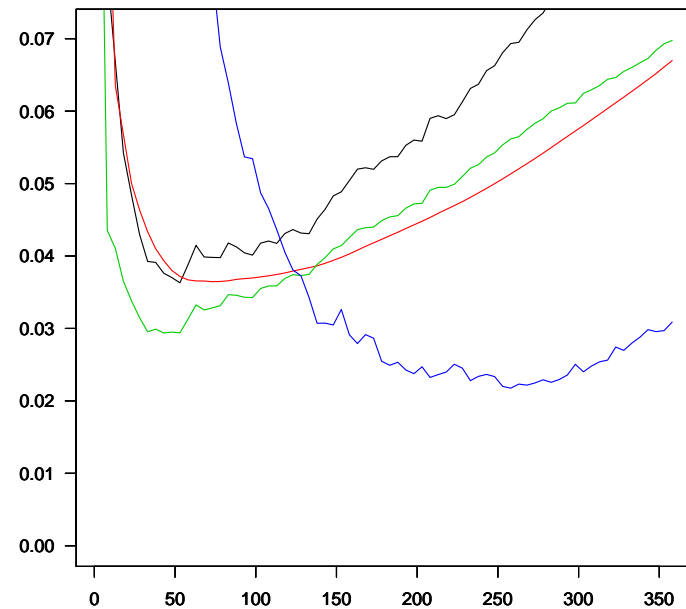
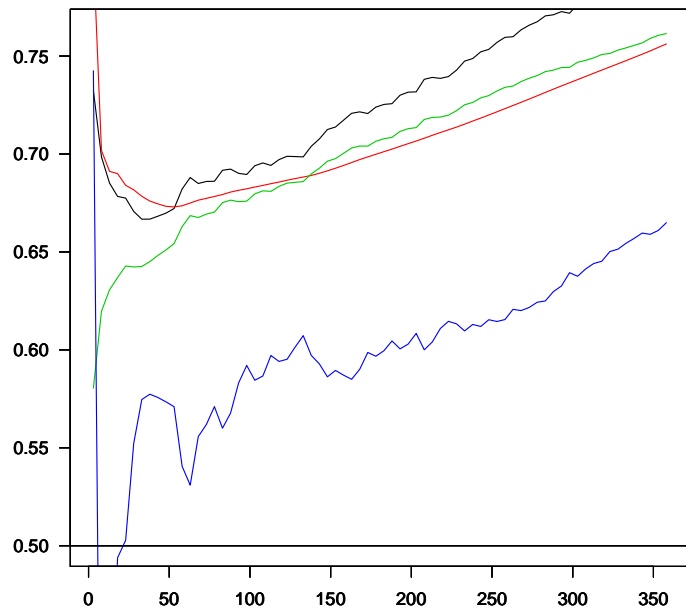
$\hat{\theta}_n^G$  (in black),  $\hat{\theta}_n^Z$  (in red),  $\hat{\theta}_n^{ML}$  (in green) and  $\hat{\theta}_n^R$  (in blue)

Gamma  $\Gamma(4, 1) \longrightarrow \theta = 1, b(x) < 0$



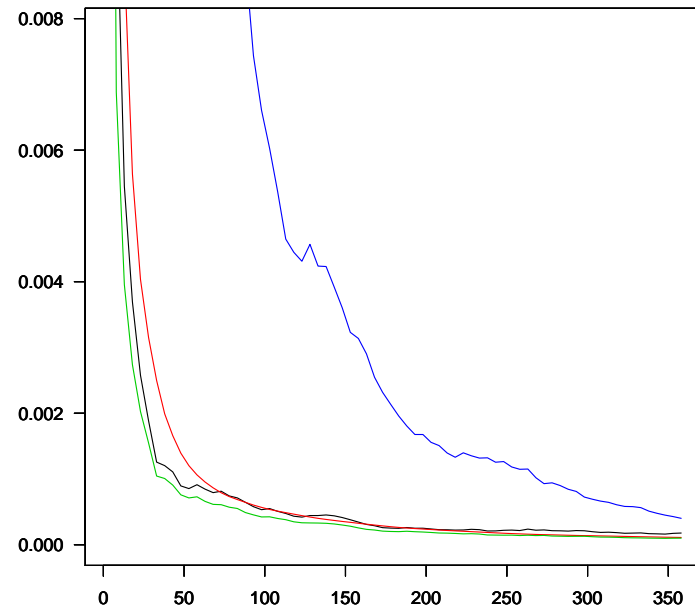
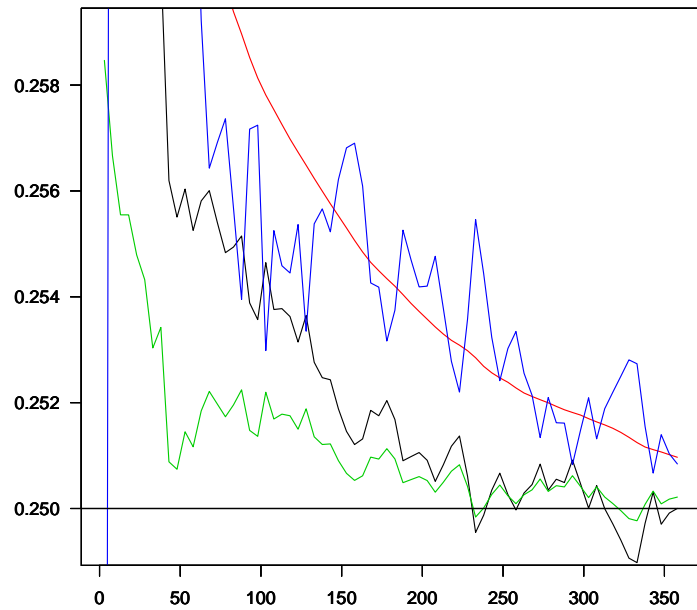
$\hat{\theta}_n^G$  (in black),  $\hat{\theta}_n^Z$  (in red),  $\hat{\theta}_n^{ML}$  (in green) and  $\hat{\theta}_n^R$  (in blue)

Absolute Gaussian  $|\mathcal{N}|(0, 1) \longrightarrow \theta = 1/2, b(x) > 0$



$\hat{\theta}_n^G$  (in black),  $\hat{\theta}_n^Z$  (in red),  $\hat{\theta}_n^{ML}$  (in green) and  $\hat{\theta}_n^R$  (in blue)

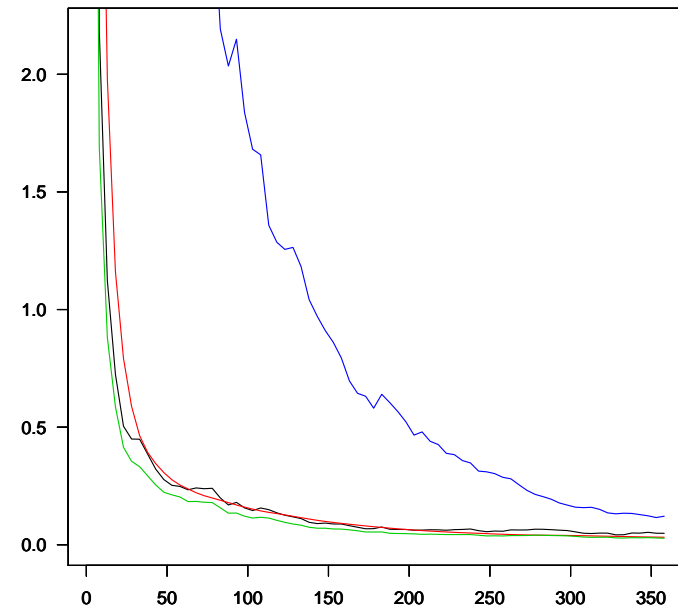
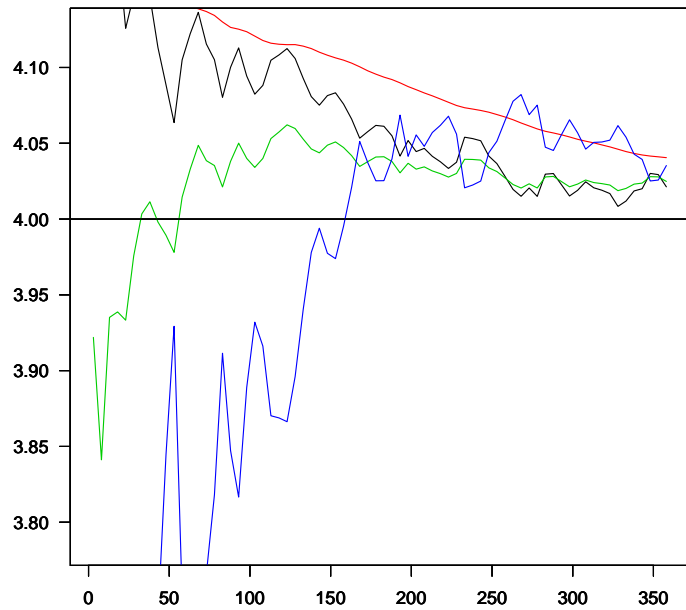
Weibull  $\mathcal{W}(4, 4) \longrightarrow \theta = 0.25, b(x) = 0$



$\hat{\theta}_n^G$  (in black),  $\hat{\theta}_n^Z$  (in red),  $\hat{\theta}_n^{ML}$  (in green) and  $\hat{\theta}_n^R$  (in blue)



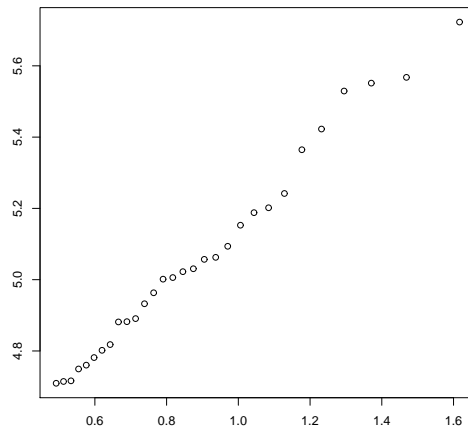
Weibull  $\mathcal{W}(0.25, 0.25) \longrightarrow \theta = 4, b(x) = 0$



$\hat{\theta}_n^G$  (in black),  $\hat{\theta}_n^Z$  (in red),  $\hat{\theta}_n^{ML}$  (in green) and  $\hat{\theta}_n^R$  (in blue)

## 6. Nidd river data.

- 154 exceedances of the level  $65 \text{ m}^3\text{s}^{-1}$  by the river Nidd (Yorkshire, England) during the period 1934-1969 (35 years).
- Widely used in extreme value studies *i.e.* Hosking, Wallis, Wood (1985) and Davison, Smith (1990).
- The  $N$ -year return level is the water level which is exceeded on average once in  $N$  years.



QQ-plot

$$\hat{k}_n = 29,$$

$$\hat{\theta}_n^{ML} \simeq 0.89,$$

Estimation of the 100-year return level:

$$\hat{x}_{p_n}(\hat{\theta}_n^{ML}) = 366 \text{ m}^3\text{s}^{-1}$$

## References

- L. Gardes & S. Girard. Estimation de quantiles extrêmes pour les lois queue de type Weibull : une synthèse bibliographique, *Journal de la Société Française de Statistique*, 154, 98–118, 2013.
- J. El Methni, L. Gardes, S. Girard & A. Guillou. Estimation of extreme quantiles from heavy and light tailed distributions, *Journal of Statistical Planning and Inference*, 142(10), 2735–2747, 2012.
- L. Gardes, S. Girard & A. Guillou. Weibull tail-distributions revisited: a new look at some tail estimators, *Journal of Statistical Planning and Inference*, 141(1), 429–444, 2011.
- J. Diebolt, L. Gardes, S. Girard & A. Guillou. Bias-reduced extreme quantiles estimators of Weibull-tail distributions, *Journal of Statistical Planning and Inference*, 138, 1389–1401, 2008.
- L. Gardes & S. Girard. Estimation of the Weibull tail-coefficient with linear combination of upper order statistics, *Journal of Statistical Planning and Inference*, 138, 1416–1427, 2008.

## References

- J. Diebolt, L. Gardes, S. Girard & A. Guillo. Bias-reduced estimators of the Weibull tail-coefficient, *Test*, 17, 311–331, 2008.
- L. Gardes & S. Girard. Comparison of Weibull tail-coefficient estimators, *REVSTAT - Statistical Journal*, 4(2):163-188, 2006.
- L. Gardes & S. Girard. Estimating extreme quantiles of Weibull tail-distributions, *Communication in Statistics - Theory and Methods*, 34, 1065-1080, 2005.
- S. Girard. A Hill type estimate of the Weibull tail-coefficient, *Communication in Statistics - Theory and Methods*, 33(2), 205-234, 2004