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Cécile Amblard, Stéphane Girard. A semiparametric family of bivariate copulas: dependence properties and estimation procedures. IMS Annual Meeting and X Brazilian School of Probability, Jul 2006, Rio de Janeiro, Brazil. hal-00985320

HAL Id: hal-00985320

<https://inria.hal.science/hal-00985320v1>

Submitted on 29 Apr 2014

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A SEMIPARAMETRIC FAMILY OF BIVARIATE COPULAS: DEPENDENCE PROPERTIES AND ESTIMATION PROCEDURES

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Joint work with Cécile Amblard.

Outline

1. Definition and basic properties.
2. First sub-family, the case $\theta(1) = 0$.
3. Second sub-family, the case $\phi(1) = 0$.
4. Inference procedures.
5. Simulation results.
6. Real data.

1. Definition and basic properties.

Definition. Let I be the unit interval. The family is defined for all $(u, v) \in I^2$ by,

$$C_{\theta, \phi}(u, v) = uv + \theta[\max(u, v)]\phi(u)\phi(v).$$

where ϕ and θ are differentiable $I \rightarrow \mathbb{R}$ functions (vanishing at most on isolated points).

Theorem. $C_{\theta, \phi}$ is a copula if and only if ϕ and θ satisfy the following conditions:

- boundary conditions: $\phi(0) = 0$ and $(\phi\theta)(1) = 0$,
- θ is non increasing on I ,
- $\phi'(u)(\theta\phi)'(v) \geq -1$ for all $0 \leq u \leq v \leq 1$.

Remark. The family can be split in two sub-families according to $\theta(1) = 0$ or $\phi(1) = 0$.

Measure of association.

Let (X, Y) a random pair with joint distribution $H(x, y) = C(F(x), G(y))$. Spearman's Rho: probability of concordance minus the probability of discordance of two random pairs with respective joint cumulative law $C(F, G)$ and FG .

$$\rho = 12 \int_0^1 \int_0^1 C(u, v) du dv - 3.$$

In the case of $C = C_{\theta, \phi}$, we have

$$\rho_{\theta, \phi} = 12 \left[\Phi^2(1)\theta(1) - \int_0^1 \Phi^2(t)\theta'(t) dt \right],$$

where $\Phi(t) = \int_0^t \phi(u) du$.

Remark.

- If $\theta(1) = 0$, then $\rho_{\theta, \phi} \geq 0$.
- If θ is a constant function, then $\rho_{\theta, \phi} = 12\theta\Phi^2(1)$.

Upper tail dependence.

The upper tail dependence coefficient is defined as

$$\lambda = \lim_{t \rightarrow 1} \mathbb{P}(F(X) > t | G(Y) > t) = \lim_{u \rightarrow 1} \frac{\bar{C}(u, u)}{1 - u},$$

where \bar{C} is the survival copula, *i.e.* $\bar{C}(u, v) = 1 - u - v + C(u, v)$.

In the case where $C = C_{\theta, \phi}$, we have

$$\lambda_{\theta, \phi} = -\phi^2(1)\theta'(1).$$

Remark.

- If $\phi(1) = 0$, then $\lambda_{\theta, \phi} = 0$.
- If θ is a constant function, then $\lambda_{\theta, \phi} = 0$.

2. First sub-family, the case $\theta(1) = 0$.

Examples.

- Fréchet upper bound. Choosing $\phi(x) = x$ and $\theta(x) = (1 - x)/x$ yields $C_{\theta,\phi}(u, v) = M(u, v) = \min(u, v)$.
- Independent copula. $\theta(x) = 0$ yields $C_{\theta,\phi}(u, v) = \Pi(u, v) = uv$.
- Cuadras-Augé family: $\phi(x) = x$ and $\theta(x) = x^{-\alpha} - 1$, $0 \leq \alpha \leq 1$ yields

$$C_{\theta,\phi}(u, v) = \min(u, v)^\alpha (uv)^{1-\alpha} = M^\alpha(u, v) \Pi^{1-\alpha}(u, v),$$

which is the weighted geometric mean of M and Π .

Remark.

- $\theta(1) = 0$ and $\theta'(u) \leq 0$ imply $\theta(u) \geq 0$ for all $u \in I$.
- $0 \leq \rho_{\theta,\phi} \leq 1$ \longrightarrow Modelling of positive dependences.
- Lower (0) and upper bounds (1) of $\rho_{\theta,\phi}$ and $\lambda_{\theta,\phi}$ are reached respectively by the Π and M copulas.

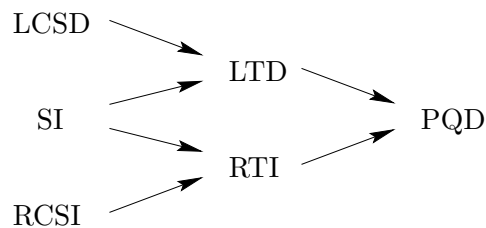
Dependence properties: definitions.

Assume X and Y are exchangeable. X and Y are

- Positively Quadrant Dependent (PQD) if $\mathbb{P}(X \leq x, Y \leq y) \geq \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y)$ for all (x, y) .
- Left Tail Decreasing (LTD) if $\mathbb{P}(Y \leq y|X \leq x)$ is non-increasing in x for all y .
- Right Tail Increasing (RTI) if $\mathbb{P}(Y > y|X > x)$ is nondecreasing in x for all y .
- Stochastically Increasing (SI) if $\mathbb{P}(Y > y|X = x)$ is nondecreasing in x for all y .
- Left Corner Set Decreasing (LCSD) if $\mathbb{P}(X \leq x, Y \leq y|X \leq x', Y \leq y')$ is non-increasing in x' and y' for all (x, y) .
- Right Corner Set Increasing (RCSI) if $\mathbb{P}(X > x, Y > y|X > x', Y > y')$ is nondecreasing in x' and y' for all (x, y) .

Theorem. X and Y are:

- PQD iff $\phi(u)$ has a constant sign on I .
- LTD or LCSD iff either $\{\phi(u)/u$ is non increasing and $\forall u \in I, \phi(u) \geq 0\}$ or $\{\phi(u)/u$ is non decreasing and $\forall u \in I, \phi(u) \leq 0\}$.
- RTI or RCSI iff $\phi(u)/(1-u)$ and $\theta(u)\phi(u)/(1-u)$ are monotone.
- SI iff either $\{\phi$ and $\theta\phi$ are concave and $\forall u \in I, \phi(u) \geq 0\}$ or $\{\phi$ and $\theta\phi$ are convex and $\forall u \in I, \phi(u) \leq 0\}$.



Implications in the general case



Implications in the sub-family

3. Second sub-family, the case $\phi(1) = 0$.

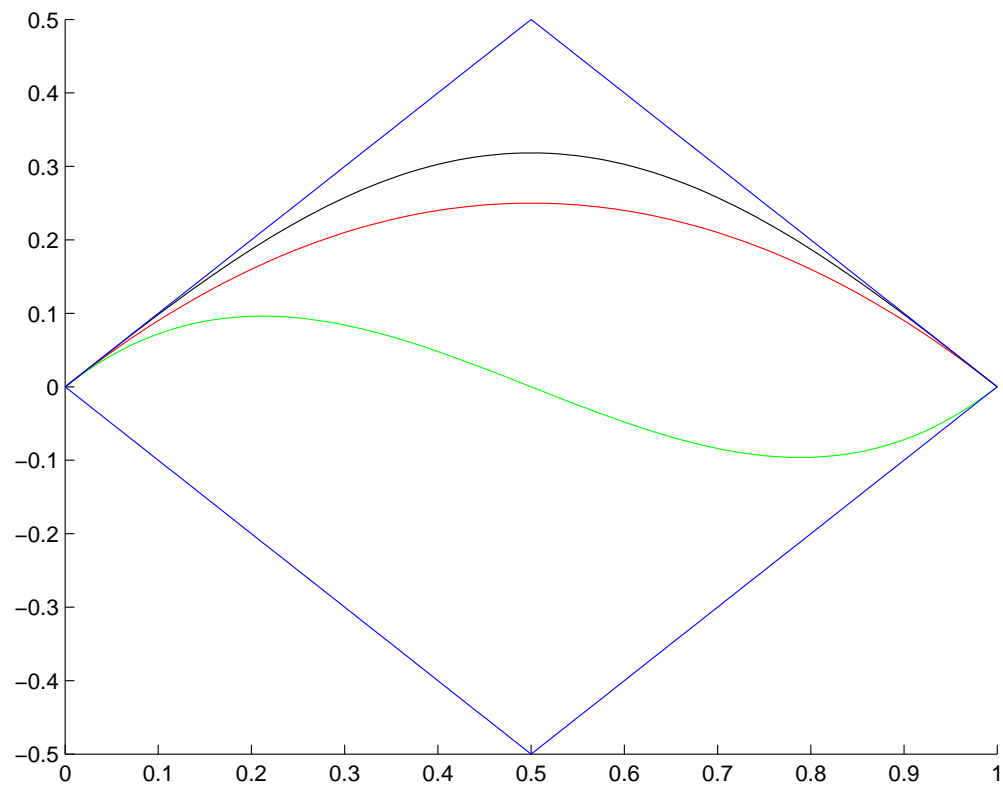
In this case, we restrict ourselves to a constant function θ , *i.e.* $\theta(x) = \theta \in [-1, 1]$.

Theorem. $C_{\theta, \phi}$ is a copula if and only if ϕ and θ satisfy the following conditions:

- boundary conditions: $\phi(0) = 0$ and $\phi(1) = 0$,
- $|\phi'(x)| \leq 1$ for all $x \in I$,
- $|\phi(x)| \leq \min(x, 1 - x)$, for all $x \in I$.

Examples.

- $\phi(x) = \min(x, 1 - x)$: upper bound of the above theorem,
- $\phi(x) = x(1 - x)$: Farlie-Gumbel-Morgenstern family of copulas (Morgenstern, 1956), which contains all copulas with both horizontal and vertical quadratic sections (Quesada-Molina, Rodríguez-Lallena, 1995)
- $\phi(x) = x(1 - x)(1 - 2x)$: symmetric copulas with cubic sections (Nelsen *et al*, 1997),
- $\phi(x) = \pi^{-1} \sin(\pi x)$.



Upper bound, Farlie-Gumbel-Morgenstern, cubic sections, sinus.

Measure of association. The Spearman's Rho can be rewritten as:

$$\rho_{\theta,\phi} = 12\theta \left(\int_I \phi(u) du \right)^2,$$

and it follows that $-3/4 \leq \rho_{\theta,\phi} \leq 3/4$ for all $\theta \in [-1, 1]$. Similar bounds hold for the Kendall's Tau: $-1/2 \leq \tau_{\theta,\phi} \leq 1/2$.

Upper tail dependence. $\rho_{\theta,\phi} = 0$.

Dependence properties. Similar to the previous family in the case $\theta > 0$.

Symmetry properties: definitions.

- X is **symmetric** about a if $(X - a)$ and $(a - X)$ are identically distributed (id).
- X and Y are **exchangeable** if (X, Y) and (Y, X) are id.
- (X, Y) is **marginally symmetric** about (a, b) if X and Y are symmetric about a and b respectively.
- (X, Y) is **radially symmetric** about (a, b) if $(X - a, Y - b)$ and $(a - X, b - Y)$ are id.
- (X, Y) is **jointly symmetric** about (a, b) if the pairs $(X - a, Y - b)$, $(a - X, b - Y)$, $(X - a, b - Y)$ and $(a - X, Y - b)$ are id.

Theorem. In the $C_{\theta, \phi}$ family:

- If X and Y are id then X and Y are exchangeable.

Besides, if (X, Y) is marginally symmetric about (a, b) then:

- (X, Y) is radially symmetric about (a, b) if and only if
 either $\forall u \in I, \phi(u) = \phi(1 - u)$ or $\forall u \in I, \phi(u) = -\phi(1 - u)$.
- (X, Y) is jointly symmetric about (a, b) if and only if $\forall u \in I, \phi(u) = -\phi(1 - u)$.

4. Inference procedures.

Assumptions.

- We restrict ourselves to the second sub-family, with constant function θ :

$$C(u, v) = uv + \theta\phi(u)\phi(v).$$

→ Estimation of θ (scalar) and ϕ (univariate function).

→ Identifiability problem: (θ, ϕ) and $(\alpha\theta, \phi/\sqrt{\alpha})$ yield the same copula for all $\alpha > 0$.

- We focus on the PQD case: $\theta > 0$ and ϕ has a constant sign.

Under these assumptions, the family can be rewritten

$$C(u, v) = uv + \psi(u)\psi(v),$$

where $\psi(x) = \sqrt{\theta}|\phi(x)|$.

→ The estimation of C reduces to the estimation of ψ (positive univariate function).

Estimation of ψ

1) Preprocessing:

- $\{(x_i, y_i), i = 1, \dots, n\}$ a sample of (X, Y) from the cdf $H(x, y) = C(F(x), G(y))$.
- Rank transformations: $u_i = \text{rank}(x_i)/n$ and $v_i = \text{rank}(y_i)/n$.
 $\{(u_i, v_i), i = 1, \dots, n\}$ an approximate sample from the copula $C(u, v)$.
- Pseudo-observations $\{w_i = \max(u_i, v_i), i = 1, \dots, n\}$ from $C(w, w) = w^2 + \psi(w)$.

2) Projection estimate: linear combination of basis functions: $\{e_k, k \geq 1\}$

$$\hat{\psi}(w) = \sum_{k \geq 1} a_k e_k(w), \quad w \in I.$$

Choice of the set of functions:

- no orthogonality condition,
- boundary conditions $e_k(0) = e_k(1) = 0$ for all $k \geq 1$ so that $\hat{\psi}(0) = \hat{\psi}(1) = 0$.

3) Optimization problem: Define

- $w_{1,n} \leq \dots \leq w_{n,n}$, the ordered pseudo-observations,
- M and M' two matrices $M_{i,k} = e_k(w_{i,n})$, $M'_{i,k} = e'_k(w_{i,n})$, $k \geq 1$, $i \in \{1, \dots, n\}$,
- a and b two vectors $b_i = (i/(n+1) - w_{i,n}^2)^{1/2}$, a_i unknown, $i \in \{1, \dots, n\}$.

Definition of the estimator.

- $\hat{\psi}(w_{i,n}) = C(w_{i,n}, w_{i,n}) - w_{i,n}^2 \simeq i/(n+1) - w_{i,n}^2$ for $i = 1, \dots, n$ can be rewritten

$$\min_a \|Ma - b\|^2,$$

- $\hat{\psi}(w_{i,n}) \geq 0$ can be rewritten $Ma \geq 0$,
- $|\hat{\psi}(w_{i,n})| \leq 1$ can be rewritten $-1 \leq M'a \leq 1$.

→ Constrained least-square problem.

Estimation of the Spearman's rho

Recall that

$$\rho_{\theta,\phi} = 12\theta \left(\int_I \phi(u) du \right)^2 = 12 \left(\int_I \psi(u) du \right)^2.$$

Replacing ψ by $\hat{\psi}$ yields the following semi-parametric estimator:

$$\hat{\rho}_{\text{SP}} = 12 \left(\sum_{k \geq 1} a_k \beta_k \right)^2,$$

where we have introduced $\beta_k = \int_I e_k(u) du$.

Another solution: adapt the nonparametric estimator of the Kendall's Tau introduced in (Genest, Rivest, 1993) to obtain

$$\hat{\rho}_{\text{NP}} = \frac{6}{n(n-1)} \sum_{i=1}^n \sum_{j=1}^n \mathbf{1}\{u_j < u_i, v_j < v_i\} - \frac{3}{2},$$

Estimation of high probability regions

Definition. The α -quantile of the copula C is defined by

$$Q_\alpha = \inf\{\lambda(S) : \mathbb{P}(S) \geq \alpha, S \subset I^2\}, \quad 0 < \alpha \leq 1,$$

where λ is the Lebesgue measure on I^2 .

Partitions. $\{I_k, k = 1, \dots, N\}$ be the equidistant N -partition of I , $K_{k,\ell} = I_k \times I_\ell$ the associated $N \times N$ grid. Denote $\delta_{k,\ell} \in \{0, 1\}$, $(k, \ell) \in \{1, \dots, N\}^2$.

Estimator: $\hat{Q}_\alpha = \bigcup_{k,\ell} K_{k,\ell} \mathbf{1}\{\delta_{k,\ell} = 1\}$.

Optimization problem. The $\delta_{k,\ell}$ are defined by

$$\min \sum_{k=1}^N \sum_{\ell=1}^N \delta_{k,\ell},$$

under the constraints $\delta_{k,\ell} \in \{0, 1\}$ and $\sum_{k=1}^N \sum_{\ell=1}^N \delta_{k,\ell} \hat{P}(K_{k,\ell}) \geq \alpha$,

where $\hat{P}(K_{k,\ell})$ is an estimation of the probability $P(K_{k,\ell})$.

Algorithm.

- First step: sort the $\widehat{P}(K_{k,\ell})$ in decreasing order to obtain the sequence \tilde{P}_τ , $\tau = 1, \dots, N^2$.
- Second step: Computation of the number of subsets of the partition:

$$J = \min \left\{ j, \sum_{\tau=1}^j \tilde{P}_\tau \geq \alpha \right\}.$$

- Third step: selection of the J first subsets: $\delta_{k,\ell} = 1$ if $1 \leq \tau(k, \ell) \leq J$,

Estimation of $P(K_{k,\ell})$. Two solutions:

- Semi-parametric estimate based on $\widehat{\psi}$

$$\widehat{P}_{\text{SP}}(K_{k,\ell}) = \frac{1}{N^2} + \left(\widehat{\psi} \left(\frac{k}{N} \right) - \widehat{\psi} \left(\frac{k-1}{N} \right) \right) \left(\widehat{\psi} \left(\frac{\ell}{N} \right) - \widehat{\psi} \left(\frac{\ell-1}{N} \right) \right).$$

- Nonparametric estimate

$$\widehat{P}_{\text{NP}}(K_{k,\ell}) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{(u_i, v_i) \in K_{k,\ell}\}.$$

5. Simulation results.

Numerical experiments on the family of copulas C_k generated by the set of functions

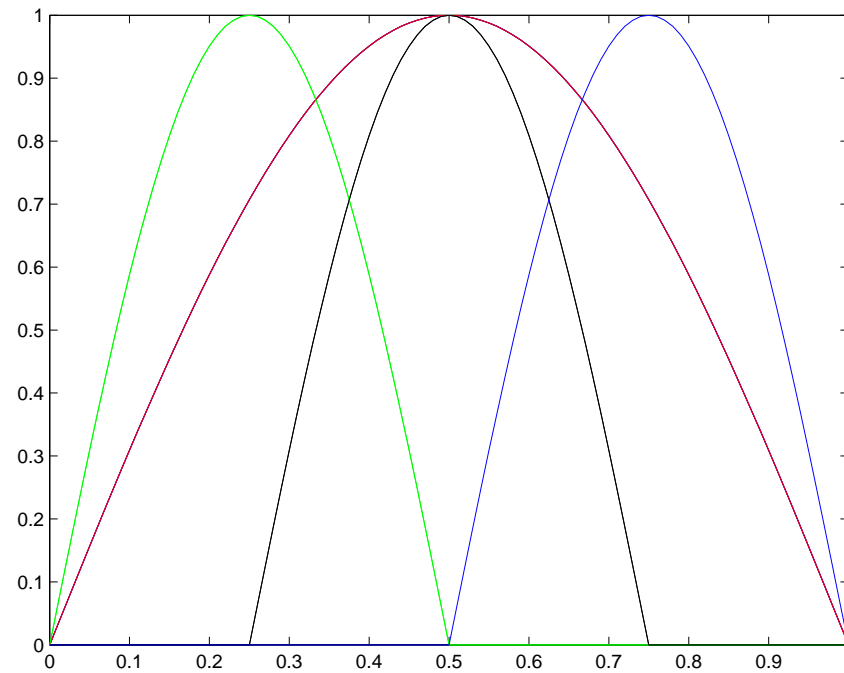
$$\forall k \geq 1, \psi_k(x) = 1 - (x^k + (1-x)^k)^{1/k}, x \in I.$$

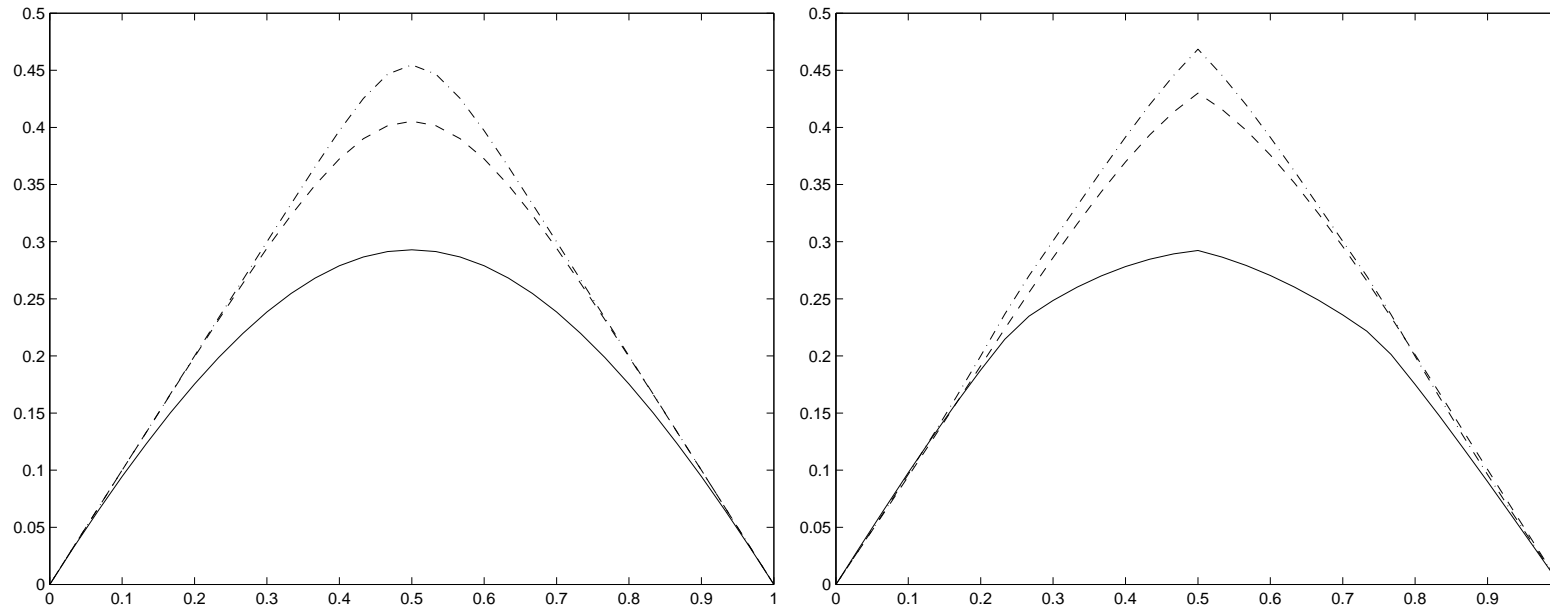
- When $k = 1$, C_1 : uniform distribution on I^2 . Spearman's Rho $\rho_1 = 0$.
- When $k \rightarrow \infty$, $\psi_k(x) \rightarrow \psi_\infty(x) = \min(x, 1-x)$ for all $x \in I$.
 C_∞ : mixture of two uniform distributions on the squares $[0, 1/2]^2$ and $[1/2, 1]^2$ with mixing parameter $1/2$. Spearman's Rho $\rho_\infty = 3/4$ (the maximum value in the sub-family).
- When $1 < k < \infty$, bivariate distribution “interpolating” between the two previous ones.

Chosen basis of functions:

$$e_{s,\ell}(x) = \sin\left(\frac{\pi}{2}(2^{s+1}x - \ell)\right) \mathbf{1}\{2^{s+1}x \in [\ell, \ell + 2]\},$$

s is a scale parameter, ℓ is a location parameter.

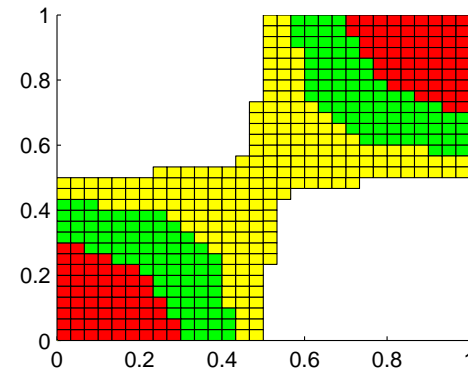
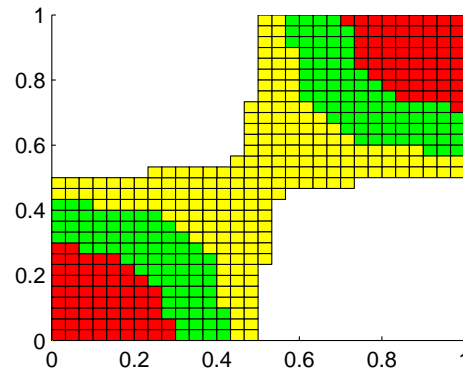
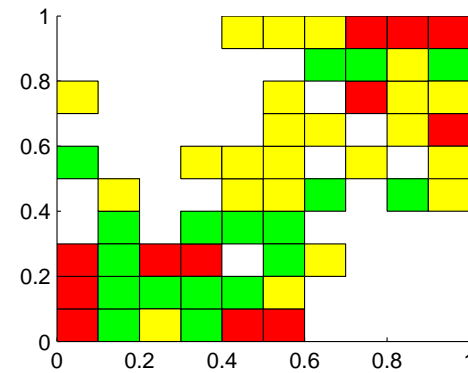
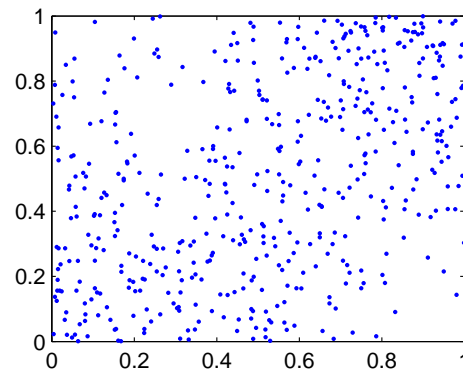




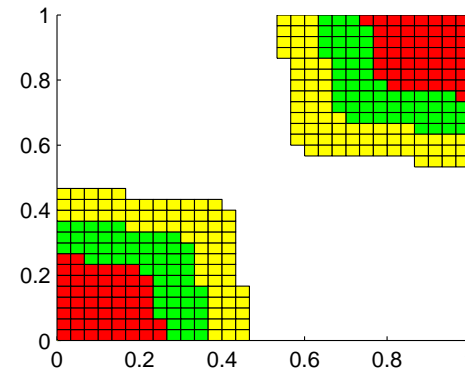
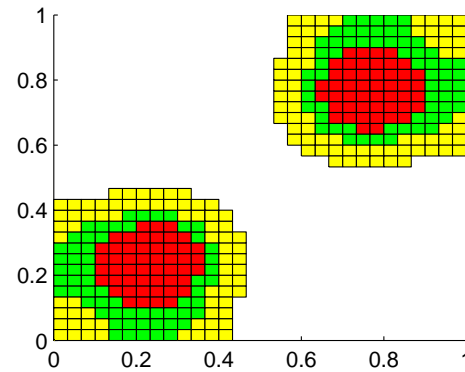
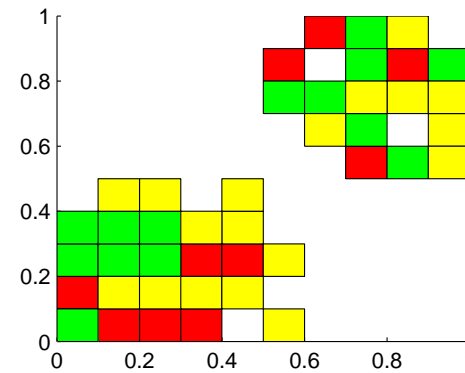
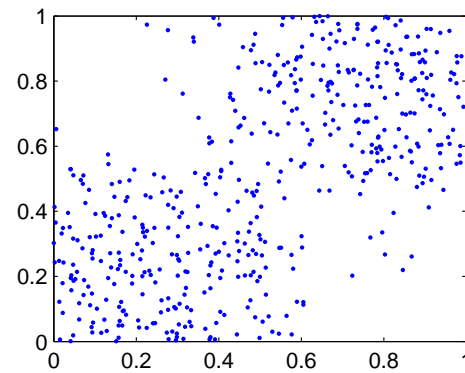
True functions $\psi_k(x)$, $k \in \{2, 4, 8\}$ – Estimated functions $\hat{\psi}_k(x)$, $k \in \{2, 4, 8\}$, $n = 100$.

k	$\rho_k \times 10^{-2}$	$\text{mean}(\hat{\rho}_{\text{SP}}) \times 10^{-2}$	$\text{mean}(\hat{\rho}_{\text{NP}}) \times 10^{-2}$
1	0	0.81	0.18
2	42.5	43.0	41.2
4	66.4	65.8	64.3
6	71.2	70.6	68.8
8	72.8	72.1	70.2

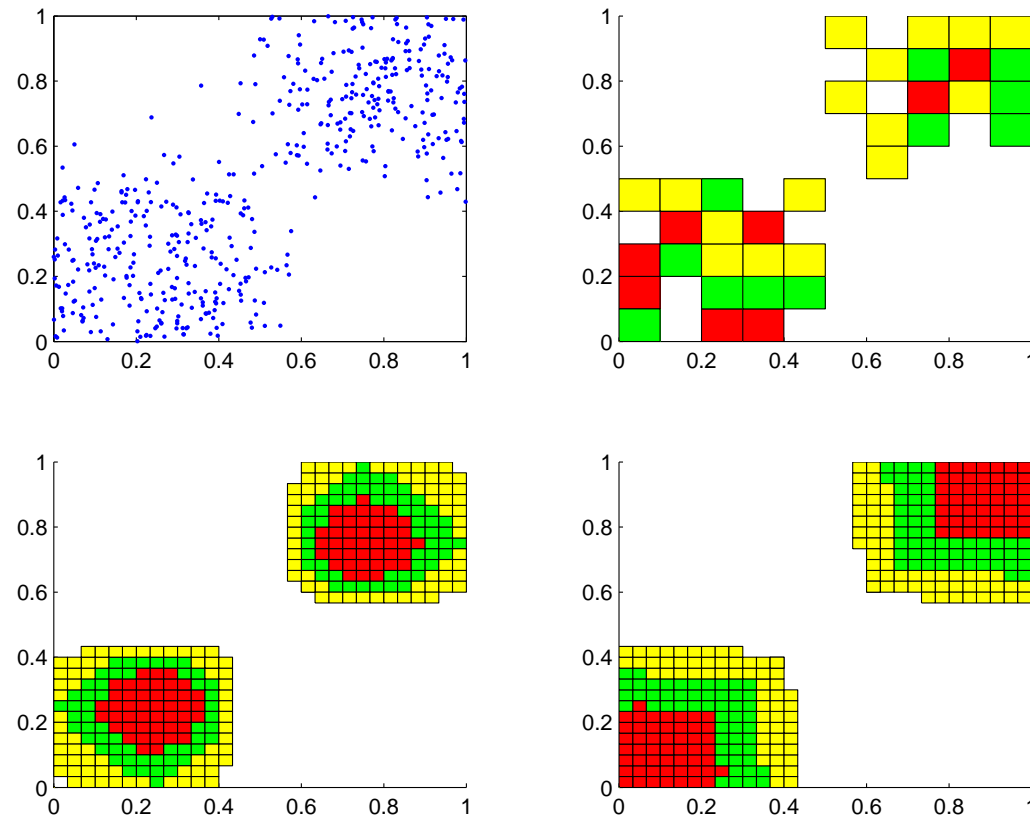
Estimation of the generating function and of the Spearman's Rho (ρ_k). The mean value of the estimates $\hat{\rho}_{\text{SP}}$ and $\hat{\rho}_{\text{NP}}$ are evaluated on 100 repetitions.



Estimation of high probability regions Q_α from C_2 . Red: $\alpha = 0.25$, green: $\alpha = 0.5$, yellow: $\alpha = 0.75$. Top left: simulated sample, top right: nonparametric estimate, bottom left: semiparametric estimate, bottom right: semiparametric estimate with the true function ψ , ($n = 500$).



Estimation of high probability regions Q_α from C_4 . Red: $\alpha = 0.25$, green: $\alpha = 0.5$, yellow: $\alpha = 0.75$. Top left: simulated sample, top right: nonparametric estimate, bottom left: semiparametric estimate, bottom right: semiparametric estimate with the true function ψ , ($n = 500$).

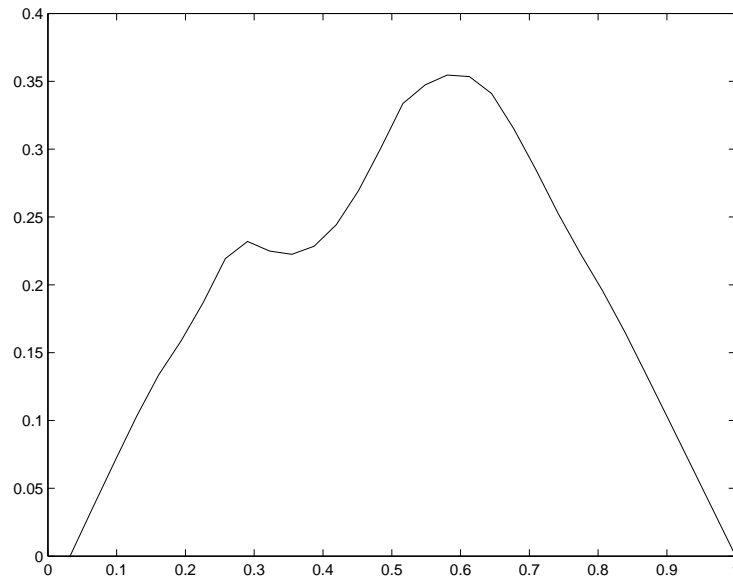


Estimation of high probability regions Q_α from C_8 . Red: $\alpha = 0.25$, green: $\alpha = 0.5$, yellow: $\alpha = 0.75$. Top left: simulated sample, top right: nonparametric estimate, bottom left: semiparametric estimate, bottom right: semiparametric estimate with the true function ψ , ($n = 500$).

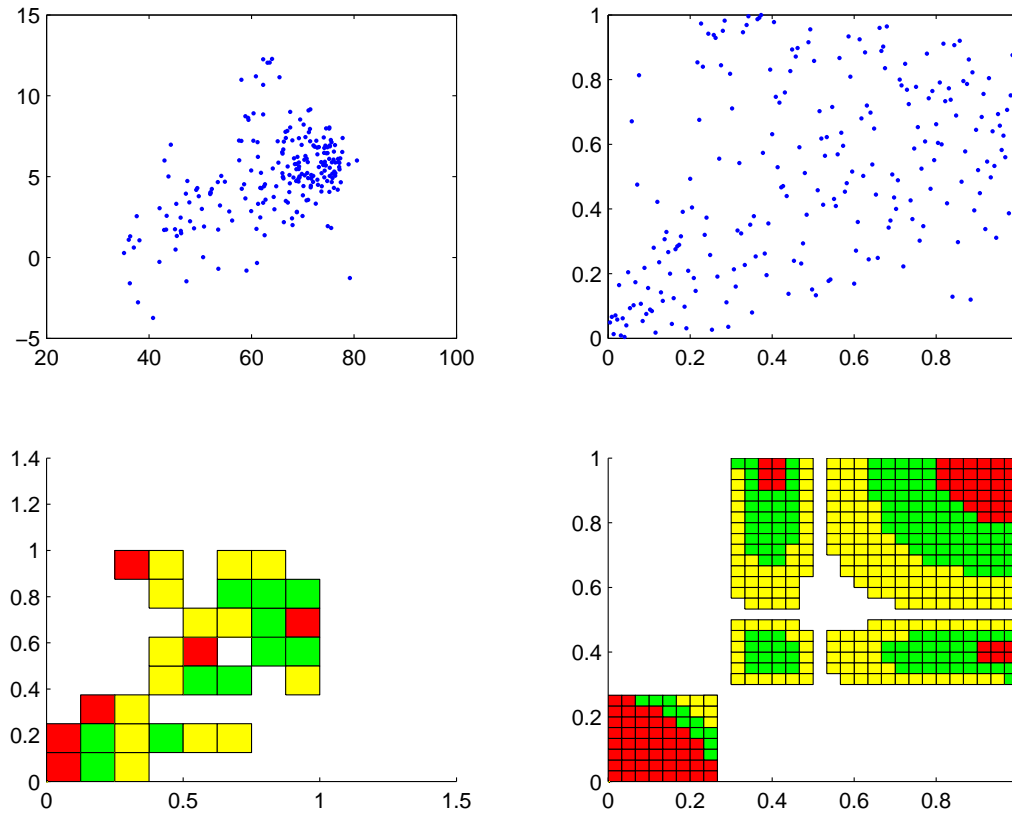
6. Real data.

$n = 225$ countries, two variables: X , the life expectancy at birth (years) in 2002 of the total population and Y , the difference between the life expectancy at birth of women and men. <http://www.odci.gov/cia/publications/factbook/>.

According to the PQD test proposed in (Scaillet, 2004), these data are PQD.



$$\hat{\rho}_{\text{NP}} = 52.4\%$$
$$\hat{\rho}_{\text{SP}} = 40.7\%$$



Estimation of high probability regions Q_α from real data. Red: $\alpha = 0.25$, green: $\alpha = 0.5$, yellow: $\alpha = 0.75$. Top left: real data, top right: real data after rank transformation, bottom left: nonparametric estimate, bottom right: semiparametric estimate.

Further work.

- Goodness of fit test.
- Study of the sub-family $\phi(1) = 0$ without the assumption that θ is a constant function.
(what is the lower bound of $\rho_{\theta, \phi}$?)
- Estimation of the function θ in the general case.

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