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To cite this version:
Andrey Polyakov. Discontinuous Lyapunov Functions for Nonasymptotic Stability Analysis. IFAC World Congress, Aug 2014, Cape Town, South Africa. hal-00977869

HAL Id: hal-00977869
https://inria.hal.science/hal-00977869
Submitted on 11 Apr 2014

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Discontinuous Lyapunov Functions for Nonasymptotic Stability Analysis

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Abstract: The paper presents mathematical tools required for finite-time stability analysis of discontinuous control systems using discontinuous Lyapunov functions. Elements of Filippov theory of differential equations with discontinuous right-hand sides and stability notions are briefly observed. Concepts of generalized derivatives and non-smooth Lyapunov functions are considered. The generalized Lyapunov theorems for stability analysis and convergence time estimation are presented and supported by examples.

1. INTRODUCTION

During whole history of control theory, a special interest of researchers was focused on systems with relay (switching) control elements (Flugge-Lotz [1953], Tsypkin [1984], Utkin et al. [2009], Boiko [2009]). Such control systems have found applications in many engineering areas. They are simple, effective, cheap and sometimes they have better dynamics than linear systems (Tsypkin [1984]). In practice both input and output of a system may be of a relay type. For example, automobile engine control systems sometimes use λ - sensor with almost relay output characteristics, i.e the only sign of a controllable output can be measured (Choi and Hedrick [1996]). In the same time, territirors can be considered as relay "actuators" for some power electronic systems (Utkin et al. [2009]).

Mathematical backgrounds for a rigorous study of relay systems were presented in the beginning of 1960s by the celebrated Filippov’s theory of differential equations with discontinuous right-hand sides, see Filippov [1988]. Following this theory, discontinuous differential equations have to be extended to differential inclusions by means of a special procedure. It is worth to stress that Filippov’s theory was severely criticized by many authors, see Neimark [1961], Aizerman and Pyatnitskii [1974], Utkin et al. [2009], since it does not describe adequately some discontinuous and relay models. That is why, extensions and specifications of this theory appear rather frequently, see, for example, Bartolini and Zolezzi [1985], Heemels and Weiland [2008].

Analysis of relay systems is frequently related to a specific property, which is called finite-time stability (Roxin [1966], Utkin et al. [2009], Bhat and Bernstein [2000], Orlov [2005], Moulay and Perruquetti [2008]). Indeed, the simplest example of a finite-time stable system is the relay system of the form: \( \dot{x} = -\text{sign}(x), x \in \mathbb{R}, x(0) = x_0 \). Any solution of this system reaches the origin in a finite time \( T(x_0) = |x_0| \) and remains there for all later time instants. Sometimes, this conceptually very simple property is hard to proof theoretically. From a practical point of view, it is also important to estimate a time of stabilization (settling time). Both these problems can be tackled by Lyapunov Function Method (Lyapunov [1992], Barbashin [1970], Bacciotti and Rosier [2005]). However, designing a finite-time Lyapunov function of a rather simple form is a difficult problem for many systems. For concrete relay systems appropriate Lyapunov functions can be nonsmooth (Polyakov and Poznyak [2009a, b]), non-Lipschitz (Moreno and Osorio [2012]) and even discontinuous Polyakov and Poznyak [2012]. Some problems of asymptotic and finite-time stability analysis using Lipschitz continuous Lyapunov functions are studied in, for example, Shevitz and Paden [1994], Bhat and Bernstein [2000], Bacciotti and Ceragioli [2006], Moulay and Perruquetti [2008]. The lower semicontinuous Lyapunov functions for asymptotic stability analysis are discussed in Chellaboina et al. [1999], Clarke et al. [1995], Zhu [2002].

This paper elaborates the non-asymptotic (finite-time) stability analysis of differential inclusions based on discontinuous Lyapunov functions. The main tool to be applied for this purpose is the theory of contingent derivatives (see Zaremba [1934], Natanson [1955], Mordukhovich [2006]). For this purpose is the theory of contingent derivatives (see Zaremba [1934], Natanson [1955], Mordukhovich [2006]). The paper is organized as follows. The next section presents notations are used in the paper. Then the elements of the Filippov theory differential equations with discontinuous right-hand sides and differential inclusions are discussed. Next the evolution of the stability notion is observed. After then the supporting results about contingent derivatives are studied in order to present a discontinuous Lyapunov function method in the last section.

The following notations are used in the paper \( \mathbb{R} \) is the set of real numbers and \( \mathbb{R} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\} \), \( \mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\} \) and \( \mathbb{R}_+ = \mathbb{R}_+ \cup \{+\infty\} \); \( I \) denotes one of the following intervals: \([a, b], (a, b], [a, b)\) or \((a, b)\), where \( a, b \in \mathbb{R}, a < b; \) the set consisting of elements \( x_1, x_2, \ldots, x_n \) is denoted by \( \{x_1, x_2, \ldots, x_n\} \); the set of all subsets of a set \( M \subseteq \mathbb{R}^n \) is denoted by \( 2^M \); the inner product of \( x, y \in \mathbb{R}^n \) is denoted by \( \langle x, y \rangle \) and \( \|x\| = \sqrt{\langle x, x \rangle} \); the sign function is defined by

\[
\text{sign}_\rho[\sigma] = \begin{cases} 
1 & \text{if } \rho > 0, \\
-1 & \text{if } \rho < 0, \\
\sigma & \text{if } \rho = 0
\end{cases}
\]

where \( \sigma \in \mathbb{R} : -1 \leq \sigma \leq 1. \) If \( \sigma = 0 \) we use the notation \( \text{sign}[\rho]; \) the set-valued modification of the sign function is
everywhere on $x$ segment $x$ continuous function and all $(t, x)$. Let $\int \Omega$ be nonempty, compact and convex for all $t, x, \delta > 0$, then there exists at least one absolutely \( \Phi(t_0, x_0) \) and a solution of (4), (7) by $x(t, t_0, x_0) \in \Phi(t_0, x_0)$.

Nomuniqueness of solutions implies two types of stability for differential inclusions (4): weak stability (a property holds for a solution) and strong stability (a property holds for all solutions) (see, for example, Filippov [1988]). Weak stability usually is not enough for robust control purposes. This section observes only strong stability properties of the system (4). All conditions presented in definitions below are assumed to be held for all solutions $x(t, t_0, x_0) \in \Phi(t_0, x_0)$.

3. STABILITY AND CONVERGENCE RATE

Consider the differential inclusion (4) for $t > t_0$ with an initial condition

$$x(t_0) = x_0,$$

where $x_0 \in \mathbb{R}^n$ is given.

Cauchy problem (4), (7) may have non-unique solution for a given $t_0 \in \mathbb{R}$ and a given $x_0 \in \mathbb{R}^n$. Let us denote the set of all solutions of Cauchy problem (4), (7) by $\Phi(t_0, x_0)$ and a solution of (4), (7) by $x(t, t_0, x_0) \in \Phi(t_0, x_0)$.

3.1 Lyapunov and asymptotic stability

The concept of stability introduced in the famous thesis of A.M. Lyapunov (see, its recent translation in Lyapunov [1992]) is the central notion of the modern stability theory. It considers some nominal motion $x^*(t, t_0, x_0)$ of a dynamic system and studies small perturbations of the initial condition $x_0$. If they imply small deviations of perturbed motions from $x^*(t, t_0, x_0)$ then the nominal motion is called stable. We study different stability forms of the zero solution (or, equivalently, the origin) of the system (4), since making the change of variables $y = x - x^*$ it is nothing to transform any problem of stability analysis for some nontrivial solution $x^*(t, t_0, x_0)$ to the same problem for the zero solution. Assume that $0 \in F(t, 0)$, $\forall t \in \mathbb{R}$, where $F$ is defined by (4).

Definition 2. (Lyapunov stability). The origin of the system (4) is said to be Lyapunov stable if for $\forall t_0 \in \mathbb{R}$ and $\forall t \in \mathbb{R}$ there exists $\delta = \delta(t, t_0) \in \mathbb{R}_+$ such that:

1) any solution $x(t, t_0, x_0)$ (with $x_0 \in B(\delta)$) of Cauchy problem (4), (7) exists for all $t > t_0$; 2) $x(t, t_0, x_0) \in B(\epsilon)$ for $t > t_0$ if $x_0 \in B(\delta)$.

If the function $\delta$ does not depend on $t_0$ then the origin is called uniformly Lyapunov stable. For instance, if $F(t, x)$ is independent of $t$ (time-invariant case) and the zero solution of (4) is Lyapunov stable, then it is uniformly Lyapunov stable. The origin, which does not satisfy any condition from Definition 2, is called unstable.

Proposition 1. If the origin of (4) is Lyapunov stable then $x(t) = 0$ is the unique solution of (4), (7) with $x_0 = 0$.

Definition 3. (Asymptotic attractivity). The origin of the system (4) is said to be asymptotically attractive if for $\forall t_0 \in \mathbb{R}$ there exists a set $U(t_0) \subseteq \mathbb{R}^n : 0 \in \text{int}(U(t_0))$ such that:

1) any solution $x(t, t_0, x_0)$ (with $x_0 \in U(t_0)$) of Cauchy problem (4), (7) exists for $t > t_0$; 2) $\lim_{t \to +\infty} \|x(t, t_0, x_0)\| = 0$. The set $U(t_0)$ is called attraction domain.
Consider the system (4), (7) exists for all parameters in the problem (4), (7). In practice, Lyapunov’s method is “weaker” than asymptotic attractivity. It is worth to stress that the presented finite-time attractivity property, introduced originally in Bhat and Bernstein [2000], is not to be confused with Lyapunov’s method.

3.2 Finite-time Stability

Introduce the functional \( T_0 : \mathbb{W}^n_{[t_0, +\infty)} \to \mathbb{R}^+ \cup \{0\} \) by the following formula \( T_0(y(t)) = \inf_{\tau \geq t_0; y(\tau) = 0} \tau \). If \( y(\tau) \neq 0 \) for all \( \tau \in [t_0, +\infty) \), then \( T_0(y(t)) = +\infty \).

The settling-time function of the system (4) is defined by
\[
T(t, t_0, x_0) = \sup_{x(t, t_0, x_0) \in \Phi(t_0, x_0)} T_0(x(t, t_0, x_0)) - t_0,
\]
where \( \Phi(t_0, x_0) \) is the set of all solutions of the Cauchy problem (4), (7).

Definition 5. (Finite-time attractivity). The origin of the system (4) is said to be finite-time attractive if for all \( t_0 \in \mathbb{R} \) there exists a set \( V(t_0) \subseteq \mathbb{R}^n : 0 \in \text{int}(V(t_0)) : 1 \) any solution \( x(t, t_0, x_0) \) (with \( x_0 \in V(t_0) \)) of Cauchy problem (4), (7) exists for all \( t > t_0 \); 2 \( T(t_0, x_0) < +\infty \) for \( x_0 \in V(t_0) \) and for \( t_0 \in \mathbb{R} \). The set \( V(t_0) \) is called finite-time attraction domain.

It is worth to stress that the presented finite-time attractivity property, introduced originally in Bhat and Bernstein [2000], is “weaker” than asymptotic attractivity. However, it is important for many control applications. For example, antismall signal control problem has to be studied only on a finite interval of time, since there is nothing to control after missile explosion. In practice, Lyapunov stability is additionally required in order to guarantee a robustness of a control system.

Definition 6. (Finite-time stability (Roxin [1966])). The origin of the system (4) is said to be finite-time stable if it is Lyapunov stable and finite-time attractive.

If \( V(t_0) = \mathbb{R}^n \) then the origin of (4) is called globally finite-time stable.

Example 1. Consider the system
\[
\dot{x} = -\frac{2}{\sqrt{\pi}} \cdot \text{sign}(x) \cdot [2tx], \quad t > t_0, \quad x \in \mathbb{R},
\]
where \( t_0 \in \mathbb{R} \). It can be shown that the origin of this system is finite-time attractive with an attraction domain \( V(t_0) = B \left( \varepsilon t_0^2 \left( 1 - \text{erf}(|t_0|) \right) \right) \), where \( \text{erf}(z) = \int_{-z}^{z} e^{-\tau^2} d\tau \), \( z \in \mathbb{R} \) is the so-called Gauss error function. Moreover, the origin of the considered system is Lyapunov stable (for \( \forall \varepsilon > 0 \) and \( \forall t_0 \in \mathbb{R} \) we can select \( \delta = \delta(t_0) = \min \left\{ \varepsilon, \varepsilon t_0^2 \left( 1 - \text{erf}(|t_0|) \right) \right\} \)), so it is finite-time stable. In particular, for \( t_0 > t_0 \) the settling-time function has the form \( T(t_0, x_0) = \text{erf}^{-1} \left( |x_0| e^{-\frac{t_0^2}{2}} + \text{erf}(t_0) \right) - t_0 \), where \( \text{erf}^{-1} \) denotes the inverse function to \( \text{erf}(\cdot) \).

The proposition 1 implies the following claim.

Proposition 2. (Bhat and Bernstein [2000]). If the origin of the system (4) is finite-time stable then it is asymptotically stable and \( x(t, t_0, x_0) = 0 \) for \( t > t_0 + T(t_0, x_0) \).

4. GENERALIZED DERIVATIVES

The celebrated Second Lyapunov Method is founded on the so-called energetic approach to stability analysis. It considers any positive definite function \( V(\cdot) \) as an possible energeic characteristic (“energy”) of a dynamic system and studies evolution of this “energy” in time. If a dynamic system has an energetic function, which is decreasing (strictly decreasing) along any trajectory of the system, then this system has a stability property and the corresponding energetic function is called Lyapunov function.

For example, to analyze asymptotic stability of the origin of the system
\[
\dot{x} = f(t, x), \quad f \in C(\mathbb{R}^{n+1}), \quad t \in \mathbb{R}_+, x \in \mathbb{R}^n.
\]
It is sufficient to find a continuous positive definite function \( V(\cdot) \) such that for any solution \( x(t) \) of the system (10) the function \( V(x(t)) \) is decreasing and tending to zero for \( t \to +\infty \). The existence of such function guarantees asymptotic stability of the origin of the system (10) due to Zubov’s theorem (see, Poznyak [2008]).

If the function \( V(x) \) is continuously differentiable then the required monotonicity property can be rewritten in the form of the classical condition Lyapunov [1992]:
\[
\dot{V}(x(t)) = \nabla V(x)f(t, x) < 0.
\]
The inequality (11) is very usable, since it does not require knowing the solutions of (10) in order to check the asymptotic stability. From the practical point of view, it is important to represent monotonicity conditions in the form of differential or algebraic inequalities like (11).

Finite-time stability analysis of relay systems is frequently based on non-smooth or even discontinuous Lyapunov functions Roxin [1966], Polyakov and Poznyak [2009b], Moreno and Osorio [2012], which require consideration of generalized derivatives and generalized gradients in order to verify stability conditions. This section presents all necessary backgrounds for the corresponding non-smooth analysis.

4.1 Derivative Numbers and Monotonicity

Let \( \mathcal{T} \) be one of the following intervals: \( [a, b], (a, b), [a, b) \) or \( (a, b] \), where \( a, b \in \mathbb{R}, a < b \).

Recall that the function \( \varphi : \mathbb{R} \to \mathbb{R} \) is called decreasing on \( \mathcal{T} \) if \( \forall t_1, t_2 \in \mathcal{T} : t_1 \leq t_2 \Rightarrow \varphi(t_1) \geq \varphi(t_2) \).

Let \( K \) be a set of all sequences of real numbers converging to zero, i.e. \( \{h_n\} \in K \iff h_n \to 0, h_n \neq 0 \).

Let a real-valued function \( \varphi : \mathbb{R} \to \mathbb{R} \) be defined on \( \mathcal{T} \).

Definition 7. (Natanson [1955], page 207). A number \( D_{h_n}(\varphi)(t) = \lim_{n \to +\infty} \varphi(t + h_n) - \varphi(t) \) is called derivative number of the function \( \varphi \) at point \( t \in \mathcal{T} \), if finite or infinite limit exists.

The set of all derivative numbers of the function \( \varphi \) at the point \( t \in \mathcal{T} \) is called contingent derivative:
\[
D_K \varphi(t) = \bigcup_{(h_n) \in K} \{D_{h_n}(\varphi)(t)\} \subseteq \mathbb{R}.
\]
A contingent derivative of a vector-valued function \( \varphi : \mathbb{R} \to \mathbb{R}^n \) can be defined in the same way. If a function \( \varphi(t) \) is differentiable at a point \( t \in I \) then \( D_{\varphi}(t) = \{ \dot{\varphi}(t) \} \).

**Lemma 1.** (Natanson [1955], page 208). If a function \( \varphi : \mathbb{R} \to \mathbb{R} \) is defined on \( I \) then
1) a set \( D_{\varphi}(t) \subseteq \mathbb{R} \) is nonempty for any \( t \in I \);
2) for any \( t \in I \) and for any sequence \( \{ h_n \} \subseteq \mathbb{R} : t + h_n \in I \) there exists a subsequence \( \{ h_{n'} \} \subseteq \{ h_n \} \) such that finite or infinite derivative number \( D_{\varphi}(t) \) exists.

Lemma 1 remains true for a vector-valued function \( \varphi : \mathbb{R} \to \mathbb{R}^n \). Inequalities \( y < 0 \), \( y \leq 0 \), \( y > 0 \), \( y \geq 0 \) for \( y \in \mathbb{R}^n \) are understood in a componentwise sense. If for \( \forall y \in D_{\varphi}(t) \) we have \( y < 0 \) then we write \( D_{\varphi}(t) < 0 \). Other ordering relations \( \leq, \geq \) for contingent derivatives are interpreted analogously.

The contingent derivative also helps to prove monotonicity of a non-differentiable function.

**Lemma 2.** (Natanson [1955], page 266). If a function \( \varphi : \mathbb{R} \to \mathbb{R} \) is defined on \( I \) and the inequality \( D_{\varphi}(t) \leq 0 \) holds for all \( t \in I \), then \( \varphi(t) \) is **decreasing function** on \( I \) and differentiable almost everywhere on \( I \).

**Example 2.** The function \( \varphi(t) = -t - \sin t \) has a negative contingent derivative for all \( t \in \mathbb{R} \) and for any \( \sigma \in (-1, 1] \), where the function \( \sin \) is defined by (1).

Indeed, \( D_{\varphi}(t) = \{-1\} \) for \( t \neq 0 \) and \( D_{\varphi}(0) = (-\infty) \) if \( \sigma \in (-1, 1) \) and \( D_{\varphi}(0) = (-\infty, -1] \) if \( \sigma \in (-1, 1] \).

The next lemma simplifies the monotonicity analysis of nonnegative functions.

**Lemma 3.** If 1) the function \( \varphi : \mathbb{R} \to \mathbb{R} \) is **nonnegative** on \( I \); 2) the inequality \( D_{\varphi}(t) \leq 0 \) holds for all \( t \in I \); \( \varphi(t) \neq 0 \); 3) the function \( \varphi(t) \) is **continuous** at any \( t \in I \); \( \varphi(t) = 0 \); then \( \varphi(t) \) is **decreasing function** on \( I \) and differentiable almost everywhere on \( I \).

The definition of Wazewski given below presents the relation between solutions of a differential inclusion (4) and its contingent derivatives.

**Lemma 4.** (Filippov [1988], page 70). A set-valued function \( F : \mathbb{R}^{n+1} \to \mathbb{R}^n \) be defined, upper-semicontinuous on a closed nonempty set \( \Omega \subseteq \mathbb{R}^n \). Let a function \( \varphi(t, x) = \varphi(t) \) be nonempty, compact and convex for all \( (t, x) \in \Omega \). Let the solution to the differential inclusion \( \dot{x} \in F(t, x) \) be determined uniquely for all \( t \in I \) and for any \( x \in \Omega \). Let \( \varphi(t) \) be defined on \( I \) and \( \varphi(t) = 0 \). Then
\[
\dot{x}(t) \in F(t, x(t)) \quad \text{almost everywhere on } I \quad \iff \quad D_{\varphi}(t) \subseteq F(t, x(t)) \quad \text{everywhere on } I.
\]

### 4.2 Generalized directional derivatives

Stability analysis based on Lyapunov functions requires calculation of derivatives of positive definite functions along trajectories of a dynamic system. If Lyapunov function is non-differentiable, a concept of generalized directional derivatives (see, for example, Clarke [1990]) can be used for this analysis. This survey introduces generalized directional derivatives by analogy with contingent derivatives for scalar functions.

Let \( \mathbb{M}(d) \) be a set of all sequences of real vectors converging to \( d \in \mathbb{R}^n \), i.e. \( \{v_n\} \in \mathbb{M}(d) \iff v_n \to d, v_n \in \mathbb{R}^n \).

Let a function \( \varphi : \mathbb{R}^n \to \mathbb{R}^p \) be defined on an open nonempty set \( \Omega \subseteq \mathbb{R}^n \) and \( d \in \mathbb{R}^p \).

**Definition 8.** A number
\[
D_{\{h_n\}, \{v_n\}} \varphi(x, d) = \lim_{n \to +\infty} \frac{\varphi(x + h_n v_n) - \varphi(x)}{h_n}, \quad h_n \in \mathbb{K}, \{v_n\} \in \mathbb{M}(d) : x + h_n v_n \in \Omega
\]

is called **directional derivative number** of the function \( \varphi(x) \) at the point \( x \in \Omega \) on the direction \( d \in \mathbb{R}^n \), if finite or infinite limit exists.

The set of all directional derivative numbers of the function \( \varphi(x) \) at the point \( x \in \Omega \) on the direction \( d \in \mathbb{R}^n \) is called **directional contingent derivative**
\[
D_{\mathbb{K}, \mathbb{M}(d)} \varphi(x) = \bigcup_{d \in \mathbb{K}, \{v_n\} \in \mathbb{M}(d)} D_{\{h_n\}, \{v_n\}} \varphi(x, d)
\]

Similarly to Lemma 1, if \( x \in \Omega \) then the set \( D_{\mathbb{K}, \mathbb{M}(d)} \varphi(x) \) is nonempty for any function \( \varphi(x) \) defined on an open nonempty set \( \Omega \subseteq \mathbb{R}^n \) and any \( d \in \mathbb{R}^p \). The next lemma gives a chain rule for the introduced contingent derivative.

**Lemma 5.** Let a function \( \varphi : \mathbb{R}^n \to \mathbb{R}^p \) be defined on an open nonempty set \( \Omega \subseteq \mathbb{R}^n \) and a function \( x : \mathbb{R}^n \to \mathbb{R}^n \) be defined on \( I \), such that \( x(t) \in \Omega \) if \( t \in I \) and the contingent derivative \( \varphi(x(t)) \) is bounded for all \( t \in I \). Then the inclusion \( \varphi(x(t)) \subseteq \bigcup_{d \in \mathbb{K}, \{v_n\} \in \mathbb{M}(d)} D_{\mathbb{K}, \mathbb{M}(d)} \varphi(x(t)) \) holds \( \forall t \in I \).

This lemma together with Lemmas 4 and 3 imply the corollary, which is useful for Lyapunov analysis.

**Corollary 1.** Let a set-valued function \( F : \mathbb{R}^{n+1} \to 2^{\mathbb{R}^n} \) be defined and upper-semicontinuous on \( I \times \Omega \) and a set \( F(t, x) \) be nonempty, compact and convex for all \( (t, x) \in \Omega \), where \( \Omega \subseteq \mathbb{R}^n \) is an open nonempty set. Let \( x(t, t_0, x_0) \) be an arbitrary solution of Cauchy problem (4), (7) defined on \([t_0, t_0 + \alpha] \), where \( t_0 \in \mathbb{I}, x_0 \in \Omega \) and \( \alpha \in \mathbb{R}^+_+ \). Let a function \( \varphi : \mathbb{R}^n \to \mathbb{R}^p \) be nonnegative on \( \Omega \). If the inequality \( D_{\mathbb{K}, \mathbb{M}(d)} \varphi(x(t, t_0, x_0)) \leq 0 \) holds for every \( t \in I \) and every \( x \in \Omega \), then the function of time \( \varphi(x(t, t_0, x_0)) \) is decreasing on \([t_0, t_0 + \alpha] \), where
\[
D_{\mathbb{K}, \mathbb{M}(d)} \varphi(x(t)) = \bigcup_{d \in \mathbb{K}, \{v_n\} \in \mathbb{M}(d)} D_{\mathbb{K}, \mathbb{M}(d)} \varphi(x(t)). \quad (12)
\]

### 5. DISCONTINUOUS LYAPUNOV FUNCTION METHOD FOR FINITE-TIME STABILITY ANALYSIS

Lyapunov function method is a very effective tool for analysis and design of both linear and nonlinear control systems. Bacciotti and Rosier [2005]. Initially, the method was presented for ”unrated” (Lyapunov and asymptotic) stability analysis Lyapunov [1992]. A development of control theory had required to study a convergence rate together with a stability properties of a control system. This section presents the discontinuous Lyapunov function method for finite-time stability analysis. Let us refund initially the Zubov’s theorems about Lyapunov and asymptotic stability using the contingent derivative approach.

#### 5.1 Analysis of Lyapunov and asymptotic stability

Recall that a continuous function \( W : \mathbb{R}^n \to \mathbb{R} \) is defined on \( \mathbb{R}^n \) and is said to be **positive definite** iff \( W(0) = 0 \) and \( W(x) > 0 \) for \( x \in \mathbb{R}^n \setminus \{0\} \).
Then the origin of the system (4) is asymptotically stable.

Theorem 3. For a given number $r \in \mathbb{R}$ and a given positive definite function $W : \mathbb{R}^n \to \mathbb{R}$, there exists a globally proper function $V : \mathbb{R}^n \to \mathbb{R}$ if
\[
V(x) \leq W(x) \quad \text{for} \quad x \in \mathbb{R}^n.
\]

Moreover, the practice shows that designing of a Lyapunov function for nonlinear and/or discontinuous system is a nontrivial problem even for a two-dimensional case. Frequently, in order to analyze stability of a sliding mode control system it is simpler to design initially a non-smooth Lyapunov function (see, for example, Utkin et al. [2009], Polyakov and Poznyak [2012], Moreno and Osorio [2012]) and after then to use some technique for its smoothing, see, for example, Clarke et al. [1998].

5.2 Lyapunov analysis of finite-time stability

Analysis of finite-time stability using the Lyapunov function method allows us to estimate of a settling time a priori. The proof of the next theorem follows the ideas introduced in Roxin [1966], Polyakov and Poznyak [2012].

Theorem 5. Let a function $V : \mathbb{R}^n \to \mathbb{R}$ be proper on an open nonempty set $\Omega \subseteq \mathbb{R}^n : 0 \in \text{int}(\Omega)$ and
\[
D_{F(t,x)}^2 V(x) \leq -q \quad \text{for} \quad t \in \mathbb{R} \quad \text{and} \quad x \in \Omega \backslash \{0\},
\]

for some $q \in \mathbb{R}^+$. Then the origin of the system (4) is finite-time stable with an attraction domain $\mathcal{U}$ defined by
\[
T(x_0) \leq V(x_0)/q \quad \text{for} \quad x_0 \in \mathcal{U}, \quad \text{where} \quad T(\cdot) \quad \text{is the settling-time function.}
\]

If a function $V$ is globally proper on $\Omega = \mathbb{R}^n$ then the inequality (15) implies global finite-time stability of the system (4).

Obviously, a settling-time function of finite-time stable system is a Lyapunov function in a generalized sense. Theorem 5 operates with a very large class Lyapunov functions. However, its conditions are still rather conservative. For example, the settling-time function from the next example does not satisfy Theorem 5. However, its simple discontinuous modification transforms the corresponding settling time function to a Lyapunov function in the sense of Theorem 5.

Example 3. Consider the so-called twisting system (Levant [1993])
\[
\begin{align*}
\dot{x}_1 &= F(x_1, x_2), \\
\dot{x}_2 &= -2(\text{sign}[x_1] + \sqrt{2} |x_2|),
\end{align*}
\]

which is finite-time stable with the settling-time function (Polyakov and Poznyak [2012]):
\[
T_{sw}(x_1, x_2) = p \sqrt{|x_1| + \frac{x_2^2}{2(2 + \text{sign}[x_1] + \sqrt{2}|x_2|)}}
\]

where $p = 4\sqrt{2}/(3 - \sqrt{3})$. The function $T_{sw}$ is globally proper, Lipschitz continuous outside the origin and continuously differentiable for $x_1x_2 \neq 0$
\[
D_{F}T_{sw} - \frac{\partial T_{sw}}{\partial x_1} x_2 - \frac{\partial T_{sw}}{\partial x_2} (2 \text{sign}[x_1] + \sqrt{2} \text{sign}[x_1] + \text{sign}[x_1] x_2) = -1.
\]

In the same time, $T_{sw}(x_1, x_2)$ does not satisfy (15), since $D_{F}T_{sw}(0, x_2) \not\equiv 0$ for $x_1 = 0$. Applying generalized Clark’s gradient (see, for example, Clarke [1990]) does not solve this problem.

However, if $p > 4\sqrt{2}/(3 - \sqrt{3})$ then the function $T_{sw}(x_1, x_2)$ is discontinuous on the line $x_1 = 0$ for $x_2 \neq 0$, but it is globally proper and satisfies the conditions of Theorem 5. Indeed, the equality (17) still holds and $D_{F}T_{sw}(0, x_2) = -1/3$ for $x_2 \neq 0$. The Fig. 3 depicts a trajectory and the discontinuous Lyapunov function for twisting system.

**Example 3.** Consider the so-called twisting system (Levant [1993])
\[
\begin{align*}
\dot{x}_1 &= F(x_1, x_2), \\
\dot{x}_2 &= -2(\text{sign}[x_1] + \sqrt{2} |x_2|),
\end{align*}
\]

which is finite-time stable with the settling-time function (Polyakov and Poznyak [2012]):
\[
T_{sw}(x_1, x_2) = p \sqrt{|x_1| + \frac{x_2^2}{2(2 + \text{sign}[x_1] + \sqrt{2}|x_2|)}}
\]

where $p = 4\sqrt{2}/(3 - \sqrt{3})$. The function $T_{sw}$ is globally proper, Lipschitz continuous outside the origin and continuously differentiable for $x_1x_2 \neq 0$
\[
D_{F}T_{sw} - \frac{\partial T_{sw}}{\partial x_1} x_2 - \frac{\partial T_{sw}}{\partial x_2} (2 \text{sign}[x_1] + \sqrt{2} \text{sign}[x_1] + \text{sign}[x_1] x_2) = -1.
\]

In the same time, $T_{sw}(x_1, x_2)$ does not satisfy (15), since $D_{F}T_{sw}(0, x_2) \not\equiv 0$ for $x_1 = 0$. Applying generalized Clark’s gradient (see, for example, Clarke [1990]) does not solve this problem.

However, if $p > 4\sqrt{2}/(3 - \sqrt{3})$ then the function $T_{sw}(x_1, x_2)$ is discontinuous on the line $x_1 = 0$ for $x_2 \neq 0$, but it is globally proper and satisfies the conditions of Theorem 5. Indeed, the equality (17) still holds and $D_{F}T_{sw}(0, x_2) = -1/3$ for $x_2 \neq 0$. The Fig. 3 depicts a trajectory and the discontinuous Lyapunov function for twisting system.
6. CONCLUSIONS

The paper presents the discontinuous Lyapunov function method for non-asymptotic stability analysis of differential inclusions. The example of discontinuous Lyapunov function is presented for twisting control system in order to demonstrate some advances of the method.

REFERENCES


