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# Simultaneous generation for zeta values by the Markov-WZ method

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By application of the Markov-WZ method, we prove a more general form of a bivariate generating function identity containing, as particular cases, Koecher's and Almkvist-Granville's Apéry-like formulae for odd zeta values. As a consequence, we get a new identity producing Apéry-like series for all  $\zeta(2n + 4m + 3)$ ,  $n, m \geq 0$ , convergent at the geometric rate with ratio  $2^{-10}$ .

**Keywords:** Riemann zeta function, Apéry-like series, generating function, convergence acceleration, Markov-Wilf-Zeilberger method, Markov-WZ pair.

## 1 Introduction

The Riemann zeta function is defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{for } \operatorname{Re}(s) > 1. \quad (1)$$

Apéry's irrationality proof of  $\zeta(3)$  [14] operates with the faster convergent series

$$\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} \quad (2)$$

first obtained by A. A. Markov in 1890 [10]. The general formula giving analogous series for all  $\zeta(2s+3)$ ,  $s \geq 0$ , was proved by Koecher [7] (and independently in an expanded form by Leshchiner [9]). It reads

$$\sum_{s=0}^{\infty} \zeta(2s+3)x^{2s} = \sum_{k=1}^{\infty} \frac{1}{k(k^2-x^2)} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} \frac{5k^2-x^2}{k^2-x^2} \prod_{m=1}^{k-1} \left(1 - \frac{x^2}{m^2}\right). \quad (3)$$

A similar identity generating fast convergent series for all  $\zeta(4s+3)$ ,  $s \geq 0$ , which for  $s > 1$  is different from Koecher's result (3), was experimentally discovered in [3] and proved by G. Almkvist and A. Granville in [1]

$$\sum_{s=0}^{\infty} \zeta(4s+3)x^{4s} = \sum_{k=1}^{\infty} \frac{k}{k^4-x^4} = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\binom{2k}{k}} \frac{k}{k^4-x^4} \prod_{m=1}^{k-1} \left(\frac{m^4+4x^4}{m^4-x^4}\right). \quad (4)$$

There exists a bivariate unifying formula for identities (3) and (4),

$$\sum_{k=1}^{\infty} \frac{k}{k^4 - x^2k^2 - y^4} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k \binom{2k}{k}} \frac{5k^2 - x^2}{k^4 - x^2k^2 - y^4} \prod_{m=1}^{k-1} \left( \frac{(m^2 - x^2)^2 + 4y^4}{m^4 - x^2m^2 - y^4} \right), \quad (5)$$

which was first conjectured by H. Cohen and then proved by D. Bradley [5] and, independently, by T. Rivoal [15]. This identity implies (3) if  $y = 0$ , and gives (4) if  $x = 0$ . The proof of (5) given in [5, 15] relies on Borwein and Bradley’s method [3] and consists of reduction of (5) to a finite non-trivial combinatorial identity which can be proved on the basis of Almkvist and Granville’s work [1].

Recently, in [6] it was shown that Koecher’s formula (3), and similarly Leschiner’s and the identities of Bailey, Borwein and Bradley [9, 4] generating accelerated series for even zeta values  $\zeta(2n + 2)$ , can be proved by means of the WZ method.

Formulas (3)-(5) generate accelerated series for odd zeta values and, in particular, series (2) for  $\zeta(3)$  which converge at a geometric rate with ratio  $1/4$ . Many other more rapidly convergent expressions for  $\zeta(3)$  can be proved on the basis of the WZ method. The following series, for example, convergent at the geometric rate with ratio  $2^{-10}$ ,

$$\zeta(3) = \sum_{n=0}^{\infty} (-1)^n \frac{n!^{10}(205n^2 + 250n + 77)}{64(2n + 1)!^5} \quad (6)$$

was obtained by T. Amdeberhan and D. Zeilberger [2] by application of WZ-pairs. There are even faster convergent representations for  $\zeta(3)$  with ratios  $10^{-5}, 10^{-8}$  (see [11]). In [6] it was shown how to get such fast convergent series explicitly for other values  $\zeta(n), n > 3$ . This can be accomplished by applying the WZ method not to the series (1) itself but to a generating function of a sequence of zeta values.

In this note, we prove a more general form of the bivariate identity (5) by application of the Markov-WZ method. We show that identity (5) and the series (6) of Amdeberhan and Zeilberger can be proved with the help of the same Markov-WZ pair, but using different summation formulas. Moreover, we get a new identity generating accelerated series for all  $\zeta(2n + 4m + 3), n, m \geq 0$ , convergent at a geometric rate with ratio  $2^{-10}$ .

## 2 Statement of the main results

We start by giving several definitions, and by reviewing known facts related to the Markov-Wilf-Zeilberger theory (see [8, 10, 11, 12]).

A function  $H(n, k)$ , in the integer variables  $n$  and  $k$ , is called *hypergeometric* or *closed form (CF)* if the quotients

$$\frac{H(n + 1, k)}{H(n, k)} \quad \text{and} \quad \frac{H(n, k + 1)}{H(n, k)}$$

are both rational functions of  $n$  and  $k$ . A hypergeometric function that can be written as a ratio of products of factorials is called *pure-hypergeometric*. A pair of CF functions  $F(n, k)$  and  $G(n, k)$  is called a *WZ-pair* if

$$F(n + 1, k) - F(n, k) = G(n, k + 1) - G(n, k). \quad (7)$$

A *P*-recursive function is a function that satisfies a linear recurrence relation with polynomial coefficients. If for a given hypergeometric function  $H(n, k)$ , there exists a polynomial  $P(n, k)$  in  $k$  of the form

$$P(n, k) = a_0(n) + a_1(n)k + \cdots + a_L(n)k^L,$$

for some non-negative integer  $L$ , and *P*-recursive functions  $a_0(n), \dots, a_L(n)$  such that  $F(n, k) := H(n, k)P(n, k)$  satisfies (7) with some function  $G$ , then a pair  $(F, G)$  is called a *Markov-WZ pair* associated with the kernel  $H(n, k)$  (MWZ-pair for short). We call  $G(n, k)$  an *MWZ mate* of  $F(n, k)$ .

In 2005, M. Mohammed [11] showed that for any pure-hypergeometric kernel  $H(n, k)$ , there exists a non-negative integer  $L$  and a polynomial  $P(n, k)$  as above such that  $F(n, k) = H(n, k)P(n, k)$  has an MWZ mate  $G(n, k) = F(n, k)Q(n, k)$ , where  $Q(n, k)$  is a ratio of two *P*-recursive functions.

From relation (7) we get the following summation formulas.

**Proposition 1 ([11, Theorem 2(b)])** *Let  $(F, G)$  be an MWZ-pair. If  $\lim_{n \rightarrow \infty} F(n, k) = 0$  for every  $k \geq 0$ , then*

$$\sum_{k=0}^{\infty} F(0, k) - \lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} G(n, k) = \sum_{n=0}^{\infty} G(n, 0), \quad (8)$$

whenever both sides converge.

**Proposition 2 ([11, Cor. 2])** *Let  $(F, G)$  be an MWZ-pair. If  $\lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} G(n, k) = 0$ , then*

$$\sum_{k=0}^{\infty} F(0, k) = \sum_{n=0}^{\infty} (F(n, n) + G(n, n+1)), \quad (9)$$

whenever both sides converge.

Formulas (8), (9) with an appropriate choice of MWZ-pairs can be used to convert a given hypergeometric series into a different rapidly converging one.

Let  $(\lambda)_{\nu}$  be the Pochhammer symbol (or the shifted factorial) defined by

$$(\lambda)_{\nu} = \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1, & \nu = 0; \\ \lambda(\lambda + 1) \cdots (\lambda + \nu - 1), & \nu \in \mathbb{N}. \end{cases}$$

Let  $a, b$  be complex numbers such that  $|a| < 1, |b| < 1$ . In Section 3, we construct a Markov-WZ pair associated with the kernel

$$H(n, k) = \frac{(1+a)_k(1-a)_k(1+b)_k(1-b)_k}{(1+a)_{n+k+1}(1-a)_{n+k+1}(1+b)_{n+k+1}(1-b)_{n+k+1}}$$

and then apply Propositions 1, 2 to get the following two theorems.

**Theorem 1** *Let  $a, b$  be complex numbers, with  $|a| < 1, |b| < 1$ . Then for arbitrary complex numbers  $A_0, B_0, C_0$  we have*

$$\sum_{k=1}^{\infty} \frac{A_0 + B_0k + C_0k^2}{(k^2 - a^2)(k^2 - b^2)} = \sum_{n=1}^{\infty} \frac{d_n}{\prod_{m=1}^n (m^2 - a^2)(m^2 - b^2)},$$

with

$$d_n = \frac{(-1)^{n-1} B_0 (5n^2 - a^2 - b^2)}{2n \binom{2n}{n}} \prod_{m=1}^{n-1} ((m^2 - a^2 - b^2)^2 - 4a^2 b^2) \\ + \frac{20n + 5}{2(5n^2 - 2a^2 - 2b^2)} L_n + \frac{35n^5 - 35n^3(a^2 + b^2) + 4n(3a^4 + 3b^4 - 4a^2 b^2)}{4(5n^2 - 2a^2 - 2b^2)} L_{n-1},$$

where  $L_n$  is a solution of the second order difference equation

$$4(4n + 3)(4n + 5)(5n^2 - 2a^2 - 2b^2)L_{n+1} + 2(n + 1)p(n)L_n \\ - n(n + 1)(5(n + 1)^2 - 2a^2 - 2b^2)q(n)L_{n-1} = 0, \quad n = 1, 2, \dots$$

with initial conditions  $L_0 = C_0$ ,

$$L_1 = \left( \frac{1}{3} - \frac{2}{15}(a^2 + b^2) \right) A_0 + \left( \frac{1}{6}(a^2 + b^2) - \frac{2}{15}(a^4 + b^4 - 4a^2 b^2) - \frac{1}{30} \right) C_0,$$

whose growth is described by the inequality

$$\lim_{n \rightarrow \infty} \left( \frac{|L_n|}{n!^4} \right)^{\frac{1}{n}} \leq \frac{1}{4},$$

and

$$p(n) = 30n^7 + 105n^6 + n^5(145 - 52(a^2 + b^2)) + n^4(100 - 130(a^2 + b^2)) \\ + n^3(35 - 124(a^2 + b^2) + 56(a^4 + b^4) - 208a^2 b^2) + n^2(5 - 56(a^2 + b^2) \\ + 84(a^4 + b^4) - 312a^2 b^2) + n(80a^2 b^2(a^2 + b^2) - 16(a^6 + b^6) + 48(a^4 + b^4 - 3a^2 b^2) \\ - 14(a^2 + b^2)) + (10(a^2 - b^2)^2 - 2(a^2 + b^2) + 40a^2 b^2(a^2 + b^2) - 8(a^6 + b^6)), \quad (10)$$

$$q(n) = n^8 - 6n^6(a^2 + b^2) + n^4(9(a^4 + b^4) + 30a^2 b^2) \\ - n^2(28a^2 b^2(a^2 + b^2) + 4(a^6 + b^6)) + 16a^2 b^2(a^2 - b^2)^2. \quad (11)$$

If in Theorem 1 we take  $B_0 = 1$ ,  $A_0 = C_0 = 0$ , then  $L_n = 0$  for all  $n \geq 0$  and we get

$$\sum_{k=1}^{\infty} \frac{k}{(k^2 - a^2)(k^2 - b^2)} \\ = \sum_{n=1}^{\infty} \frac{(5n^2 - a^2 - b^2)(1 + a + b)_{n-1}(1 + a - b)_{n-1}(1 - a + b)_{n-1}(1 - a - b)_{n-1}}{2(-1)^{n-1} n \binom{2n}{n} (1 + a)_n(1 - a)_n(1 + b)_n(1 - b)_n}. \quad (12)$$

If we now put

$$a^2 = \frac{x^2 + \sqrt{x^4 + 4y^4}}{2}, \quad b^2 = \frac{x^2 - \sqrt{x^4 + 4y^4}}{2}, \quad (13)$$

we get the bivariate identity (5) conjectured by H. Cohen.

If  $A_0 = 1, B_0 = C_0 = a = b = 0$ , we get the following series for  $\zeta(4)$  mentioned by Markov in [10, p.18]:

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^{14}} \left( \frac{4n+1}{2n^2} L_n + \frac{7n^3}{4} L_{n-1} \right),$$

where  $L_0 = 0, L_1 = 1/3$ , and

$$4(4n+3)(4n+5)L_{n+1} + 2(n+1)^3(6n^3 + 9n^2 + 5n+1)L_n - n^7(n+1)^3L_{n-1} = 0, \quad n \geq 1.$$

**Theorem 2** *Let  $x, y$  be complex numbers such that  $|x|^2 + |y|^4 < 1$ . Then*

$$\sum_{k=1}^{\infty} \frac{k}{k^4 - x^2k^2 - y^4} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} r(n)}{n \binom{2n}{n}} \frac{\prod_{m=1}^{n-1} ((m^2 - x^2)^2 + 4y^4)}{\prod_{m=n}^{2n} (m^4 - x^2m^2 - y^4)}, \quad (14)$$

where

$$r(n) = 205n^6 - 160n^5 + (32 - 62x^2)n^4 + 40x^2n^3 + (x^4 - 8x^2 - 25y^4)n^2 + 10y^4n + y^4(x^2 - 2).$$

Since

$$\sum_{k=1}^{\infty} \frac{k}{k^4 - x^2k^2 - y^4} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{n+m}{n} \zeta(2n+4m+3) x^{2n} y^{4m},$$

the formula (14) generates Apéry-like series for all  $\zeta(2n+4m+3)$ ,  $n, m \geq 0$ , convergent at the geometric rate with ratio  $2^{-10}$ . So, for example, if  $x = y = 0$ , we get Amdeberhan and Zeilberger's series (6) for  $\zeta(3)$ ,

$$\zeta(3) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (205n^2 - 160n + 32)}{n^5 \binom{2n}{n}^5}.$$

If  $y = 0$ , we recover Theorem 4 from [6]. If  $x = 0$ , we find, in particular, the following expression for  $\zeta(7)$ :

$$\begin{aligned} \zeta(7) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n (25n^2 - 10n + 2)}{n^9 \binom{2n}{n}^5} \\ - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n (205n^2 - 160n + 32)}{n^5 \binom{2n}{n}^5} \left( \sum_{m=1}^{2n} \frac{1}{m^4} + \sum_{m=1}^{n-1} \frac{3}{m^4} \right). \end{aligned}$$

### 3 Proof of Theorem 1.

Let  $a, b$  be complex numbers such that  $|a| < 1, |b| < 1$ . Let

$$H(n, k) = \frac{(1+a)_k (1-a)_k (1+b)_k (1-b)_k}{(1+a)_{n+k+1} (1-a)_{n+k+1} (1+b)_{n+k+1} (1-b)_{n+k+1}}$$

We are interested in finding a Markov-WZ pair associated with  $H(n, k)$ . For this purpose, we define the function  $F(n, k) = H(n, k)P(n, k)$ , where  $P(n, k)$  is a polynomial in  $k$  of degree  $L_1$  with unknown coefficients as functions of  $n$ . Then

$$F(n+1, k) - F(n, k) = \frac{(1+a)_k(1-a)_k(1+b)_k(1-b)_k}{(1+a)_{n+k+2}(1-a)_{n+k+2}(1+b)_{n+k+2}(1-b)_{n+k+2}} P_1(n, k), \quad (15)$$

where  $P_1(n, k)$  is a polynomial in  $k$  of degree  $L_1 + 4$ . From (15) it follows that we can determine a MWZ mate of  $F(n, k)$  in the form  $G(n, k) = H(n, k)Q(n, k)$ , where  $Q(n, k)$  is a polynomial in  $k$  of degree  $L_2$  with unknown coefficients as functions of  $n$ . Indeed, for such a choice we have

$$G(n, k+1) - G(n, k) = \frac{(1+a)_k(1-a)_k(1+b)_k(1-b)_k}{(1+a)_{n+k+2}(1-a)_{n+k+2}(1+b)_{n+k+2}(1-b)_{n+k+2}} Q_1(n, k),$$

where  $Q_1(n, k)$  is a polynomial in  $k$  of degree  $L_2 + 3$ . Therefore,  $(F, G)$  is a Markov-WZ pair if and only if

$$P_1(n, k) = Q_1(n, k) \quad \text{identically for all } n, k. \quad (16)$$

This implies that  $L_2 = L_1 + 1$ . On the other hand, equating coefficients of powers of  $k$  on both sides of (16), we get a system of  $L_1 + 5$  linear homogeneous equations with  $L_1 + L_2 + 2 = 2L_1 + 3$  unknowns. In order to guarantee a solution, we should at least have that  $2L_1 + 3 \geq L_1 + 5$  and hence  $L_1 \geq 2, L_2 \geq 3$ .

We now show that there is a non-zero solution of (16) with the optimal choice  $L_1 = 2, L_2 = 3$ . To see this, define two functions

$$\begin{aligned} F(n, k) &= H(n, k)(A(n) + B(n)(k+1) + C(n)(k+1)^2), \\ G(n, k) &= H(n, k)(D(n) + E(n)k + K(n)k^2 + L(n)k^3), \end{aligned}$$

with 7 unknown coefficients  $A(n), B(n), C(n), D(n), E(n), K(n), L(n)$  as functions of  $n$ . We require that

$$F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k). \quad (17)$$

Substituting  $F, G$  into (17) and cancelling common factors we get that (17) is equivalent to the following equation of degree 6 in the variable  $k$  :

$$\begin{aligned} &((n+k+2)^2 - a^2)((n+k+2)^2 - b^2)(A(n) + B(n)(k+1) + C(n)(k+1)^2) \\ &- A(n+1) - B(n+1)(k+1) - C(n+1)(k+1)^2 = ((n+k+2)^2 - a^2) \\ &\times ((n+k+2)^2 - b^2)(D(n) + E(n)k + K(n)k^2 + L(n)k^3) - ((k+1)^2 - a^2) \\ &\times ((k+1)^2 - b^2)(D(n) + E(n)(k+1) + K(n)(k+1)^2 + L(n)(k+1)^3). \end{aligned} \quad (18)$$

To satisfy condition (17), all the coefficients of the powers of  $(k+1)$  in the equation (18) must be identically zero. This leads to a system of first order linear recurrence equations with polynomial coefficients for  $A(n), B(n), C(n), D(n), E(n), K(n), L(n)$

$$C(n) = (4n_1 - 3)L(n), \quad B(n) = (4n_1 - 2)K(n) - (10n_1^2 - 3)L(n), \quad n_1 = n + 1, \quad (19)$$

$$A(n) = (4n_1 - 1)E(n) - (10n_1^2 - 1)K(n) + (20n_1^3 + 2n_1(a^2 + b^2) - 1)L(n), \quad (20)$$

$$4D(n) = 10n_1E(n) - (20n_1^2 + 2a^2 + 2b^2)K(n) + (35n_1^3 + 11n_1(a^2 + b^2))L(n), \quad (21)$$

$$2(4n_1 + 1)L(n + 1) = 2n_1(5n_1^2 - 2a^2 - 2b^2)E(n) - 2n_1^2(15n_1^2 - 6(a^2 + b^2))K(n) \\ + n_1(63n_1^4 - 17n_1^2(a^2 + b^2) - 4(a^4 + b^4))L(n), \quad (22)$$

$$2(4n_1 + 2)K(n + 1) - 2(10n_1^2 + 20n_1 + 7)L(n + 1) \\ = 2n_1^2(5n_1^2 - 2(a^2 + b^2))E(n) - 2(16n_1^5 - 8n_1^3(a^2 + b^2) + n_1(a^2 - b^2)^2)K(n) \\ + (70n_1^6 - 31n_1^4(a^2 + b^2) + n_1^2(3a^4 + 3b^4 - 14a^2b^2))L(n), \quad (23)$$

$$4(20(n_1 + 1)^3 + 2(n_1 + 1)(a^2 + b^2) - 1)L(n + 1) - 4(10n_1^2 + 20n_1 + 9)K(n + 1) \\ + 4(4n_1 + 3)E(n + 1) = (6n_1^5 - 6n_1^3(a^2 + b^2) + 16a^2b^2n_1)E(n) \\ - (20n_1^6 - 22n_1^4(a^2 + b^2) + 2n_1^2(a^4 + b^4 + 22a^2b^2))K(n) \\ + (45n_1^7 - 48n_1^5(a^2 + b^2) + n_1^3(3a^4 + 3b^4 + 86a^2b^2) + 8a^2b^2n_1(a^2 + b^2))L(n). \quad (24)$$

Now multiplying equation (22) by  $n_1$  and subtracting from (23), we get

$$2K(n + 1) - 7(n_1 + 1)L(n + 1) = -\frac{n_1((n_1^2 - a^2 - b^2)^2 - 4a^2b^2)}{2(2n_1 + 1)}(2K(n) - 7n_1L(n)),$$

which yields

$$K(n) - \frac{7}{2}n_1L(n) = \frac{(-1)^n(2K(0) - 7L(0))n!}{2^{n_1}} \prod_{m=1}^n \left( \frac{(m^2 - a^2 - b^2)^2 - 4a^2b^2}{2m + 1} \right).$$

From (19) it follows that  $2K(0) = B(0) + 7L(0)$ , and therefore we have

$$K(n) = \frac{7}{2}n_1L(n) + \frac{(-1)^nB(0)n!}{2^{n_1}} \prod_{m=1}^n \left( \frac{(m^2 - a^2 - b^2)^2 - 4a^2b^2}{2m + 1} \right). \quad (25)$$

Substitution of (25) into (22) yields the formula

$$E(n) = \frac{4n_1 + 1}{n_1(5n_1^2 - 2a^2 - 2b^2)}L(n + 1) + \frac{42n_1^4 - 25n_1^2(a^2 + b^2) + 4(a^4 + b^4)}{2(5n_1^2 - 2a^2 - 2b^2)}L(n) \\ + \frac{3B(0)(-1)^nn_1!}{2^{n_1}} \prod_{m=1}^n \left( \frac{(m^2 - a^2 - b^2)^2 - 4a^2b^2}{2m + 1} \right). \quad (26)$$

Substitution of (25) and (26) into (21) gives the formula

$$D(n) = \frac{(40n_1 + 10)L(n + 1) + (35n_1^5 - 35n_1^3(a^2 + b^2) + 4n_1(3a^4 + 3b^4 - 4a^2b^2))L(n)}{4(5n_1^2 - 2a^2 - 2b^2)} \\ + \frac{(-1)^nB(0)n!(5n_1^2 - a^2 - b^2)}{2^{n_1+1}} \prod_{m=1}^n \left( \frac{(m^2 - a^2 - b^2)^2 - 4a^2b^2}{2m + 1} \right). \quad (27)$$



Finally, substitution of (25), (26) into (24) gives the second-order difference equation

$$4(4n+3)(4n+5)(5n^2-2a^2-2b^2)L(n+1)+2(n+1)p(n)L(n) \\ -n(n+1)(5(n+1)^2-2a^2-2b^2)q(n)L(n-1)=0, \quad n=1,2,\dots,$$

with initial conditions  $L(0) = C(0)$ ,

$$L(1) = \left(\frac{1}{3} - \frac{2}{15}(a^2+b^2)\right)A(0) + \left(\frac{1}{6}(a^2+b^2) - \frac{2}{15}(a^4+b^4-4a^2b^2) - \frac{1}{30}\right)C(0),$$

derived from (19), (20), (22), and polynomials  $p(n)$ ,  $q(n)$  defined in (10), (11).

If we put  $l(n) = L(n)/(n!)^4$ ,  $n = 0, 1, 2, \dots$ , then it is easily seen that the sequence  $l(n)$  satisfies the following recurrence equation:

$$4(4n+3)(4n+5)(5n^2-2a^2-2b^2)n^3(n+1)^3l(n+1)+2n^3p(n)l(n) \\ -(5(n+1)^2-2a^2-2b^2)q(n)l(n-1)=0, \quad n=1,2,\dots \quad (28)$$

Its characteristic polynomial  $64\lambda^2 + 12\lambda - 1 = 0$  has two different zeros  $\lambda_1 = -1/4$ ,  $\lambda_2 = 1/16$ . Then, by Poincaré's theorem [13], for each solution  $l(n)$ ,  $n = 0, 1, 2, \dots$ , of (28), either  $l(n) = 0$  for all sufficiently large  $n \geq n_0$ , or the limit  $\lim_{n \rightarrow \infty} l(n+1)/l(n)$  exists and equals one of the roots of the characteristic polynomial. Therefore, in both cases we get that the limit  $\lim_{n \rightarrow \infty} |l(n)|^{1/n}$  exists and does not exceed  $1/4$  or

$$\lim_{n \rightarrow \infty} \left(\frac{|L(n)|}{(n!)^4}\right)^{\frac{1}{n}} \leq \frac{1}{4}. \quad (29)$$

The limit inequality (29) implies

$$\lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} G(n, k) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} F(n, k) = 0 \quad \text{for every } k \geq 0,$$

and therefore, by Proposition 1, we have

$$\sum_{k=0}^{\infty} F(0, k) = \sum_{n=0}^{\infty} G(n, 0),$$

yielding the theorem with  $d_n = D(n-1)$ ,  $L_n = L(n)$ ,  $A_0 = A(0)$ ,  $B_0 = B(0)$ ,  $C_0 = C(0)$ .  $\square$

## 4 Proof of Theorem 2.

To deduce (14) from (17), take  $A(0) = C(0) = 0$ ,  $B(0) = 1$ , and apply Proposition 2 to obtain

$$\sum_{k=0}^{\infty} F(0, k) = \sum_{n=0}^{\infty} (F(n, n) + G(n, n+1)). \quad (30)$$

Since in this case  $L(n) = 0$  for all  $n \geq 0$ , an easy computation of the right-hand side of (30) by (19), (20), (25)–(27), and substitution (13) lead to the desired conclusion.  $\square$

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